The Semi-center of an Enveloping Algebra is Factorial.

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84-03

January, 1984.

(★) work supported ny an NFWO/FNRS-grant.

THE SEMI-CENTER OF AN ENVELOPING ALGEBRA IS FACTORIAL

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INTRODUCTION.

Throughout this note, L will be a nonzero finite dimensional Lie algebra over a field k of characteristic zero. Let U(L) be the universal enveloping algebra of L with center Z(U(L)), D(L) will be the division ring of quotients of U(L) with center Z(D(L)). For each $\lambda \in L^*$, we denote by $D(L)_\lambda$ the set of those $u \in D(L)$ such that $xu-ux = \lambda(x)u$ for all $x \in L$. Its elements are called the semi-invariants of D(L) relative to λ. Clearly, $D(L)_\lambda . D(L)_\mu \subset D(L)_{\lambda+\mu}$ for all $\lambda, \mu \in L^*$. We denote by $\Lambda_D(L)$ the subgroup of L^* consisting of those $\lambda \in L^*$ such that $D(L)_\lambda \neq 0$. The sum of the $D(L)_\lambda$ is direct and is a $\Lambda_D(L)$-graded

(*) work supported by an NFWO/FNRS-grant

subalgebra Sz(D(L)) of D(L), called the semi-center of D(L). If we put $U(L)_\lambda$ = $U(L) \cap D(L)_\lambda$, then the sum Sz(U(L)) of the $U(L)_\lambda$ is called the semi-center of U(L). It is a commutative subalgebra of U(L), which never reduces to k and which is stable under all automorphisms of U(L).

If k is algebraically closed, C. Moeglin has shown that Sz(U(L)) is a unique factorization domain [9]. The main aim of this note is to extend this result to any base field of characteristic zero. We wish to thank Prof. F. Van Oystaeyen for his helpful comments.

1. FACTORIALITY OF Sz(D(L)).

Using the result of M.P. Malliavin [8] that each nonzero two-sided ideal of U(L) contains a nonzero semi-invariant, it is fairly easy to check that w is a semi-invariant of D(L) if and only if w is a quotient of two semi-invariants of U(L) [6, Prop. 1.8]. It follows that Sz(D(L)) is the localization of Sz(U(L)) at the multiplicative system of nonzero semi-invariants of U(L).

PROPOSITION 1. The semi-center Sz(D(L)) of D(L) is a factorial domain (UFD).

PROOF. Clearly, Sz(D(L)) is graded by the torsion-free Abelian (hence totally ordered) group $\Lambda_D(L)$. By [3, Cor. 3.4 (1)] we have to verify that Sz(D(L)) satisfies the ascending chain condition on principal ideals. So, let $Sz(D(L))a_1 \subset Sz(D(L))a_2 \subset \ldots$ be an ascending chain with $a_i \in Sz(D(L))$ for all i. Let k' be an algebraic closure of k and put $L' = L \otimes k'$. Then Sz(U(L')) is factorial [9] and thus also its localization Sz(D(L')). Therefore, there exists a natural number N such that for all $m \geqslant N$ we have $Sz(D(L'))a_N = Sz(D(L'))a_m$. Hence, we can find a unit $w \in Sz(D(L'))^*$ such that $a_N = w.a_m$. Now, $\Lambda_D(L')$ being totally ordered, every unit w of Sz(D(L')) is homogeneous, i.e. a semi-invariant of D(L'). But $w \in D(L)$, so w is a nonzero semi-invariant of D(L) and thus a unit of Sz(D(L)). Consequently, $Sz(D(L))a_N = Sz(D(L))a_m$ for all $m \geqslant N$.

2. THE SEMI-CENTER OF U(L) IS A KRULL DOMAIN.

A domain A with field of fractions Q(A) and graded by a torsion free Abelian group is a Krull domain if and only if the following conditions are satisfied :

(1) $(I:I) = \{x \in Q(A) : Ix \subset I\} = A$ for every graded ideal I of A,

(2) A satisfies the ascending chain condition on graded divisorial ideals of A,

(3) $Q^g(A)$, the localization of A at the multiplicatively closed set of nonzero homogeneous elements, is a Krull domain.

[1], [3] and [4. Th. 2.8].

Let k' be an algebraic closure of k and put $L'=L \otimes k'$. Then $U(L') = U(L) \otimes k'$ and each $\sigma \in Gal(k'/k) = Aut_k(k')$ can be extended to a k-automorphism, also denoted by σ, of U(L') and D(L'). The orbit $\{\sigma(u) : \sigma \in Gal(k'/k)\}$ of u is finite for every $u \in U(L')$. Furthermore, $u \in U(L)$ iff $\sigma(u) = u$ for every $\sigma \in Gal(k'/k)$. If $u \in U(L')$ is a semi-invariant with weight λ, then σ(u) is a semi-invariant of U(L') with weight $\sigma\circ\lambda\circ\sigma^{-1}$. Therefore, Gal(k'/k) acts on Sz(U(L')).

LEMMA 2. Let u, v be nonzero elements of U(L). If uv and v belong to Sz(U(L)), then so does u.

PROOF. By assumption $v=v_1+\ldots+v_n$ and (*) $uv=w_1+\ldots+w_s$ where $v_j \in U(L)_{\mu_j}$, $w_r \in U(L)_{\gamma_r}$ with $\mu_j, \gamma_r \in L^*$. Each v_j(resp. w_r) is also a semi-invariant of

$U(L')$ with weight μ'_j (resp. γ'_r). Because $uv \in Sz(U(L'))$ and k' is algebraically closed, $u \in Sz(U(L'))$ by [9]. So, $u=u_1+\ldots+u_m$ where $0\neq u_i \in U(L')_{\lambda_i}$ with $\lambda_i \in (L')^*$. We show that $u \in Sz(U(L))$ by induction on m. This is clear if $m=1$, because then u is a semi-invariant of $U(L)$ as $u \in U(L)$. Now, let $m > 1$. Take $\sigma \in Gal(k'/k)$, then

$$u = \sigma(u) = \sigma(u_1)+\ldots+\sigma(u_m)$$

where $\sigma(u_i)$ is a semi-invariant with weight $\sigma\circ\lambda_i\circ\sigma^{-1}$. By the uniqueness of the decomposition of u, we see that $\{\sigma(u_1),\ldots, \sigma(u_m)\} = \{u_1,\cdots,u_m\}$ and $\{\sigma\circ\lambda_1\circ\sigma^{-1},\ldots, \sigma\circ\lambda_m\circ\sigma^{-1}\} = \{\lambda_1,\ldots, \lambda_m\}$. On the other hand, $uv = \Sigma\, u_i v_j$ where $u_i v_j$ is a semi-invariant with weight $\lambda_i+\mu'_j$.
Regrouping the terms and comparing with (*) we conclude that $\gamma'_1 = \lambda_i+\mu'_j$ for some i,j. Hence, $\lambda_i | L = (\gamma'_1-\mu'_j)| L = \gamma_1-\mu_j \in L^*$. Consequently, $\sigma\circ\lambda_i\circ\sigma^{-1} = \lambda_i$ since these are two k'-linear functionals, taking the same value on each $x\in L$. Hence $\sigma(u_i)=u_i$ for all $\sigma \in Gal(k'/k)$ yielding that u_i is a semi-invariant of $U(L)$. So, $(u-u_i)v = uv-u_iv \in Sz(U(L))$ whence by induction $u-u_i \in Sz(U(L))$ and therefore finally $u \in Sz(U(L))$.

COROLLARY 3. $Sz(D(L)) \cap U(L) = Sz(U(L))$

PROOF. Let $0\neq u \in Sz(D(L)) \cap U(L)$, then $u=w_1+\ldots+w_m$ where each w_i is a nonzero semi-invariant of $D(L)$. Now, $w_i=u_i.v_i^{-1}$ for some semi-invariants u_i,v_i of $U(L)$. We may assume that $v_1=\ldots=v_m=v$. Then $0\neq v \in Sz(U(L))$ and $uv=u_1+\ldots+u_m \in Sz(U(L))$. By the previous lemma $u \in Sz(U(L))$ finishing the proof.

PROPOSITION 4. The semi-center of $U(L)$ is a Krull domain.

PROOF. $Sz(U(L))$ is $\Lambda_D(L)$-graded and $Q^g(Sz(U(L))) = Sz(D(L))$ is a Krull domain by

Proposition 1. Let I be a graded ideal of Sz(U(L)), then I is generated by semi-invariants of U(L), hence by normalizing elements of U(L). It follows that U(L)I = IU(L) is a twosided ideal of U(L). Let $q \in Q(Sz(U(L)))$ be an element of (I:I), then $Sz(D(L))q = Sz(D(L))Iq \subset Sz(D(L))I = Sz(D(L))$ yielding that $q \in Sz(D(L))$. Also, $U(L)Iq \subset U(L)I$. This forces $q \in U(L)$ since U(L) is a maximal order of which U(L)I is a two-sided ideal [5]. Therefore, for any graded ideal I of Sz(U(L)), $(I:I) \subset Sz(D(L)) \cap U(L) = Sz(U(L))$ by Corollary 3. Let $I_0 \subset I_1 \subset \ldots$ be an ascending chain of graded divisorial ideals of Sz(U(L)). Because U(L) is Noetherian there exists a natural number N such that $U(L)I_m = U(L)I_N$ for all $m \geqslant N$. Let $q \in (I_m:I_N) = \{x \in Q(Sz(U(L))) : I_N x \subset I_m\}$ then $q \in Sz(D(L))$ since $Sz(D(L))I_m = Sz(D(L))I_N = Sz(D(L))$. Further, $U(L)I_N q \subset U(L)I_m = U(L)I_N$ whence $q \in U(L)$. Therefore, $I_m * I_N^{-1} = (I_m:I_N) \subset Sz(D(L)) \cap U(L) = Sz(U(L))$ and so $I_m \subset I_N$ whence $I_m = I_N$ for all $m \geqslant N$, finishing the proof.

3. FACTORIALITY OF THE EXTENDED SEMI-CENTER.

As we noticed above, Gal(k'/k) acts on Sz(U(L')). Its fixed ring under this action, $Sz(U(L')) \cap U(L)$, will be called the extended semi-center of U(L) and will be denoted by Esz(U(L)). In [6] it was shown that Esz(U(L)) is a factorial domain. Using Proposition 4 we will now give a more ringtheoretical proof of this result, based on Samuel's theory on descent of class groups, cfr. e.g. [7].

PROPOSITION 5. The extended semi-center of U(L) is a factorial domain.

PROOF. Let K be a finite Galois extension of k with corresponding Galois group G, then as before G acts on $Sz(U(L \otimes K))$. By Proposition 4 $Sz(U(L \otimes K))$ is a

Krull domain, hence so is its fixed ring $Sz(U(L \otimes K))^G$. By Samuel's descent theory we know that the kernel of the natural morphism :

$$\varphi : Cl(Sz(U(L \otimes K))^G) \rightarrow Cl(Sz(U(L \otimes K))$$

embeds injectively into $H^1(G, Sz(U(L \otimes K))^*) \cong H^1(G, K^*)$ which is trivial by Hilbert 90, cfr. e.g. [11, ch X, Prop. §1.1], hence φ is monomorphic.

The family of all finite Galois extensions of k form a directed system and because class groups conmute with these direct limits, we obtain that the class group $Cl(Esz(U(L))) \cong \varinjlim Cl(Sz(U(L \otimes K))^G)$ embeds injectively into $\varinjlim Cl(Sz(U(L \otimes K))) \cong Cl(Sz(U(L'))) = 1$, finishing the proof.

COROLLARY 6. [6] Let u be a nonzero semi-invariant of U(L). Then there exist unique irreducible, pairwise non-associated, semi-invariants $u_1, \ldots, u_n$ and natural numbers m_i, $a \in k^*$ such that $u = a.u_1^{m_1} \ldots u_n^{m_n}$. Moreover, each u_i is prime in Sz(U(L)).

PROOF. Clearly, $u \in Esz(U(L))$ which is a UFD. So, let $u = a.u_1^{m_1} \ldots u_n^{m_n}$ be a unique factorization of u into irreducible factors. As u is a semi-invariant, so are $u_1, \ldots, u_n$ [6]. Next, suppose u_i divides vw in Sz(U(L)), where v,w are nonzero elements of Sz(U(L)). In Esz(U(L)) u_i is prime and divides vw. Therefore u_i divides v (or w) in Esz(U(L)), i.e. $v = u_i y$ for some nonzero $y \in Esz(U(L))$. Now, $u_i y$ and u_i belong to Sz(U(L)). By lemma 2, this implies that $y \in Sz(U(L))$ and thus u_i is prime in Sz(U(L)).

The following shows that it may happen that $Esz(U(L)) \neq Sz(U(L))$.

EXAMPLE 7. [6]

Let L be the Lie algebra over $\mathbb{R}$ with basis x,y,z such that $[x,y] = y+z$, $[x,z] = -y+z$

and $[y,z]=0$. One verifies that $Sz(U(L))=\mathbb{R}[y^2+z^2]$. In $L\otimes\mathbb{C}$ we consider the basis $x, u_1=y+iz$, $u_2=y-iz$. Then $Sz(U(L\otimes\mathbb{C})) = \mathbb{C}[u_1,u_2]$ and $Esz(U(L))=\mathbb{R}[y,z]$.

4. FACTORIALITY OF Sz(U(L)).

We are now in a position to state the main result of this note.

THEOREM 8. The semi-center Sz(U(L)) of U(L) is a unique factorization domain.

PROOF. From Corollary 6 we know that Sz(U(L)) is a graded UFD, that is, a graded integral domain such that each nonzero, nonunit homogeneous element is a product of prime elements [1, Definition 4.1]. Moreover, the localization of Sz(U(L)) at the multiplicative system of nonzero homogeneous elements, $Q^g(Sz(U(L)))=$ Sz(D(L)) is a UFD by Proposition 1. Invoking [1, Theorem 4.4], we may conclude that Sz(U(L)) is also a UFD.

5. FACTORIALITY OF Sz(U($\mathfrak{g}$)).

We shall now take a brief look at a more general situation. Let R be a Noetherian Krull domain of characteristic zero with quotient field k. Let $\mathfrak{g}$ be a Lie algebra over R such that $\mathfrak{g}$ is a free R-module of finite rank and put $L=\mathfrak{g}\otimes k$. Recently, D. Reynaud has studied U($\mathfrak{g}$) and its semi-center A=Sz(U($\mathfrak{g}$)), which is defined analogously [10]. The results we shall quote from this work do not require that k be algebraically closed. For instance, $Sz(U(\mathfrak{g}))\otimes k = Sz(U(L))$ [10, lemma 8, p. 47]. Therefore Sz(U(L)) is precisely the localization of A at the multiplicative set S of nonzero elements of R. Also, by [10, lemma 1, p. 3] it is easy to see that if u and v are nonzero elements of U($\mathfrak{g}$) such that uv is an element of R, then so are u and v.

LEMMA 9. (1) Let I be a prime ideal of R. Then IA is a prime ideal of A and $IA \cap R = I$.

(2) Let P be a height-one prime ideal of A. Then $P \cap R$ has height at most one, i.e. $R \subset A$ satisfies (PDE).

PROOF. (1) A/IA is a subring of $U(\mathfrak{g})/IU(\mathfrak{g})$, which contains no zero divisors [10, proposition 4, p. 24]. Hence IA is prime in A. Also, $I \subset IA \cap R \subset IU(L) \cap R = I$.

(2) Suppose $P \cap R \neq 0$ and let I be a nonzero prime ideal of R, contained in $P \cap R$. Then P contains IA, which is prime in A by (1). Therefore P=IA and $I = IA \cap R = P \cap R$.

The following generalizes theorem 10 of [10, p. 48].

THEOREM 10. $Cl(R) \cong Cl(Sz(U(\mathfrak{g})))$. In particular, R is factorial if and only if $Sz(U(\mathfrak{g}))$ is factorial.

PROOF. Again, we put $A = Sz(U(\mathfrak{g}))$ and $S = R \setminus 0$. By Nagata's theorem [7, p. 36], there is a short exact sequence $0 \to K \to Cl(A) \to Cl(S^{-1}A) \to 0$ where K is the subgroup of Cl(A) generated by the classes [P] of the height-one primes P of A which meet S, i.e. $P \cap R \neq 0$. By theorem 8, $S^{-1}A = Sz(U(L))$ is factorial, so $Cl(S^{-1}A) = 0$ and thus $K = Cl(A)$. Since the extension $R \subset A$ satisfies (PDE), the natural map $f: Cl(R) \to Cl(A)$, sending [I] into $[\overline{IA}]$ where $\overline{IA} = (A:(A:IA))$, is a homomorphism [3, p. 93].
Take $[P] \in K = Cl(A)$ where P is a height-one prime of A such that $P \cap R \neq 0$. Using lemma 9, it is easy to verify that $f([P \cap R]) = [P]$, showing that f is onto.
So, it remains to prove that f is injective. Let $[I] \in \ker f$, where I is a nonzero divisorial ideal of A. Clearly, $\overline{IA} = Ab$ for some $b \in A$. Take $0 \neq x \in I$. As $I \subset Ab$, x is of the form ab for some $a \in A$. Since $x \in R$, so is b. Now, we claim that $\overline{IA} \cap R = I$. Take $x \in \overline{IA} \cap R$. Then $x(A:IA) \subset A$ and thus $x(R:I) \subset A$ as $(R:I) \subset (A:IA)$. But (R:I) is a fractional R-ideal, so $s(R:I) \subset R$ for some nonzero $s \in R$.

Next, take $0 \neq z \in x(R:I)$. Then $sz \in xs(R:I) \subset xR \subset R$ which implies that $z \in R$. Therefore $x(R:I) \subset R$ and thus $x \in \overline{I}=I$. This establishes the claim. Finally, $I=\overline{IA} \cap R=Ab \cap R=Rb$, showing that ker f is trivial.

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