

Trace Rings of Generic Matrices
are Unique Factorization Domains.

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The Functional Equation for Poincaré Series
of Trace Rings of Generic 2x2-Matrices.

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Trace rings of generic matrices are U.F.D.

A.W. Chatters and D.A. Jordan defined in [0] a unique factorization ring to be a prime ring in which every height one prime ideal is principal. In this note we will prove that the trace ring of m generic $n \times n$ -matrices satisfies this condition.

Throughout this note, k will be a field of characteristic zero. Consider the polynomial ring $S = k[t_{ij}^l; 1 \leq i, j \leq n, 1 \leq l \leq m]$ and the $n \times n$ matrices $X_l = (t_{ij}^l)$ in $M_n(S)$. The k -subalgebra of $M_n(S)$ generated by $\{X_l; 1 \leq l \leq m\}$ is called the ring of m generic $n \times n$ matrices $G_{m,n}$. Adjoining to it the traces of all its elements we obtain the trace ring $\overline{\mathbb{T}}_{m,n}$ of m $n \times n$ generic matrices, cfr. e.g. [1].

The main aim of this note is to prove :

Theorem 1. : Height one prime ideals of $\overline{\mathbb{T}}_{m,n}$ are cyclic

M. Artin and A. Schofield [2] have proved that $\overline{\mathbb{T}}_{m,n}$ is always a maximal order, cfr. [3]. The following two results liberate their proof from Hilbert-Mumford theory.

Lemma 2. : Let B be a unique factorization domain and G a group of automorphisms of B s.t. $H^1(G, B^*) = 1$. If A is the fixed ring of B under G , then :

- (a) $A \subset B$ satisfies no blowing up
- (b) A is a unique factorization domain.

Proof.

(a) Let $P = Bp$ be a height one prime of B s.t. $P \cap A \neq 0$. Then P has a finite orbit under G , say $\{Bp, Bp_1, \dots, Bp_k\}$ (take an element $a \in P \cap A$ write $a = p^k q_1^{l_1} \dots q_m^{l_m}$, then $\sigma(Bp) \in \{Bp, Bq_i\}$). This shows that for every $\sigma \in G$ there exists a unit $f_\sigma \in B^*$ s.t. $\sigma(p.p_1 \dots p_k) = f_\sigma.p.p_1 \dots p_k$. $\{f_\sigma; \sigma \in G\}$ is clearly a 1-cocycle, so by assumption there exists a unit $\alpha \in B^*$ s.t. $f_\sigma = \sigma(\alpha).\alpha^{-1}$ for every $\sigma \in G$. Replace p by $p' = \alpha^{-1}.p$, then $p'.p_1 \dots p_k \in A$. Therefore, any element $0 \neq a \in P \cap A$ can be written as $a = (p'.p_1 \dots p_k)^l q_1^{l_1} \dots q_m^{l_m}$, i.e., $a \in (p'.p_1 \dots p_k)A$. So, $P \cap A = (p'.p_1 \dots p_k)A$ and therefore $A \subset B$ satisfies no blowing up

(b) By (a) and [4] we know that the natural map $Cl(A) \rightarrow Cl(B)$ is a homomorphism. Suppose Q is a nonprincipal height one prime of A , then $(BQ)^{**} = Bp_1^{l_1} \dots p_m^{l_m}$ for irreducible elements $p_i \in B$. Clearly, $Q = Bp_1 \cap A$ which is a principal ideal by the proof of part (a), done.

Let us return to trace rings. Procesi [6] has shown that there exists an action of $PGL_n(k)$ on S and $M_n(S)$ s.t.

- (1). $PGL_n(k)$ acts trivially on k
- (2). The fixed ring of $M_n(S)$ under $PGL_n(k)$ equals $\overline{\mathbb{T}}_{m,n}$
- (3). The fixed ring of S under $PGL(k)$ equals $R_{m,n}$, the center of $\overline{\mathbb{T}}_{m,n}$.

Corollary 3. The extension $R_{m,n} \subset S$ satisfies no blowing up and $R_{m,n}$ is a

unique factorization domain.

Proof.

Because $PGL_n(k)$ acts trivially on k , $H^1(PGL_n(k), S^*) = H^1(PGL_n(k), k^*) = Hom(PGL_n(k), k^*) = 1$ because $PGL_n(k)$ is a simple group.

It follows immediately from this result that $\mathbb{T}_{m,n}$ is reflexive as an $R_{m,n}$ -module. A reflexive order Λ over a normal domain R is said to be a reflexive Azumaya algebra, cfr. e.g. [8], if the natural map :

$$\phi : (\Lambda \otimes_R \Lambda^{opp})^{**} \rightarrow End_R(\Lambda)$$

is an isomorphism. It is fairly easy to show that for every divisorial Λ -ideal I (i.e. a fractional Λ -ideal which is reflexive as an R -module) : $I = \Lambda(I \cap R)^{**}$.

We are now in a position to prove Theorem 1.

Proof of Theorem 1.

The proof of the Artin-Schofield theorem shows that the localization of $\mathbb{T}_{m,n}$ at every central height one prime ideal p is an Azumaya algebra (except if $m = n = 2$). This shows that ϕ_p is an isomorphism for every $p \in X^{(1)}(R)$. $\mathbb{T}_{m,n}$ being a reflexive $R_{m,n}$ -module, this yields that $\mathbb{T}_{m,n}$ is a reflexive Azumaya algebra and the theorem follows from Corollary 3 and the remark above.

If $m = n = 2$, then the only height one prime which is not centrally generated is $\mathbb{T}_{2,2}(XY - YX)$ which is cyclic, done.

We will give two applications of this result :

Theorem 4. : (Montgomery) Every automorphism of $G_{m,n}$ which leaves the center invariant is the identity.

Proof.

By the Skolem-Noether theorem such an automorphism is given by conjugation with a normalizing element of $G_{m,n}$, $h.G_{m,n} \subset \mathbb{T}_{m,n}$ being a central extension, h is also a normalizing element of $\mathbb{T}_{m,n}$, i.e. $\mathbb{T}_{m,n}.h$ is a divisorial $\mathbb{T}_{m,n}$ -ideal.

If m or $n \neq 2$, this entails that $h = \gamma.c$ for some $\gamma \in \mathbb{T}_{m,n}^* = k$ and c in the field of fractions of $R_{m,n}$, done.

If $m = n = 2$, the only noncentral normalizing element of $\mathbb{T}_{m,n}$ is $XY - YX$. This element does not normalize $G_{2,2}$, done.

If Λ is a maximal order over a normal domain R in some central simple algebra Σ , we denote by $h(\Lambda)$ the (pointed) set of left Λ -module isomorphism classes of left fractional Λ -ideals which are reflexive R -modules and with $t_R(\Sigma)$ we denote the conjugacy classes of maximal R -orders in Σ .

Theorem 5. : There is a one-to-one correspondence between $h(\mathbb{T}_{m,n})$ and $t_{R_{m,n}}(Q(\mathbb{T}_{m,n}))$. (m and n not both equal to 2).

Proof.

The correspondence is given by assigning to an (isomorphism class) of a left fractional $\mathbb{T}_{m,n}$ -ideal L , its right order

$$O_r(L) = \{X \in Q(\mathbb{T}_{m,n}) : Lx \subset L\}$$

This map is well defined and epimorphic because for any maximal $R_{m,n}$ -order in $Q(\mathbb{T}_{m,n})$ we can take :

$$L = (\Lambda :_l \mathbb{T}_{m,n}) = \{x \in Q(\mathbb{T}_{m,n}) : x\Lambda \subset \mathbb{T}_{m,n}\}$$

cfr., e.g., [4]. Now suppose L and L' are left fractional $\mathbb{T}_{m,n}$ -ideals s.t. $O_r(L) = \alpha^{-1} \cdot \Theta_r(L') \cdot \alpha$, then replacing L' by $L'\alpha$ we may assume that $\Theta_r(L) = \Theta_r(L') = \Lambda$.

Let $M = ((\Lambda :_r \mathbb{T}_{m,n})L)^{**}$ and $M' = ((\Lambda :_r \mathbb{T}_{m,n})L')^{**}$, then M and M' are twosided divisorial Λ -ideals, i.e., $M = M' \cdot c$ for some element c in the field of fractions of $R_{m,n}$ (because Λ is also a reflexive Azumaya algebra and therefore every twosided divisorial Λ -ideal is generated by a central element).

Finally

$$L = ((\mathbb{T}_{m,n} :_r \Lambda)M)^{**} = ((\mathbb{T}_{m,n} :_r \Lambda)M'c)^{**} = L'c$$

finishing the proof.

Remark. : Even if one restricts attention to projective left $\Delta_{m,n}$ -ideals, some of them are not free (cfr. [7] in $m = n = 2$ case and similarly in $m = 3, n = 2$ case [5]), so there are maximal orders over $R_{m,n}$ not conjugated to $\mathbb{T}_{m,n}$.

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