Maximal Orders having a Discrete Normalizing Class Group.

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If Λ is a maximal orden over a normal (noetherian !) domain R is some central simple algebra Σ over its field of fractions K, then one may define the normalizing class group $NCl(\Lambda)$ of Λ to be the quotient group of the free abelian group of twosided divisorial Λ-ideals by the subgroup of those divisorial Λ-ideals which are generated by a normalizing element (i.e. an element $x \in \Sigma$ such that $x\Lambda = \Lambda x$), cf. e. g. [2,8,13] . Since twosided fractional Λ- ideals extend to twosided fractional $\Lambda[[t]]$- ideals and $\Lambda[[t]]$ is again a maximal order, cf. [9] , one has a natural morphism $NCl(\Lambda) \to NCl(\Lambda[[t]])$. In this note we first prove that this morphism splits, thus generalizing V.I. Danilov's "Key Lemma". The proof we give is based on a similar proof in the commutative case, due to W. Melchior's [10] . We say that Λ has a discrete normalizing class group if the above morphism is actually isomorphic. We then study whether having a normalizing class group is a central property, i.e. we would like to answer the following questions :

A. If R has a discrete class group, does it follow that Λ has a discrete class group?

B. If Λ has a discrete normalizing class group, does it follow that R has a discrete class group ?

Problem A will be answered affirmatively, as well as Problem B in case Λ is an Azumaya algebra over R. Even if Λ is a reflexive Azumaya algebra it is not clear to the authors whether Problem B should be true.

1. The natural splitting.

Throughout this note, Λ denotes a maximal order over a normal R; $X^{(1)}(R)$ is the set of height one prime ideals of R. Let $p \in X^{(1)}(R)$ and denote by (p,t) the prime ideal in $R[[t]]$ generated by p and t; the subscript

$(\ )_{(p,t)}$ denotes localization at $R[[t]]-(p,t)$. Since $R_p[[t]]$ is the (t)-adic completion of its subring $R[[t]]_{(p,t)}$, it follows from the fact that $R[[t]]_{(p,t)}$ is a Zariski ring that $R_p[[t]]$ is a faithfully flat $R[[t]]_{(p,t)}$-algebra.

As $\Lambda_p[[t]]$ is a quasi-local maximal order over the regular local ring $R_p[[t]]$, every left divisorial $\Lambda_p[[t]]$-ideal is principal, cfr. [12, Prop. 5.4.].

<u>Lemma 1</u> : If Λ is a maximal order over a normal domain R, then $\mathrm{NCl}(\Lambda[[t]]_{(p,t)}) = 1$ for every $p \in X^{(1)}(R)$.

<u>Proof.</u>

Consider the commutative diagram :

$$\begin{array}{ccc} R[[t]]_{(p,t)} & \xrightarrow{\ u\ } & R_p[[t]] \\ \downarrow & & \downarrow \\ \Lambda[[t]]_{(p,t)} & \xrightarrow[\ v\ ]{} & \Lambda_p[[t]] \end{array}$$

Because Λ is finitely generated over R, we have that $\Lambda[[t]]_{(p,t)} = \Lambda \otimes_R R[[t]]_{(p,t)}$ and $\Lambda[[t]] = \Lambda \otimes_R R_p[[t]]$; since we also have that $\Lambda_p[[t]] = R_p[[t]] \otimes_{R[[t]]_{(p,t)}} \Lambda[[t]]_{(p,t)}$, it follows that $v = \Lambda[[t]]_{(p,t)} \otimes_{R[[t]]_{(p,t)}} u$ is faithfully flat and therefpre v is an extension which satisfies PDE for twosided height one prime ideals.

Therefore, v induces a morphism $\mathbb{D}(\Lambda[[t]]_{(p,t)}) \to \mathbb{D}(\Lambda_p[[t]]$ by sending I to $I\Lambda_p[[t]] = \Lambda_p[[t]]I$. Assume that $I \in \mathbb{D}(\Lambda[[t]]_{(p,t)})$ is mapped to a principal ideal of $\Lambda_p[[t]]$, then from $I/(t)I \cong I\Lambda_p[[t]]/(t)I\Lambda_p[[t]]$ and $\Lambda[[t]]_{(p,t)}/(t)\Lambda[[t]]_{(p,t)} \cong \Lambda_p[[t]]/(t)\Lambda_p[[t]]$ it follows $I/(t)I$ is a free $\Lambda[[t]]_{(p,t)}/(t)\Lambda[[t]]_{(p,t)}$- module (e.g. on the left).

The map

$$(t)\ \Lambda[[t]]_{(p,t)} \otimes I \to I$$

is injective, since it becomes so, after tensoring with $\Lambda[[t]]$ and because divisorial $\Lambda[[t]]$ -ideals are principal. Applying Prop. 5 in II.3.2. of [12] yields that I is a principal left ideal. □

This is the main ingredient of the proof of the following noncommutative version of Danilov's key lemma.

Theorem 2 : If Λ is a maximal order over a normal domain R, then $NCl(\Lambda)$ is a direct factor of $NCl(\Lambda[[t]])$.

Proof.

Let (t) be the princpal) divisorial ideal of $\Lambda[[t]]$ generated by t. Every element of $NCl(\Lambda[[t]])$ can be represented by a divisorial ideal I not contained in (t). If $j : \Lambda[[t]] \to \Lambda$ is the natural ringmorphism defined by sending t to 0, then we define a map

$$\varphi : NCl(\Lambda[[t]] \to NCl(\Lambda)$$

by $\varphi([I]) = [j(I)^{**}]$. This map is well defined and splits the canonical map $NCl(\Lambda) \to NCl(\Lambda[[t]])$.

We have to check that φ is a groupmorphism. Let I_1, I_2 be divisorial ideals in $\Lambda[[t]]$ not contained in (t), then $\varphi([I_1].[I_2]) = [j(I_1I_2^{**})^{**}]$ and $\varphi([I_1]).\varphi([I_2]) = [(j(I_1)^{**}.j(I_2)^{**})^{**}]$. Note that the biduals are taken in different rings, this should not cause any ambiguity, however. Since, $(j(I_1)^{**}.j(I_2)^{**})^{**} = j(I_1I_2)^{**}$, it is sufficient to show that for all $p \in X^{(1)}(R)$, we have :

$$j(I_1I_2)^{**})_p = j(I_1I_2)_p$$

to derive the conclusion. But as $NCl(\Lambda[[t]]_{(p,t)}) = 1$ by the lemma, it follows that $((I_1I_2)^{**})_{(p,t)} = (I_1I_2)_{(p,t)}$ for all $p \in X^{(1)}(R)$, so for any $f \in (I_1I_2)^{**}$ we may find $g \in R[[t]] - (p,t)$ s. t. $f\,g \in I_1I_2$ and it follows that $j(f) \in j(I_1I_2)_p$. Therefore, $j((I_1I_2)^{**})_p = j(I_1I_2)_p$. □

2. Some applications.

In analogy with the commutative case, we say that a maximal order Λ has a discrete normalizing class group if $\mathrm{NCl}(\Lambda) \cong \mathrm{NCl}(\Lambda[[t]])$ under the canonical map.

Proposition 3 : If Λ is a maximal order over a normal domain with discrete classgroup, then Λ has a discrete normalizing classgroup.

Proof.

The central classgroup of Λ, $\mathrm{CCl}(\Lambda)$, is the quotient group of $\mathbb{D}(\Lambda)$ modulo the subgroup of divisorial Λ-ideals generated by a central element. We have the exact sequence, [7] :

$$1 \to \mathrm{Cl}(R) \overset{\mu}{\to} \mathrm{CCl}(\Lambda) \to \bigoplus_{i=1}^{m} \mathbb{Z}/e_i\,\mathbb{Z} \to 1$$

where $\mathrm{Coker}(\mu)$ is a finite group determined by the height one primes of Λ. The diagram below is commutative

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Cl}(R) & \xrightarrow{\mu} & \mathrm{CCl}(\Lambda) & \longrightarrow & \bigoplus_{i=1}^{m} \mathbb{Z}/e_i\,\mathbb{Z} & \longrightarrow & 1 \\ & & \cong\downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Cl}(R[[t]]) & \xrightarrow{\mu'} & \mathrm{CCl}(\Lambda[[t]]) & \longrightarrow & \bigoplus_{j=1}^{m'} \mathbb{Z}/f_j\,\mathbb{Z} & \longrightarrow & 1 \end{array}$$

Now, let $c \in R$ be an element in the Formanék-center of Λ, then Λ_c is an Azumaya algebra over R_c. This entails that $\Lambda[[t]]_c \cong \Lambda_c \otimes_{R_c} R[[t]]_c$ is an Azumaya algebra over $R[[t]]_c$. Therefore, the only ramified height one primes of $\Lambda[[t]]$ are of the form $P[[t]]$ with P a ramified height one prime of Λ. Therefore, $\mathrm{Coker}(\mu) \cong \mathrm{Coker}(\mu')$ and hence $\mathrm{CCl}(\Lambda) \cong \mathrm{CCl}(\Lambda[[t]])$.

If Outcent (÷) is the group of Outer-automorphism, then the diagram below is commutative and exact :

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \text{Outcent}(\Lambda) & \longrightarrow & \text{CCl}(\Lambda) & \longrightarrow & \text{NCl}(\Lambda) & \longrightarrow & 1 \\
 & & & & \downarrow \cong & & \downarrow \alpha & & \\
1 & \longrightarrow & \text{Outcent}(\Lambda[[t]] & \longrightarrow & \text{CCl}(\Lambda[[t]] & \longrightarrow & \text{NCl}(\Lambda[[t]] & \longrightarrow & 1
\end{array}$$

Whence α is epimorphic. Theorem 2 finishes the proof.

<u>Proposition 4</u> : If Λ is an Azumaya algebra, with a discrete normalizing class group, over a normal domain R, then R has a discrete class group.

<u>Proof.</u>

For any azumaya algebra Γ over a normal domain S one has $\text{CCl}(\Gamma) \cong \text{Cl}(S)$, whence :

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \text{Outcent}(\Lambda) & \longrightarrow & \text{Cl}(R) & \longrightarrow & \text{NCl}(\Lambda) & \longrightarrow & 1 \\
 & & \downarrow \alpha & & \downarrow \beta & & \cong \downarrow & & \\
1 & \longrightarrow & \text{Outcent}(\Lambda[[t]]) & \longrightarrow & \text{Cl}(R[[t]]) & \longrightarrow & \text{NCl}(\Lambda[[t]]) & \longrightarrow & 1
\end{array}$$

is exact and commutative.

For any Azumaya algebra Γ over a normal domain S, $\text{Outcent}(\Gamma)$ consists of torsion elements of the Picard group of S.

Further, for any normal doamin R, the natural morphism

$$\text{Pic}(R) \to \text{Pic } R[[t]]$$

is an isomorphism, showing that α is epimorphic, hence so is β. Injectivity of β follows from Danilov.

It is not known to the authors whether this proposition can be extended to general maximal orders. The problem seens to be that even for a reflexive azumaya algebra Γ, $\text{Outcent}(\Gamma)$ can contain any torsion element of Cl(S) :

Recall from [4] that an algebra Λ over a normal domain R is said to be a reflexive azumaya algebra if the natural map

$$(\Lambda \otimes \Lambda^{opp})^{**} \qquad \mathrm{End}_R(\Lambda)$$

is an isomorphism of R-algebras. For reflexive azumaya algebras one can also prove that $CCl(\Lambda) \cong Cl(R)$ but the difficulty in extending the foregoing proposition to reflexive azumaya algebras is that $\mathrm{Ker}(Cl(R) \to Cl(\Lambda)) \cong \mathrm{Outcent}(\Lambda)$ does not necessarily consist of elements of the Picard group of R. Let us give an example of such a situation :

Example 1 : (cf. [8] also. Let R be a normal domain and let I be a representant of a 2-torsion element in Cl(R). Consider :

$$\Lambda = \mathrm{End}_R(R \oplus I) \cong \begin{pmatrix} R & I \\ I^{-1} & R \end{pmatrix}$$

then Λ is a reflexive Azumaya algebra over R(Λ is Azumaya if and only if $[I] \in \mathrm{Pic}(R)$). This entails that there is a well defined isomorphism :

$$\psi : \mathbb{D}(R) \to \mathbb{D}(\Lambda);\ \psi(A) = (\Lambda.A)^{**}$$

showing that every divisorial Λ-ideal is of the form :

$$\begin{pmatrix} A & I \star A \\ I^{-1} \star A & A \end{pmatrix}$$

where $A \in \mathbb{D}(R)$. ψ induces a morphism

$$\Phi : Cl(R) \to Cl(\Lambda)$$

which is really epimorphic. The class of an ideal A is killed under Φ if and only if :

$$A \oplus (I \star A) \cong R \oplus I$$

the isomorphism being one of R-modules. So, in particular, if we take $A = I$ then $\Phi([I]) = 1$ since

$$\begin{pmatrix} I & I \star I \\ I^{-1} \star I & I \end{pmatrix} = \begin{pmatrix} I & R\alpha \\ R & I \end{pmatrix} = \begin{pmatrix} R & I \\ I^{-1} & R \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$$

Where the element on the right is redily checked to be a nontrivial normalizing element of Λ.

Thus, any 2-torsion element of the class group (resp. of the Picard group) of R can be killed in the normalizing class group of a reflexive Azumaya algebra (resp. Azumaya algebra) over R of p.i. degree 2.

This construction can of course be extended to higher torsion elements

Take $[I] \in Cl(R)_n$, then let

$$\Lambda = End_R(R \oplus I \oplus I^2 \oplus \dots \oplus I^{n-1}$$

and the class of a divisorial ideal A is killed in $Cl(\Lambda)$ if and only if:

$$A \oplus (A \star I) \oplus \dots \oplus (A \star I^{n-1}) \cong R \oplus I \oplus \dots \oplus I^{n-1}$$

So, in particular, taking $A = I$ and $I \star I^{n-1} = R\alpha$ we know that $(\Lambda.I)^{\star\star}$ is generated by the normalizing element

$$\begin{pmatrix} 0 & 0 & \dots & \alpha \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{pmatrix}$$

Conversely, it is of course easy (taking reduced norms) that the kernel of the natural morphism $Cl(R) \to Cl(\Lambda)$ consists of n-torsion elements if n = p.i. degree (Λ).

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