The Functional Equation for Poincaré Series of Trace Rings of Generic 2x2-Matrices.

ABSTRACT :

In this note we give a rational expression for the Poincaré series of $\mathbb{T}_{m,2}$, the trace ring of m generic 2x2-matrices. This result extends the computations of E. Formanek for $m \leqslant 4$.

As a consequence, we prove that the Poincaré series satisfies the functional equation :

$$P(\mathbb{T}_{m,2};\frac{1}{t}) = -t^{4m}.P(\mathbb{T}_{m,2},t)$$

supporting the conjecture that $\mathbb{T}_{m,2}$ is a Gorenstein ring.

1. THE RATIONAL EXPRESSION.

Throughout this note, k will be a field of characteritic zero. With R we will denote the polynomial ring $k[x_{ij}(l);1 \leqslant i,j \leqslant n;\ 1 \leqslant l \leqslant m]$.

The sub k-algebra of $M_n(R)$ generated by the elements $\{X_l = (x_{ij}(l))_{i,j}\}$ is $\mathbb{G}_{m,n}$, the ring of m generic nxn-matrices. Adjoining to it the traces of all its elements we obtain the trace ring $\mathbb{T}_{m,n}$ of m generic nxn-matrices, see for example [1], [5]. If $\deg(x_{ij}(l)) = 1$ for all i,j and l, then $\mathbb{T}_{m,n}$ is a positively graded k-algebra $\overset{\infty}{\underset{i=0}{\oplus}}(\mathbb{T}_{m,n})_i$. Its Poincaré series is then the formal power series over $\mathbb{Z}$:

$$P(\mathbb{T}_{m,n};t) = \sum_{i=0}^{\infty} \dim_k((\mathbb{T}_{m,n})_i).t^i$$

Similarly, if $\deg(X_{ij}(l)) = (0,\dots,1,\dots,0) = e_l$, then $\mathbb{T}_{m,n}$ is a $\mathbb{N}^{(m)}$-graded k-algebra. Its Poincaré series in this multigradation is then :

$$P(\mathbb{T}_{m,n};t_1,\dots,t_m) = \sum_{(i_1,\dots,i_m)} \dim_k((\mathbb{T}_{m,n})_{(i_1,\dots,i_m)}).t_1^{i_1} \dots t_m^{i_m}$$

For n = 2, a power series expansion for the Poincaré series was given by C. Procesi [7] and in the multigradation by E. Formanek [2].

We will now give a rational expression for $P(\mathbb{T}_{m,2};t_1,\dots,t_m)$ using some 60 years old results due to H. Weyl [10, p. 11 and p. 17] and I. Schur [8].

Let $R_{m,n}$ denote the center of $\Pi_{m,n}$. In [6] Procesi has proved that the map

$$f \vdash Tr(f.X_{m+1})$$

defines a monomorphism from $\Pi_{m,n}$ onto the subspace of $R_{m+1,n}$ consisting of all elements of degree one in X_{m+1}, i.e. $\sum_{(i_1,\ldots,i_m)} (R_{m+1,n})_{(i_1,\ldots i_m,1)}$.

Translating to Poincaré series, this means that $P(\Pi_{m,n};t_1,\ldots,t_m)$ is the coefficient of t_{m+1} in the power series expansion of $P(R_{m+1,n};t_1,\ldots,t_m,t_{m+1})$ or equivalently.

$$P(\Pi_{m,n};t_1,\ldots,t_m) = \frac{\partial}{\partial t_{m+1}} P(R_{m+1,n};t_1,\ldots,t_m,t_{m+1})\Big|_{t_{m+1}=0}$$

If n = 2 one can give a fairly precise description of $R_{m+1,2}$, [7]. Recall from [7] that $R_{m+1,2}$ is the ring of invariant polynomial mapping from m+1 copies of $M_2(k)$ under the componentwise action by conjugation of $GL_2(k)$. Now, $M_2(k)$ decomposes in the direct sum $k \oplus M^0$ with M^0 the 3-dimensional vector space of trace zero matrices. Therefore, $R_{m+1,2}$ is the polynomial ring in the elements $Tr(X_1),\ldots,Tr(X_{m+1})$ over the ring of invariants of m+1 copies of M^0 under induced action of $GL_2(k)$, $R^0_{m+1,2}$.

M is endowed with the nongenerate quadratic form $Tr(A^2)$, thus $GL_2(k)$ acts on M^0 inducing the full group $SO(M^0)$ of special orthogonal transformations for the form $Tr(A^2)$. Therefore, $R^0_{m+1,2}$ is the ring of special orthogonal invariants of m+1 copies of the standard representation.

The composition of the Poincaré series in the multigradation of this ring was carries out by H. Weyl [10,p.17] and I. Schur [8].

They found the following rational expression :

$$P(R^{o}_{m+1,2};t_1,\dots,t_{m+1}) = \frac{[1+t^{2m-1},\dots,t^{m-2}+t^{m+1};t^{m-1},t^{m}]}{\prod\limits_{i<k}^{m+1}(t_k-t_i)\cdot\prod\limits_{i\leq k}^{m+1}(1-t_it_k)}$$

where the numerator of this expression denotes the determinant of the following $(m+1)\times(m+1)$-matrix :

$$\begin{bmatrix} 1+t_1^{2m-1} & 1+t_2^{2m-1} & \dots & 1+t_{m+1}^{2m-1} \\ t_1+t_1^{2m-2} & t_2+t_2^{2m-2} & \dots & t_{m+1}+t_{m+1}^{2m-2} \\ \vdots & \vdots & & \vdots \\ t_1^{m-2}+t_1^{m+1} & t_2^{m-2}+t_2^{m+1} & \dots & t_{m+1}^{m-2}+t_{m+1}^{m+1} \\ t_1^{m-1} & t_2^{m-1} & \dots & t_{m+1}^{m-1} \\ t_1^{m} & t_2^{m} & \dots & t_{m+1}^{m} \end{bmatrix}$$

Combining the facts we get :

$$P(\Pi_{m,2});t_1,\dots,t_m) = \frac{\partial}{\partial t_{m+1}} \left. \frac{[1+t^{2m-1},\dots,t^{m-2}+t^{m+1};t^{m-1},t^{m}]}{\prod\limits_{j=1}^{m+1}(1-t_j)\prod\limits_{i<k}^{m+1}(t_k-t_i)\prod\limits_{i\leq k}^{m+1}(1-t_it_k)} \right|_{t_{m+1}=0} \qquad (\star)$$

Calculating the numerator of the right hand side of $(\star)$ gives :

$$\prod_{j=1}^{m}(1-t_j)\cdot\prod_{i<k}^{m}(t_k-t_i)\cdot(-1)^m\cdot\prod_{j=1}^{m}t_j\cdot\prod_{i\leq k}^{m}(1-t_it_k)\cdot M_1 -$$

$$\{-\prod_{j=1}^{m}(1-t_j)\cdot\prod_{i<k}^{m}(t_k-t_i)\cdot(-1)^m\cdot\prod_{j=1}^{m}t_j\cdot\prod_{i\leq k}^{m}(1-t_it_k)$$

$$+\prod_{j=1}^{m}(1-t_j)\cdot\prod_{i<k}^{m}(t_k-t_i)\cdot(\sum_{j=1}^{m}(-1)^{m-1}t_1\dots\check{t}_j\dots t_m)\cdot\prod_{i\leq k}^{m}(1-t_it_k)$$

$$+\prod_{j=1}^{m}(1-t_j)\cdot\prod_{i<k}^{m}(t_k-t_i)\cdot(-1)^m\cdot\prod_{j=1}^{m}t_j\cdot\prod_{i\leq k}^{m}(1-t_it_k)\cdot(\sum_{j=1}^{m}-t_j)\}\cdot M_2$$

where M_1 and M_2 are defined to be :

$$M_1=\det\begin{bmatrix} 1+t_1^{2m-1} & \dots & 1+t_m^{2m-1} & 0 \\ \vdots & & \vdots & 1 \\ & & & 0 \\ & & & \vdots \\ t_1^{m-1}+t_1^{m+1} & \dots & t_m^{m-1}+t_m^{m+1} & 0 \\ t_1^{m-1} & \dots & t_m^{m-1} & 0 \\ t_1^{m} & \dots & t_m^{m} & 0 \end{bmatrix} = (-1)^{m+2}.\det\begin{bmatrix} 1+t^{2m-1} & \dots & 1+t_m^{2m-1} \\ t_1^2+t_1^{2m-3} & \dots & t_m^2+t_m^{2m-3} \\ \vdots & & \vdots \\ t_1^{m-1}+t^{m+1} & \dots & t_m^{m-1}+t_m^{m+1} \\ t_1^{m-1} & \dots & t_m^{m-1} \\ t_1^{m} & \dots & t_m^{m} \end{bmatrix} = (-1)^{m+2}.\Delta_1$$

$$M_2=\det\begin{bmatrix} 1+t_1^{2m-1} & \dots & 1+t_m^{2m-1} & 1 \\ \vdots & & \vdots & 0 \\ & & & \vdots \\ t_1^{m-1}+t_1^{m+1} & \dots & t_m^{m-1}+t_m^{m} & 0 \\ t_1^{m-1} & \dots & t_m^{m-1} & 0 \\ t_1^{m} & \dots & t_m^{m} & 0 \end{bmatrix} = (-1)^{m+1}.\det\begin{bmatrix} t_1+t^{2m-2} & \dots & t_m+t_m^{2m-1} \\ \vdots & & \vdots \\ t_1^{m-1}+t_1^{m+1} & \dots & t_m^{m-1}+t_m^{m+1} \\ t_1^{m-1} & \dots & t_m^{m-1} \\ t_1^{m} & \dots & t_m^{m} \end{bmatrix} = (-1)^{m+1}.\Delta_2$$

Therefore, the numerator is equal to :

$$\prod_{j=1}^{m}(1-t_j).\prod_{i<k}^{m}(t_k-t_i).\prod_{i\leq k}^{m}(1-t_it_k)\,[\,e_m.\Delta_1-(e_m+e_{m-1}+e_1e_m).\Delta_2\,]$$

where e_i denotes the i^{th} elementary symmetric function in m variables.

This finishes the proof of :

THEOREM 1. The Poincaré series of the trace ring of m generic 2x2 matrices has the following rational expression :

$$P(\Pi_{m,2},t_1,\dots,t_m) = \frac{e_m.\Delta_1-(e_m+e_1.e_m+e_{m-1}).\Delta_2}{e_m^2.\prod\limits_{j=1}^{m}(1-t_j).\prod\limits_{i<k}^{m}(t_k-t_i).\prod\limits_{i\leq k}^{m}(1-t_it_k)}$$

2. THE FUNCTIONAL EQUATION.

The main result of this section is :

THEOREM 2 : The Poincaré series of the trace ring of m generic 2x2-matrices satisfies the functional equation :

$$P(\Pi_{m,2},\frac{1}{t}) = -t^{4m}.P(\Pi_{m,2};t)$$

PROOF :

We note that :

$$e_m^{2m-1}.\Delta_1(\frac{1}{t_1},\dots,\frac{1}{t_m}) = -\Delta_1(t_1,\dots,t_m)$$

$$e_m^{2m-1}.\Delta_2(\frac{1}{t_1},\dots,\frac{1}{t_m}) = -\Delta_2(t_1,\dots,t_m)$$

$$e_m^2.(e_m+e_1e_m+e_{m-1}(\frac{1}{t_1},\dots,\frac{1}{t_m})) = e_m+e_1.e_m+e_{m-1}$$

and therefore we get :

$$P(\Pi_{m,2};\frac{1}{t_1},\dots,\frac{1}{t_m})$$

$$= \frac{-e_m^{-2m-1}.\{e_m\Delta_1-(e_m+e_1e_m+e_{m-1}).\Delta_2\}}{e_m^{-2m-5}.\{\prod_{j=1}^{m}(t_j-1).\prod_{i<h}^{m}(t_i-t_k).\prod_{i\leq h}^{m}(t_it_k-1)\}}$$

$$= -e_m^4.(-1)^{2m+m^2-m}.P(\Pi_{m,2};t_1,\dots,t_m)$$

Finally, specializing $t_1 = t_2 = \dots = t_m = t$, we get the desired result

In [3] we have shown that there exists an iterated Ore-extension Γ_m and a natural morphism

$$\pi_m : \Gamma_m \twoheadrightarrow \Pi_{m,2}$$

In analogy with the commutative case, we say that $\Pi_{m,2}$ is a Gorenstein ring iff

$$\mathrm{Ext}^i_{\Gamma_m}(\Pi_{m,2},\Gamma_m) = 0 \quad \text{for } i \neq 0, \frac{(m-2)(m-3)}{2}$$

$$\mathrm{Ext}^j_{\Gamma_m}(\Pi_{m,2},\Gamma_m) \cong \Pi_{m,2} \quad \text{for } j = \frac{(m-2)(m-3)}{2}$$

In [4] it is shown that Gorensteinness of $\Pi_{m,2}$ is equivalent to the following :

CONJECTURE : $\Pi_{m,2}$ is a Cohen-Macauley module.

A fact which is very plausible since $\Pi_{m,2}$ is the fixed module of a free module over a regular domain under a reductive group.

As in the commutative case [9], it would follow from the fact that $\Pi_{m,2}$ is Gorenstein that its Poincaré series satisfies the functional equation

$$P(\Pi_{m,2};\frac{1}{t}) = (-1)^{\mathrm{kdim}(\Pi_{m,2})}.t^{\alpha}.P(\Pi_{m,2};t)$$

for some $\alpha \in \mathbb{Z}$

Because $\mathrm{Kdim}(\Pi_{m,2}) = 4m-3$, Theorem 2 supports the above conjecture.

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