The Ramification Divisor of Regular Tame Orders, I.

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Introduction. This paper was inspired by Artin's paper [1] about maximal orders over 2 dimensional regular local rings. The techniques in that paper however are more geometrical than ring theoretical.

In this paper we basically study tame orders of global dimension 2. It appears that it is easier to study a more general class of orders : orders that satisfy the A2 condition. The main theorem is that these orders are recognisable by their ramification divisor. However this ramification divisor can still be quite bad. We therefore introduce "smooth" orders. These orders have by definition a nice ramification divisor. For these orders we can prove nice structure theorems.

The techniques used in this paper are easy order theory and graded ring theory, in particular generalized Reesrings as introduced by F. Van Oystaeyen, cfr. [10] .

Acknowledgement : While writing this note we received a preprint of a paper of M. Artin [2] that contains much more specific results than ours. Our methods however are basically different and we think they are interesting enough to justify publication of our results.

1. Some generalities.

Throughout this paper, R will denote a (commutative) integrally closed Noetherian domain with field of fractions K. An R-order Λ in a central simple K-algebra Σ is said to be tame [6] iff :

(T1) : Λ is a reflexive R-module

(T2) : Λ_p is hereditary for all $p \in X^{(1)}(R)$

where $X^{(1)}(R)$ denotes the set of all height one prime ideals of R.

A twosided fractional ideal I of Λ is said to be a (Weil) divisor of Λ if I is a reflexive R-module and if $\mathrm{End}^{l}_{\Lambda}(I) = \Lambda = \mathrm{End}^{r}_{\Lambda}(I)$. It is clear

that I is a divisor iff I is a reflexive R-module and I_p is an invertible Λ_p-ideal for all $p \in X^{(1)}(R)$. The set of all Weil divisors of Λ, $\mathbb{D}(\Lambda)$, forms an Abelian group under the multiplication law : $I\star J = (I.J)^{\star\star}$ where $(-)^{\star\star}$ denotes the bidual Λ-module $\mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(-,\Lambda),\Lambda)$ which is clearly equal to $\cap(-)_p$ where the intersection is taken over all $p \in X^{(1)}(R)$.
From [6] we recall that $\mathbb{D}(\Lambda)$ is the free Abelian group generated by the elements of :
$X^{(1)}(\Lambda)$: $\{J(\Lambda_p) \cap \Lambda ; p \in X^{(1)}(R)\}$
where J(-) denotes the Jacobson radical.
If $D = \{I_1,\dots,I_n\}$ is a set of divisors of Λ and if $g = \{g_1,\dots,g_n\}$ is a set of natural numbers, we will denote with $\Lambda[D,g]$ the $\mathbb{Z}^{(n)}$-graded subring of $\Sigma[X_1,X_1^{-1},\dots,X_n,X_n^{-1}]$, where $\deg(X_i) = (0,\dots,1,\dots,0)$, whose part of degree $(m_1,\dots,m_n)$ is given by :

$$\Lambda[D,g]_{(m_1,\dots,m_n)} = (I_1^{[\frac{m_1}{g_1}]} \star \dots \star I_n^{[\frac{m_n}{g_n}]})\, X_1^{m_1} \dots X_n^{m_n}$$

where $[\frac{a}{b}]$ is the least natural number $\geqslant \frac{a}{b}$. If $g = e = \{1,\dots,1\}$ we will simply write $\Lambda[D]$. Of course, the rings $R[D,g]$ are defined similarly.

From [8], [9] we recall that the orders $\Lambda[D]$ are again tame orders and that $R[D,g]$ is normal if and only if $D = \{i_1,\dots,i_n\}$ is a set of semiprime ideals of R. If Λ is a tame order over R, there are only finitely many "prime"-divisors $P \in X^{(1)}(\Lambda)$ such that $P \neq \Lambda.(P \cap R)$. This finite set, which we will denote by P, is called the ramification divisor of Λ.
Further, for every $P_i \in P$ there exists a natural number e_i such that $(P_i^{e_i})^{\star\star} = (\Lambda.(P_i \cap R))^{\star\star}$. The set $e = \{e_i ; P \in P\}$ is said to be the central ramification divisor.
An order Λ is said to satisfy (Et_1) iff for every $p \in X^{(1)}(R)$ there exists an étale extension $R_p \subset S(p)$ such that S(p) splits Σ. Note that this condition is s tisfied if the residue fields $\mathbb{K}(p)$, $p \in X^{(1)}(R)$, are all perfect [9] .

The Key Result can then be stated as :

Theorem 1: [8] , [9] .

If Λ is a tame order over R such that Λ satisfied (Et_1), then $\Lambda [P]$ is a reflexive Azumaya algebra over the normal domain $R [P_c, e]$.

A reflexive Azumaya algebra Γ, cfr. [8], [9], is an order over a normal domain S such that :

$$\Gamma \perp_S \Gamma^{opp} \cong \mathrm{End}_S(\Gamma)$$

where $A \perp_S B = (A \otimes_S B)^{**}$ for all (reflexive) S-modules A and B.
For every set of divisors D of Λ, there is an equivalence of categories between Λ-ref, the category of all left Λ-modules which are reflexive R-modules, and $\Lambda [D]$-gref, the category of all $\mathbb{Z}^{(n)}$-graded left $\Lambda[D]$-modules which are reflexive modules over the center of $\Lambda [D]$.
If every element of D is an invertivble Λ-ideal, then Λ-mod is equivalent with $\Lambda [D]$-gr, the category of all $\mathbb{Z}^{(n)}$-graded left $\Lambda [D]$-modules. In this case, $\Lambda [D]$ is said to be a Rees ring, [10].
Let R be a normal domain. Suppose that $D = \{p_1,\dots,p_n\} \subset X^{(1)}(R)$ are torsion elements in the classgroup of R with orders $\{l_1,\dots,l_n\}$ and let $g = \{g_1,\dots,g_n\}$ be some set of natural numbers. It is easy to see that $R [D,g]$ contains a $\mathbb{Z}^{(n)}$-graded Rees ring :

$$T = \bigoplus_{(j_1,\dots,j_n) \in \mathbb{Z}^n} (p_1^{j_1 l_1} \star \dots \star p_n^{j_n l_n}) X_1^{j_1 l_1 q_1} \dots X^{j_n l_n g_n}$$

By assumption, $p_i^{l_i} = R\, a_i$ for some element $a_i \in R$ and we define the "roll up" of $R [D,g]$ to be the ring :

$$R [D,g]_! = R[D,g] / (1-a_1 X^{l_1 g_1},\dots, 1-a_n X^{l_n g_n})$$

Clearly, $R [D,q]$, is $\mathbb{Z}/l_1 q_1 \mathbb{Z} \times \dots \times \mathbb{Z}/l_n g_n \mathbb{Z}$-graded in the natural way.

Further, one easily verifies that there is an equivalence of categories between $R[\mathcal{D},g]$-gr and $R[\mathcal{D},g]$-gr.

2. Auslanders condition A_2.

Definition 1 : A normal domain R is said to satisfy Auslanders condition A_2 iff there are only finitely many isomorphism classes of indecomposable reflexive R-lattices. A tame order Λ over a normal domain R is said to satisfy Auslanders condition A_2 iff there are only finitely many isomorphism classes of indecomposable left Λ-modules which are reflexive R-lattices. The usefullness of the A_2 condition is established in the following lemma.

Lemma 0. Let R be a normal noetherian local henselian domain of K dim 2 Let Λ be a reflexive order over R. Then there exists a reflexive lattice over Λ such that gl. dim $\mathrm{End}_\Lambda(M) = 2 \Leftrightarrow \Lambda$ satisfies A_2.

Proof. This is well known.

In this section we aim to show how one can construct tame orders satisfying Auslanders condition A_2 and having a nasty central ramification divisor.
In this section we assume that all rings are reflexive modules over some local normal Noetherian Henselian domain R (and finitely generated as modules). We need R to be Henselian because we like to have a Krull-Remak-Schmidt-Azumaya decomposition for reflexive modules over R.
Let us begin with some easy remarks.

Lemma 1. Let $R \subset S$ be a finite extension of normal domains such that R is an R-module direct summand of S. If S satisfies A_2, then so does R.

Proof. Let M be an indecomposable reflexive R-lattice and let $S \perp_R M = \oplus M_i$ be the decomposition of $S \perp_R M$ in indecomposable reflexive S-modules.

$M \lhd S \perp_R M$ implies that $M \lhd M_i$ for some i. So, there are only a finite number of possibilities for M.

Lemma 2. Suppose that the map $S \perp_R S \to S$ splits as S - S-bimodules. If R satisfies A_2 then S satisfies A_2.

Proof. Let M be an indecomposable reflexive reflexive S-lattice and let $M = \oplus M_i$ be the decomposition of M in indecomposable reflexive R-lattices. Because $S \perp_R M = S \perp_R S \perp_S M$, $M \lhd S \perp_R M$ as S-modules. Therefore, $M \lhd S \perp_R M_i$ for some i and so there are only a finite number of possibilities for M.

Lemma 3. If Λ is a reflexive Azumaya algebra over R, then Λ satisfies A_2 if R does.

Proof. Let M be an indecomposable left Λ-module which is reflexive as an R-module, then $M \perp_R \Lambda$ is a reflexive Λ-Λ bimodule. Let $M \perp_R \Lambda = \oplus M_i$ be the decomposition of $M \perp_R \Lambda$ in indecomposable reflexive Λ-Λ-bimodules. Then $M \lhd M_i$ for some i. Because there is an equivalence of categories between the reflexive R-lattices and the reflexive Λ-Λ lattices, there are only a finite number of different M_i's, whence a finite number of possibilities for M.

We will now give some results for rings which are graded by a finite group G. We use a technique which was first introduced in [13] . Let R be any G-graded ring and form the groupring RG which we equip with a G-gradation given by the formula : $\deg(R_\sigma . \tau) = \tau$.
The ring $S = \sum_{\sigma \in G} R_\sigma . \sigma$ is a G-graded subring of RG which is graded isomorphic to R. There exists a Maschke-type theorem between S and RG.

Lemma 4. If $|G|^{-1} \in R$, then for all M,N in RG-gr there exists a canonical map :

$$\sim \, : \mathrm{Hom}_{S\text{-}gr}(M,N) \to \mathrm{Hom}_{RG\text{-}gr}(M,N)$$

with the following properties :

(1) : if $\varphi \in \mathrm{Hom}_{RG\text{-}gr}(M,N)$, then $\tilde{\varphi} = \varphi$

(2) : if we have a commutative diagram :

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ M' & \xrightarrow{\varphi'} & N' \end{array}$$

of graded RG-modules such that φ, φ' are S-linear and α, α' RG-linear, then the diagram below is also commutative :

$$\begin{array}{ccc} M & \xrightarrow{\tilde{\varphi}} & N \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ M' & \xrightarrow[\tilde{\varphi}']{} & N' \end{array}$$

There exists another nice correspondence between S-modules and RG-modules. If M is a graded left S-module and $\tau \in G$, then $M(\tau)$ is defined to be the graded S-module such that $M(\tau)_\mu = M_{\tau\mu}$ for all $\mu \in G$.

Lemma 5. [13] Suppose that M is a graded left S-module, then

$RG \otimes_S M \cong \sum_{\sigma \in G} M(\tau)$ as graded left S-modules.

As a direct application of this construction we get :

<u>Proposition 1</u>. If $|G|^{-1} \in R$ and if R has finite graded global dimension, then R has finite global dimension.

<u>Proof.</u> Suppose that R is graded regular, then so is S. A standard argument combined with the foregoing two lemmas then yield that RG is a strongly graded ring, [10] so there is an equivalence of categories between RG-gr and R-mod, yielding that R is regular.

<u>Proposition 2 :</u> If $|G|^{-1} \in R$ and if R satisfies a graded version of A_2, then R satisfies A_2.

<u>Proof.</u> Suppose that R satisfies graded A_2, then so does S. Now, the reader may easily verify that the natural map

$$RG \otimes_S RG \to RG$$

is graded split. Therefore, by a graded version of Lemma 2, RG satisfies graded A_2. Again, using that RG is strongly graded, this entails that R satisfies A_2.

Let R be a local normal Henselian domain such that $D = \{p_1,\dots,p_n\} \subset X^{(1)}(R)$ are torsion elements of the class group of R. We aim to investigate the relation between the A_2-conditions on R, R $[D,g]$ and the roll-up R $[D,g]_!$ An R $[D,g]_!$-module is said to be R-reflexive if it is reflexive considered and an R-module. The equivalence of categories between R $[D,g]$-gr and R $[D,g]_!$-gr maps graded reflexive R $[D,g]$-modules to R-reflexive graded R $[D,g]_!$-modules and vice versa.

The roll-up R $[D,g]_!$ is said to satisfy (graded) A_2 iff it has only a finite number of isomorphism classes of R-reflexive indecomposable (graded) modules. By a similar argument as in Prop. 2 one can prove that graded A_2 implies A_2 for R $[D,g]_!$ if l.c.m. $(g_1 l_1,\dots,g_n l_n)$ (in the notation of § 1) is a unit in R.

We can now state and prove the main result of this section :

<u>Theorem 2</u> : If Λ is a tame order, over a local Henselian domain R, satisfying (Et_1). With notations as in § 1, Λ satisfies A_2 iff R $[P_c,e]$ satisfies A_2, if l.c.m.(e) $\in R^*$.

<u>Proof.</u> Suppose that Λ satisfies A_2, then $\Lambda [P]$ satisfies graded A_2 by the equivalence of categories between Λ-ref and $\Lambda [P]$-gref. By a graded version of lemma 3 and theorem 1 this entails that R $[P_c,e]$ satisfies graded A_2. By the equivalence of categories between R $[P_c,e]$-gr and R $[P_c,e]_!$-gr, the roll-up satisfies graded A_2 whence A_2 by proposition 2. Conversely, if R $[P_c,e]_!$ satisfies A_2 then R $[P_c,e]$ satisfies A_2 then R $[P_c,e]$ satisfies graded A_2 whence so does $\Lambda [P]$ by a graded version

of lemma 3. Finally, $\Lambda[\mathcal{P}]$-gref being equivalent to Λ-ref, Λ satisfies A_2.

It is now fairly easy to construct example of tame orders Λ satisfying Auslanders condition A_2 which have a nasty ramification divisor.

Example 1. $S = k[[x,y]]$ where $\mathrm{char}(k) \neq 2,3$.
$S_3 = \langle \sigma,\tau : \sigma^2 = 1, \tau^3 = 1, \sigma\tau\sigma^{-1} = \tau^2 \rangle$ acts in a natural way on $k[[x,y]]$ by $\sigma \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\tau \to \begin{pmatrix} \xi & 0 \\ 0 & \xi^2 \end{pmatrix}$ where ξ is a 3rd root of unity
Then :

$$S^{S_3} = R = k[[x^3+y^3, xy]]$$

hence R is regular. If we take $u = x^3+y^3$ and $v = xy$, then :

$$du = 3x^2\,dx + 3y^2 dy$$

$$dv = ydx + xdy$$

The ramification divisor of S/R is given by

$$\begin{vmatrix} 3x^2 & 3y^2 \\ y & x \end{vmatrix} = 3x^2-3y^2 = 0$$

Therefore, $v^3 = x^6 = \frac{u^2}{4}$ and the ramification divisor of S/R has a cusp.
The order $\Lambda = S \star S_3$ is an order in a matrix ring. Λ/R has the same ramification divisor as S/R and is regular.

3. Regularizable domains.

In this section we aim to characterize those normal domains R such that for some subset $\mathcal{D} \subset X^{(1)}(R)$ and some set of natural numbers $g \in \mathbb{N}^{(n)}$, the ring $R[\mathcal{D},g]$ defined in § 1 is a graded regular domain, i.e. such that every graded $R[\mathcal{D},g]$-module has a finite resolution in graded projective modules.
We will begin by investigating the connection between regularity of R and graded regularity of $R[\mathcal{D},g]$.

Lemma 6. Let R be a Noetherian gr-local domain with unique maximal graded ideal m. Then R is graded regular if and only if $\text{gr-dim}_{R/m}(m/m^2) = \text{gr-Kdim}(R)$.

Proof. It is clearly sufficient to prove that R_m is a regular domain. In order to do this we have to prove that P is graded projective if P is a f.g. graded R-module s.t. P_m is R_m-projective. So, we have to show that $\text{Hom}_R(P,-)$ is exact in R-gr. Let $f: M \to N$ be an epimorphism of graded R-modules and let T be the cokernel of the induced map $\text{Hom}_R(P,M) \to \text{Hom}_R(P,N)$, then it is clear that $T_m = 0$. Let t be an homogeneous element of T. Then there exists an element $\mu \in R/m$ s.t. $\mu . t = 0$ Let $\mu = \mu_{\sigma_1} + \dots + \mu_{\sigma_h}$ be an homogeneous decomposition of μ, then at least one of the μ_{σ_i} is not in m. This implies that μ_{σ_i} is a unit in R and since $\mu_{\sigma_i} . t = 0$ we conclude $t = 0$. So, $T = 0$.

Lemma 7 : If R is a regular local domain, then R $[\mathcal{D}]$ is graded regular for every set of divisors $\mathcal{D}$.

Proof. This follows trivially from the equivalence of categories between R-mod and R $[\mathcal{D}]$ -gr which exists because every element of $\mathcal{D}$ is invertible.

We can now give a complete answer to the question : when is R $[\mathcal{D}, g]$ graded regular if R is regular ?

Theorem 3 : Let R be a regular local domain. Let $\mathcal{D} = \{p_1, \dots, p_n\} \subset X^{(1)}(R)$ and $g = \{g_1, \dots, g_n\}$ with all $g_i > 1$. Then R $[\mathcal{D}, g]$ is regular if and only if the generators of the principal prime ideals p_i form part of a regular system of parameters of R.

Proof. Define $S_o = R$; $S_{i+1} = \sum_{j \in \mathbb{Z}} p_{i+1}^{\left[\frac{j}{g_{i+1}}\right]} . S_i . X_{i+1}^j$. Let M_{i+1} be the unique maximal graded ideal of S_{i+1}. As the reader may easily verify :

$$M_{i+1} = \sum_{j \in \mathbb{Z}} I_j . p_{i+1}^{[\frac{j}{g_{i+1}}]} . S_i . X_{i+1}^j$$

where $I_j = M_i$ if $g_{i+1} \mid j$ and $I_j = S_i$ if $g_{i+1} \nmid j$. Calculating the graded dimension of M_{i+1}/M_{i+1}^2 over S_{i+1}/M_{i+1} gives us :

$$1 + \dim_{S_i/M_i}(M_i/M_i^2 + p_{i+1}S_i)$$

and therefore, S_{i+1} is graded regular if and only if S_i is graded regular and $p_{i-1} \not\subset (M_i^2)_o$. Calculating $(M_{i+1}^2)_o$ gives us $(M_i^2)_o + p_{i+1}$, so $(M_i^2)_o = M_o^2 + \sum_{j=1}^{i-1} p_j$. Therefore, S_{i+1} is graded regular if and only if S_i is graded regular and $p_{i+1} \not\subset M_o^2 + p_1 + \dots + p_i$.

However, as we will see below, there exist non-regular normal domains s.t. $R[\mathcal{D},g]$ is graded regular for some suitable choice of $\mathcal{D}$ and g. The next result chracterizes those normal domains :

<u>Theorem 4.</u> Let R be a normal local domain containing an algebraically closed field of characteristic zero.
There exists a set of divisors $\mathcal{D} \subset X^{(1)}(R)$ representing torsion elements in Cl(R) and a set of natural numbers g such that $R[\mathcal{D},g]$ is graded regular if and only if there exists a regular overring S of R, finitely generated over R and a finite Abelian group G acting on S s.t. $S^G = R$.

<u>Proof.</u> Let S the roll-up of $R[\mathcal{D},g]$, then S is graded regular by the category-equivalence of $R[\mathcal{D},g]$ and $R[\mathcal{D},g]_!$. Because R contains a field of characteristic zero it follows from prop. 1 that S is regular. Since R contains all roots of unity, the gradation on S may be changed into a group action having the required properties, [3].
Conversely, because S is a regular ring, it is a direct sum of regular domains each containing R. Therefore, we can replace S by a domain and G by some subgroup of G. R containing an algebraically closed field, the group action of G may be turned into a gradation of S by $G^{\star}$.

Therefore, $S = \bigoplus_{\sigma \in G^\star} I_\sigma$ with $I_e = R$. S being a reflexive R-lattice, every I_σ is a reflexive R-lattice. Further, S being a domain entails that every I_σ has either rank zero or one. If $\mathrm{rank}_R(S) = n$, then the classgroup of R is n-torsion. Let $\mathcal{D} = \{p_1,\dots,p_k\}$ be the set of prime factors occuring in the decomposition of the I_σ, $\sigma \in G^\star$, and let $\mathcal{D}' = \{(p_1 S)^{\star\star},\dots,(p_k S)^{\star\star}\}$.
Each $(p_i S)^{\star\star}$ being invertible, it is claer from the category-equivalence that $S[\mathcal{D}']$ is graded regular. Furrther, $S[\mathcal{D}']$ is a graded free extension of $R[\mathcal{D}]$, entailing that $R[\mathcal{D}]$ is graded regular.

In dimension two, we can give an intrinsic characterization of regularizable domains.

Theorem 5. Let R be a normal local domain of Krull dimension two. There exists a set of divisors $\mathcal{D} = \{p_1,\dots,p_n\} \subset X^{(1)}$ of torsion elements of Cl(R) and a set of natural numbers $g = \{g_1,\dots,g_n\}$ s.t. $R[\mathcal{D},g]$ is graded regular if and only if R has only a finite number of isomorphism classes of indecomposable reflexive modules which are all of rank one.

Proof. Because each of the p_i's is torsion in Cl(R), $R[\mathcal{D},g]$ has graded Krull dimension two, whence every graded reflexive $R[\mathcal{D},g]$-module is free. So, let M_o be an indecomposable reflexive R-module. Then $(M_o R[\mathcal{D},g])^{\star\star}$ is graded free, yielding that $M_o = (M_o R[\mathcal{D},g])^{\star\star}_e$ is a direct sum of divisors which are products of the p_i's. Therefore, every indecomposable reflexive R-module has rank one. Since every p_i is torsion, there are only a finite number of isomorphism classes.
Conversely, if R has only a finite number of isomorphism classes of indecomposable reflexive ideals, Cl(R) is finite. Let $\mathcal{D} = \{p_1,\dots,p_k\}$ be the prime factors of these indecomposable reflexives. Because there is an equivalence of categories between R-ref and $R[\mathcal{D}]$-gref, every graded reflexive $R[\mathcal{D}]$-module is free, yielding that $R[\mathcal{D}]$ has graded global

dimension two.

Let us give an example of a regularizable domain which is not regular.

Example 2 : Let R be the affine cone $\mathbb{C}[X,Y,Z]/(XY-Z^2)$ then $Cl(R) \cong \mathbb{Z}/2\mathbb{Z}$ and is generated by the ruling $p = (Y,Z)$.

Take $R[D] : \dots \oplus (Y^{-1})X_1^{-2} \oplus p^{-1}X_1^{-1} \oplus R \oplus pX_1 \oplus (Y)X_1^2 \oplus p.(Y)X_1^3 \oplus \dots$

Then, using lemma 6, it is fairly easy to check that $R[D]$ is regular.

4. Smooth Orders.

Whereas tame orders satisfying Auslanders condition A_2 can still have a very nasty ramification divisor, we will study in this section the Zariski and étale local structure of regular tame orders with an extremely nice ramification divisor : smooth orders.

Definition 2 : A tame order Λ over a normal domain R is said to be smooth if there exists a set of divisors D of Λ such that $\Lambda[D]$ is an Azumaya algebra over a graded regular center and every element of D is an invertible Λ-ideal.

Clairly, in view of the equivalence of categories between Λ-mod and $\Lambda[D]$-gr and because $\Lambda[D]$ is graded regular, it follows that Λ has finite global dimension. Furthermore, it is trivial to verify that smooth orders are closed under taking matrix rings and polynomial extensions.

In most applications, one takes $D = P$ the ramification divisor of Λ, but we will give an example of a smooth order s.t. $\Lambda[P]$ is not Azumaya.

Example 3 : Let R be the affine cone $\mathbb{C}[X,Y,Z]/(XY-Z^2)$ and let $p = (Y,Z)$ be the ruling which generates the class group. Let Λ be the reflexive Azumaya algebra :

$$\Lambda = \mathrm{End}_R(R \oplus p) = \begin{pmatrix} R & p \\ p^{-1} & R \end{pmatrix}$$

and let $D = \{\Lambda.\begin{pmatrix} 0 & Y \\ 1 & 0 \end{pmatrix}\}$ which is clearly an invertible two-sided Λ-ideal. Then $\Lambda[D]$ is the $\mathbb{Z}$-graded ring :

$$\Lambda[D] \cong \Lambda[\begin{pmatrix} 0 & Y \\ 1 & 0 \end{pmatrix}X_1, \begin{pmatrix} 0 & 1 \\ Y^{-1} & 0 \end{pmatrix}X_1^{-1}]$$

which is easily checked to be an Azumaya algebra with the graded regular center :

$$\ldots \oplus (Y^{-1})X_1^{-2} \oplus p^{-1}X_1^{-1} \oplus R \oplus pX_1 \oplus (Y)X_1^2 \oplus p(Y)X_1^3 + \ldots$$

Therefore, Λ is smooth through $\Lambda[P] = \Lambda$ is not an Azumaya algebra.

From now on, we restrict attention to smooth orders s.t. $D = P$ in a p^2-dimensional division algebra, p being a prime number.

It turns out that in this case one has a good hold upon the structure of such orders. Furthermore, we will always work over a normal local domain R and assume that the ramified height one prime ideals of Λ are generated by a normalizing element. We do not know whether this condition is always satisfied.

Working inside p^2-dimensional divison rings puts severe restrictions on the number of ramified primes.

<u>Lemma 8 :</u> If Λ is a smooth order over a local normal domain R in a p^2-dimensional division ring Σ, then $\# P \leqslant 2$.

<u>Proof.</u> Let $n = \# P$, then, because $\Lambda[P]$ is a $\mathbb{Z}^{(n)}$-graded Azumaya algebra over $R[P]$, which is a graded local domain with unique maximal graded ideal :

$$m[P] = \sum_{\sigma \in H} m.R[P]_\sigma \oplus \bigoplus_{\sigma \in G \setminus M} R[P]_\sigma$$

where $G = \mathbb{Z}^{(n)}$ and $M = p\mathbb{Z} \oplus \ldots \oplus p\mathbb{Z}$. We must have that $\Lambda[P]/\Lambda[P].m[P]$ is a graded central simple algebra of dimension p^2 over the $\mathbb{Z}^{(n)}$-graded field [ ]

$$R[P]/m[P] = R/m[X_1^p, X_1^{-p}, \ldots, X_n^p, X_n^{-p}]$$

Because we have assumed that every prime ideal P_i, $1 \leqslant i \leqslant n$, is generated by a normalizing element, an easy calculation shows :

$$\Lambda[P]/\Lambda[P].m[P] = \bigoplus_{0 \leqslant i_j < p} \Lambda/(\Lambda m + P_1 + \dots + P_n) X_1^{i_1} \dots X_n^{i_n}$$

the isomorphism being one of the graded $R/m\,[X_1^p, X_1^{-p}, \dots, X_n^p, X_n^{-p}]$ modules.
Calculating dimensions on both sides yields :

$$(\star) \quad p^2 = p^n . \dim_{R/m}(\Lambda/(\Lambda m + P_1 + \dots + P_n))$$

This immediately implies that $n \leqslant 2$.

Combining $(\star)$ above with theorem 3 we get :

Theorem 6 : If Λ is a maximal order over a regular domain R in a p^2-dimensional division algebra Σ, then Λ is smooth if

(a) R has a regular ramification divisor with normal crossings in Σ

(b) For every $m \in \mathrm{Spec}(R)$ one of the following cases occur :

CASE 0 : $N_m = 0$, i.e. Λ_m is an Azumaya algebra over R_m

CASE 1 : $N_m = 1$ and $\dim_{R/m}(\Lambda/\Lambda m+P) = p$

CASE 2 : $N_m = 2$ and $\dim_{R/m}(\Lambda/\Lambda m+P+a) = 1$

where P resp. Q) is the height one prime of Λ lying over the ramified central prime p (resp. q).

Let us give a geometric interpretation of this result for a maximal order over a smooth surface. Let $\{p_1, \dots, p_n\}$ be the ramified height one primes, then each p_i can be viewed as a curve on the surface. $N_m \leqslant 2$ then says that there are no three such curves intersecting at one point. Furthermore, each curve must be nonsingular and in an intersection point of two curves the tangent lines may not coincide.

As we will see later, CASE 0 (resp. 1,2) corresponds to CASE 1.1. (i) (resp. (ii), (iii) of [1] , whereas CASE 1.1 (iv) cannot occur as a smooth order. Let us give some examples of smooth maximal orders.

Example 4 : Let $\Lambda = ¢ [X,-]$ be the skew polynomial ring over ¢, where - denotes conjugation. It is clear that Λ is a maximal order with center $R = \mathbb{R}[t]$ where $X^2 = t$. It follows that $P = \{(X)\}$, so $\Lambda[P]$ is the $\mathbb{Z}$-graded ring :

$$\dots (X^{-2})X_1^{-2} \oplus (X^{-1})X_1^{-1} \oplus \Lambda \oplus (X)X_1 \oplus (X^2)X_1^2 \oplus \dots$$

and $R[P]$ is the $\mathbb{Z}$-graded ring :

$$\dots \oplus (t^{-1})X_1^{-2} \oplus RX_1^{-1} \oplus R \oplus (t)X_1 \oplus (t)X_1^2 \oplus (t^2)X_1^3 \oplus \dots$$

For every prime ideal $q \neq (t) \in \text{Spec } \mathbb{R}[t]$, Λ_q is an Azumaya algebra over R_q, whence $\Lambda[P]_q$ is Azumaya over $R[P]_q$. In the ramified prime we have

$$\dim_{\mathbb{R}[t]/(t)} (¢[X,-]/(x)) = \dim_{\mathbb{R}} (¢) = 2$$

i.e. CASE 1, so Λ is smooth over R.

Further, $R[P]_{(t)}/t[P]_{(t)} \cong \mathbb{R}[t_1,t_1^{-1}]$ where $t_1 = t X_1^2$ and $\Lambda[P]_{(t)}/t[P]\Lambda[P]_{(t)}$ is the $\mathbb{Z}$-graded central simple algebra : $¢[Y_1,Y^{-1},-]$ with $Y_1 = X.X_1$ over $\mathbb{R}[t_1,t_1^{-1}]$.

Example 5 : Let R be a regular local domain of dimension two and suppose that x and y generate the maximal ideal m. Let Σ be the quaternion-algebra $(\frac{x,y}{K})$ and let $\Lambda = R[1,i,j,ij]$, i.e. Λ is R-free with generators 1,i,j,ij and relations :

$$i^2 = x, j^2 = y \text{ and } ij = -ji$$

In [16] it is shown that Λ is a maximal R-order. Clearly, $P_e = \{(X),(Y)\}$ which is a set of regular ramification divisors with normal crossings. Further, $\Lambda[P]$ is a $\mathbb{Z} \oplus \mathbb{Z}$-graded ring which can be visualized (omitting powers of X_1 and X_2) as :

$$\begin{array}{ccccccccccccc}
\cdots & \oplus & (i^{-2}j^{2}) & \oplus & (i^{-1}j^{2}) & \oplus & (j^{2}) & \oplus & (ij^{2}) & \oplus & (i^{2}j^{2}) & \oplus & \cdots \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\cdots & \oplus & (i^{-2}j) & \oplus & (i^{-1}j) & \oplus & (j) & \oplus & (ij) & \oplus & (i^{2}j) & \oplus & \cdots \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\cdots & \oplus & (i^{-2}) & \oplus & (i^{-1}) & \oplus & \Lambda & \oplus & (i) & \oplus & (i^{2}) & \oplus & \cdots \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\cdots & \oplus & (i^{-2}j^{1}) & \oplus & (i^{-1}j^{-1}) & \oplus & (j^{-1}) & \oplus & (ij^{-1}) & \oplus & (i^{2}j^{-1}) & \oplus & \cdots \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\cdots & \oplus & (i^{-2}j^{-2}) & \oplus & (i^{-1}j^{-2}) & \oplus & (j^{-2}) & \oplus & (ij^{-2}) & \oplus & (i^{2}j^{-1}) & \oplus & \cdots \\
 & & & & & & \oplus & & \oplus & & \oplus & & \\
 & & & & & & \vdots & & \vdots & & \vdots & &
\end{array}$$

and its center $R[P]$ is the $\mathbb{Z}\oplus\mathbb{Z}$-graded ring which looks like :

$$\begin{array}{ccccccccc}
(x^{-1}y) & \oplus & (y) & \oplus & (y) & \oplus & (xy) & \oplus & (xy) \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
(x^{-1}y) & \oplus & (y) & \oplus & (y) & \oplus & (xy) & \oplus & (xy) \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
(x^{-1}) & \oplus & R & \oplus & R & \oplus & (x) & \oplus & (x) \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
(x^{-1}) & \oplus & R & \oplus & R & \oplus & (x) & \oplus & (x) \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
(x^{-1}y^{-1}) & & (y^{-1}) & & (y^{-1}) & & (xy^{-1}) & & (xy^{-1})
\end{array}$$

and $\dim_{R/m}(\Lambda/\Lambda m + (i) + (j)) = \dim_{R/m}(R/m$ so Λ is a smooth maximal R-order of case 2.

Further, $R[P]/m[P] = R/m[Y_1^2, Y_1^{-2}, Y_2^2, Y_2^{-2}]$ where $Y_1 = iX_1$ and $Y_2 = jX_2$. Whereas $\Lambda[P]/\Lambda[P]m[P]$ is the $\mathbb{Z}\oplus\mathbb{Z}$-graded central simple algebra : $R/m[Y_1, Y_1^{-1}, Y_2, Y_2^{-1}]$ with $Y_1Y_2 = -Y_2Y_1$. Its part of degree $(0,0)$ equals R/m corresponding to the fact that Λ is quasi-local with maximal ideal $M = (i,j)$.

Example 6 : Let F be any field with characteristic unequal to 2. Let $R = F[X,Y]_{(X,Y)}$ where X and Y are indeterminates over F. Then R is regular local of dimensions two and has field of fractions $K = F(X,Y)$. Let Σ be the quaternion algebra $(X,1+Y)_K$ and let Λ be the R-free order $R[1,i,j,ij]$ then Λ is a maximal order [16, p. 471]. Then $P = \{(i)\}$, $P_c = \{(x)\}$ and $(x) \notin m^2$. Because $\dim_{R/m} (\Lambda/\Lambda m+(i)) = \dim_F(F \oplus F) = 2$, Λ is smooth. Further $R[P]/m[P] \cong F[Y_1^2, Y_1^{-2}]$ where $Y_1 = iX_1$ and $\Lambda[P] / \Lambda[P] m[P]$ is the $\mathbb{Z}$-graded algebra

$$(F \oplus F\epsilon)[Y_1, Y_1^{-1}, \varphi] \overset{\alpha}{\underset{\approx}{\to}} M_2(F[Y_1^2, Y_1^{-2}])$$

where $\varphi(a \oplus b\,E) = a \oplus -b\,E$ and α is given by

$$\alpha(1 \oplus 0) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \alpha(0 \oplus E) = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

$$\alpha(Y_1) = \begin{pmatrix}0 & 1\\ Y_1^2 & 0\end{pmatrix}$$

Therefore, $\Lambda[P]m[P]$ is a $\mathbb{Z}$-graded central simple algebra over $F[Y_1^2, Y_1^{-2}]$ However, the part of degree 0 of $\Lambda[P] / \Lambda[P]m[P]$ is semisimple $F \oplus F\epsilon$. corresponding to the fact that Λ is not quasi-local. Each factor corresponds to one of the two maximal ideals of Λ lying over $m = (X,Y)$:

$$M_1 = \Lambda(i, j-1) \quad , \quad M_2 = \Lambda(i, j+1)$$

Having characterized smooth maximal orders over a regular center, we will now study their Zariski local structure, i.e. the number of conjugacy classes over a regular local domain. One of the basic ingredients in this study is a result of Grothendieck on descent of modules. For convenience, we state this theorem here.

Theorem [7,2.5.8] Let R be a Noetherian (semi) local ring, Λ a finite R-algebra and let M_1, M_2 be finite left Λ-modules. Let $R \to S$ be a faithfully

flat morphism with S Noetherian. If $M_1 \otimes S \cong M_2 \otimes S$ as left $\Lambda \otimes S$-modules., the $M_1 \cong M_2$ as left Λ-modules.
Using this theorem, we will have to compute the conjugacy classes of the extended orders $\Lambda \otimes R^{sh}$ where Λ is smooth over R and R^{sh} denotes the strict Henselization of R cfr. [12].

CASE 0 is easy : if one (maximal) order Λ over R in Σ is Azumaya, then every smooth order, say Λ is Azumaya too. Because $Br(R^{sh}) = 0$, $\Lambda \otimes R^{sh} \cong M_n(R^{sh}) \cong \Gamma \otimes R^{sh}$ and by descent $\Lambda \cong \Gamma$ as R-algebras, yielding that Λ and Γ are conjugated.
Before treating other cases, let us recall the definition of the graded Brauer group as introduced by F. Van Oystaeyen in the $\mathbb{Z}$-graded case in [14] .
If T is any $\mathbb{Z}^{(n)}$-graded ring, then a graded Azumaya algebra over T is an Azumaya algebra over T admitting a $\mathbb{Z}^{(n)}$-gradation extending the gradation of the center. Two graded algebras Γ and Ω are said to be gr-equivalent if there exist finitely generated graded projective T-modules P and Q such that there exists a degree preserving isomorphism

$$\Gamma \otimes_T \mathrm{END}_T(P) \cong \Omega \otimes_T \mathrm{END}_T(Q)$$

where the rings $\mathrm{END}_T(-)$ and the tensorproducts are equippped with the natural gradation, cfr. e.g. [14]. The set of gr-equivalence classes of graded Azumaya algebras forms a group with respect to the tensorproduct $Br^g(T)$, called the graded Brauer group of T.
If T is a $\mathbb{Z}$-graded Krull domain it was shown by S. Caenepeel, M. Van den Bergh and F. Van Oystaeyen, [5] that the natural (i.e. gradation-forgetting) morphism $Br^g(T) \to Br$ [4] is monomorphic. Their argument can easily be extended to the $\mathbb{Z}^{(n)}$-graded case.
S. Caenepeel [4] calles a graded local ring R (i.e. having a unique maximal graded ideal) gr-Henselian if every finite graded commutative R-algebra B is graded decomposed, i.e. when it is the direct sum of graded

local rings. In the $\mathbb{Z}$-graded case it tursn out that a graded local ring is gr-Henselian iff its part of degree zero is Henselian, [4] . This result can be generalized to $\mathbb{Z}^{(n)}$-graded rings. Furthermore, if R is gr-Henselian with maximal graded ideal m then the natural map :

$$Br^g R \to Br^g R/m$$

is monomorphic by a similar argument as in the ungraded case.

<u>Theorem 7 :</u> If Λ is a smooth order over a local normal domain in a p^2-dimensional division algebra, p a prime number, then :

(a) : If Λ is in CASE 1, R^{sh} splits Δ

(b) : If Λ is in CASE 2, R^{sh} does not split Δ

if char $(R^{sh}/m^{sh}) \neq p$.

<u>Proof.</u> $\Lambda[P] \otimes_R R^{sh}$ is a graded Azumaya algebra over $R^{sh}[P] \otimes_R R^{sh}$ equipped with the natural gradation. $R^{sh}[P]$ is graded Henselian because its part of degree (0,-,0) is Henselian. The unique maximal graded ideal of $R^{sh}[P]$ will be denoted by $m^{sh}[P]$.

(a) : In this case $R^{sh}[P]/m^{sh}[P]$ is the graded field

$$R^{sh}/m^{sh}[Y_1^p, Y_1^{-1}]$$

where $Y_1^p = \pi X_1^p$, (π) being the ramified central prime.

Now,

$$\overline{\Gamma} = (\Lambda[P] \otimes R^{sh})/(\Lambda[P] \otimes R^{sh})m^{sh}[P]$$

is a $\mathbb{Z}$-graded central simple algebra of dimension p^2 over $R^{sh}/m^{sh}[Y_1^p, Y_1^{-p}]$.
Calculating the part of degree zero of $\overline{\Gamma}$ it turns out that $\overline{\Gamma}_0$ need to be an algebra of dimension p over R^{sh}/m^{sh}, using the formula $(\star)$.
Because R^{sh}/m^{sh} is separably closed and char$(R^{sh}/m^{sh}) \neq p$ we must have that :

$$\overline{\Gamma}_0 = R^{sh}/m^{sh} \oplus \dots \oplus R^{sh}/m^{sh} \quad \text{(p copies)}$$

Therefore, $\overline{\Gamma}$ contains zero divisors and hence :

$$\overline{\Gamma} = M_p(R^{sh}/m^{sh}[Y_1^p, Y_1^{-p}])$$

with an appropriate gradation (cf. example 6).

Finally, using the injectivity of the morphism :

$$Br^g(R^{sh}[P]) \to Br^g(R^{sh}[P]/m^{sh}[P])$$

it follows that :

$$(S1) : \Lambda[P] \otimes R^{sh} \cong END_{R^{sh}[P]}(P)$$

for some graded finitely generated projective $R^{sh}[P]$-module P. Calculating the parts of degree zero on both sides yields that :

$$\Lambda \otimes_R R^{sh} \to M_p(Q(R^{sh}))$$

finishing the proof of part (a).

(b) : In the second case, $R^{sh}[P]/m^{sh}[P]$ is the $\mathbb{Z} \oplus \mathbb{Z}$-graded field :

$$R^{sh}/m^{sh}[Y_1^p, Y_1^{-1}, Y_2^p, Y_2^{-p}]$$

where $Y_1^p = \pi X_1^p$ and $Y_1^p = \pi' X_2^p$ where (π) and (π') are two central ramified primes and $\deg(Y_1^p) = (p,0)$ and $\deg(Y_2^p) = (0,p)$. Now,

$$\overline{\Gamma} = \Lambda[P] \otimes R^{sh}/(\Lambda[P] \otimes R^{sh})m^{sh}[P]$$

is a $\mathbb{Z} \oplus \mathbb{Z}$-graded central simple algebra of dimension p^2 over $R^{sh}/m^{sh}[Y_1^p, Y_1^{-p}, Y_2^p, Y_2^{-p}]$. Using formula (★) it is easy to see that all homogeneous parts should be one-dimensional. In particular,

$$\overline{\Gamma}_{(o,o)} = R^{sh}/m^{sh}$$

Let X be a generator of the $\overline{\Gamma}_{(o,o)}$-module $\overline{\Gamma}_{(1,o)}$ and Y a generator of the $\overline{\Gamma}_{(o,o)}$-module $\overline{\Gamma}_{(o,1)}$, then it turns out that $\overline{\Gamma}$ is a graded cyclic algebra determined by the relations :

$$\begin{cases} X^P = a\, Y_1^P & a \in R^{sh}/m^{sh} \\ Y^P = b\, Y_1^P & b \in R^{sh}/m^{sh} \\ XY = \xi^i XY & 1 \leqslant i \leqslant p-1 \end{cases}$$

where ξ is any primitive p^{th} root of unity. Because R^{sh}/m^{sh} is separately closed and $char(R^{sh}/m^{sh}) \neq p$, this algebra does not depend upon the choice of a and b, showing that $\overline{\Gamma}$ is graded isomorphic to the $\mathbb{Z} \oplus \mathbb{Z}$-graded cyclic algebra determined by :

$$S2 : \begin{cases} X^P = Y_1^p \\ Y^P = Y_2^P \\ XY = \xi YX \end{cases}$$

and calculating the norm, it follows that this algebra is a domain. Again, using injectivity of the morphism :

$$Br^g(R^{sh}[\mathcal{P}]) \to Br^g(R^{sh}[\mathcal{P}]/m^{sh}[\mathcal{P}])$$

it follows that $\Lambda[\mathcal{P}] \otimes R^{sh}$ represents a non-trivial element in $Br^g(R^{sh}[\mathcal{P}])$, so its part of degree (o,o) cannot be an order in a matrixring since $R[\mathcal{P}] \otimes R^{sh}$ is regular and

$$Br^g(R^{sh}[\mathcal{P}]) \to Br(R^{sh}[\mathcal{P}]) \to Br(Q(R^{sh}[\mathcal{P}]))$$

finishing the proof.

<u>Theorem 8 :</u> All smooth orders over a local normal domain in a p^2-dimensional division algebra are conjugated.

<u>Proof.</u>

CASE 1 : Let Λ be a smooth order in Δ. By (S1) in the proof of Theorem 9 we know that there exists an étale extension $R \to S$ s.t.

$$\Lambda[\mathcal{P}] \otimes S \cong END_{S[\mathcal{P}]}(P)$$

where P is a graded finitely generated projective $S[\mathcal{P}]$-module. Because

$S\,[P]$ is graded local, P is graded free, i.e. of the form :

$$P \cong S\,[P](\sigma_1) \oplus \ldots \oplus S\,[P](\sigma_p)$$

where $\sigma_i \in \mathbb{Z}$ and $S\,[P]\,(\sigma_i)$ is the $\mathbb{Z}$-graded $S\,[P]$- module determined by taking for its homogeneous part of degree α :

$$S\,[P](\sigma_i)_\alpha = S\,[P]_{\alpha+\sigma_i}$$

Therefore, $\Lambda\,[P] \otimes S \cong M_p(S\,[P])(\sigma_1,\ldots,\sigma_p)$, where the homogeneous part of degree α of the ring on the right side is given by

$$\begin{pmatrix} S\,[P]_\alpha & & \vdots^{\,j} & \\ & \ddots & \vdots & \\ i \cdots & \cdots & S\,[P]_{\alpha+\sigma_i+\sigma_j} & \\ & & & \ddots \\ & & & & S\,[P]_\alpha \end{pmatrix}$$

An easy computation shows that up to conjugation in the part of degree zero, all σ_i may be chosen to be elements of the set $\{0,1,\ldots,p-1\}$. Further, we may assume that $\sigma_1 \leqslant \sigma_2 \leqslant \ldots \leqslant \sigma_p$ for otherwise one simply has to conjugate with a permutation matrix.

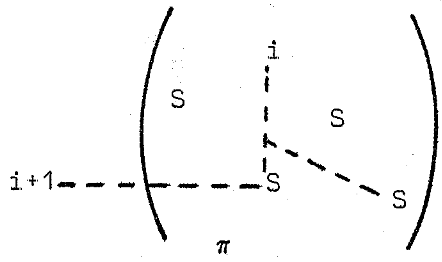
Because all isomorphisms occuring above are gradation preserving we have :

$$\Lambda \otimes R^{sh} \cong \begin{pmatrix} S & & \vdots^{\,j} & \\ i \cdots & \ddots & S\,[P]_{\sigma_i\sigma_j} & \\ & & \ddots & \\ & & & S \end{pmatrix} (N) = M_n(S)(\sigma_1,\ldots,\sigma_p)$$

Only very few rings $M_n(S)(\sigma_1,\ldots,\sigma_p)$ can occur.

E.g. if $\sigma_1 = \sigma_2 = \ldots = \sigma_p$, $M_p(S)(\sigma_1,\ldots,\sigma_p) = M_p(S)$ an Azumaya algebra, contradicting the fact that Λ is ramified.

Further, if $\sigma_1 \leqslant \sigma_2 \leqslant \dots \leqslant \sigma_p$ and if $\sigma_i = \sigma_{i+1}$ for some i, then $M_p(S)(\sigma_1,\dots,\sigma_p)$ is no longer a tame order because it has the form :



where the upper-triangular entries are S, the lower triangular are either (π) or S and at least one of them, namely $(i+1,i)$ equals S. Localizing at (π) and using the characterization of hereditary orders given in [1] it follows that $(\Lambda \otimes S)_{(\pi)}$ is not hereditary, whence $\Lambda \otimes S$ is not tame. Because $R \to S$ is étale this cannot occur.

Therefore, the only possibility is $M_p(S)(0,1,2,\dots,p-1)$ and Grothendieck descent finishes the proof.

CASE 2 : From Theorem 7 we retain that $\Lambda[P]$ cannot be split by an étale extension of R. Nevertheless, mimicking the ungraded case, $\Lambda[P]$ can be split by a graded étale extension of $R[P]$ because $R[P]$ is graded local, here graded étale is defined in the obvious way).

Denote $S = R[X]/(X^p-\pi)$ then $S(\Phi)$ will be defined to be the $\mathbb{Z} \oplus \mathbb{Z}$-graded ring :

$$
\begin{array}{cccccccccc}
(\pi' X^{-2}) & \oplus & (\pi' X^{-1}) & \oplus & (\pi') & \oplus & (\pi' X) & \oplus & (\pi' X^2) & \oplus \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus & \\
(\pi' X^{-2}) & \oplus & (\pi' X^{-1}) & \oplus & (\pi') & \oplus & (\pi' X) & \oplus & (\pi' X^2) & \oplus \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus & \\
(X^{-2}) & \oplus & (X^{-1}) & \oplus & S & \oplus & (X) & \oplus & (X^2) & \oplus \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus & \\
(X^{-2}) & \oplus & (X^{-1}) & \oplus & S & \oplus & (X) & \oplus & (X^2) & \oplus \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus & \\
(\pi'^{-1} X^{-2}) & \oplus & (\pi'^{-1} X^{-1}) & \oplus & (\pi'^{-1}) & \oplus & (\pi'^{-1} X) & \oplus & (\pi'^{-1} X^2) & \oplus
\end{array}
$$

This is such a graded étale splitting ring for $\Lambda[P]$. Again, mimicking the argument of case 1 above

$$\Lambda[P] \underset{R[P]}{\otimes} S(\Phi) \cong M_p(S(\Phi))(\sigma_1,\dots,\sigma_p)$$

where the $\sigma_i \in \mathbb{Z} \oplus \mathbb{Z}$ may be chosen to lie in the set :

$$\{(0,\alpha) \mid 0 \leqslant \alpha < p\}$$

It can be shown that all rings $M_p(S(\Phi))(\sigma_1,\dots,\sigma_p)$ are graded isomorphic to

$$M_p(S(\Phi))((0,0),\dots,(0,0))$$

Applying a graded version of Grothendieck's descent theorem, it follows that whenever Λ and Γ are two smooth R-orders in Δ, then

$$\Lambda[P] \cong \Gamma[P]$$

as graded $R[P]$-algebras. As always, this isomorphism is given by conjugation with a unit

$$\alpha \in \Delta[X_1, X_1^{-1}, X_2, X_2^{-1}]$$

and because this ring is a $\mathbb{Z} \oplus \mathbb{Z}$-graded domain, α is homogeneous, i.e.

$$\alpha = \delta\, X_1^{l_1} X_2^{l_2}$$

with $\delta \in \Delta^{\star}$. Finally, computing parts of degree (0,0) gives us that

$$\Lambda = \delta^{-1}.\Gamma.\delta$$

finishing the proof and this paper.

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