

**HOMOLOGICAL PROPERTIES
OF TRACE RINGS
OF GENERIC MATRICES**

by

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1. Introduction and motivation .

The trace ring $T_{m,n}$ of $m \times n$ generic matrices arises naturally in rather different algebraic disciplines :

affine p.i.-theory : up to some extent, affine p.i.-theory is the study of the rings $G_{m,n}$ of $m \times n$ generic matrices. However, these rings are rather difficult to handle and it turns out to be convenient to study the overring $T_{m,n}$ of $G_{m,n}$ and consequently to pull the obtained information back to $G_{m,n}$, cfr. [1],[4].

representation theory : in [2],[4] and [12] it is shown that the study of finite dimensional representations of the free associative algebra $k \langle x_1, \dots, x_m \rangle$ in m variables is essentially the study of the maximal ideals of the trace rings $T_{m,n}$, $n \in \mathbb{N}$.

invariant theory : Procesi [14] proved that the ring of invariant polynomial mappings from m copies of $M_n(k)$ to $M_n(k)$ under the natural action (i.e. by conjugation) of $GL_n(k)$ is precisely the trace ring $T_{m,n}$.

maximal orders : if Λ is an affine maximal [5] or tame [9] order over a normal domain R , then there is a sufficiently large ring of generic matrices $G_{m,n}$ and a specialization morphism $\phi : G_{m,n} \rightarrow \Lambda$ which factorizes through $T_{m,n}$. Artin and Schofield [3] have proved that $T_{m,n}$ is a maximal order. Therefore, trace rings can be viewed as generic maximal orders.

It was this last property that motivated me to look for analogies between these trace rings and commutative polynomial rings. In particular we were interested in the following :

Question A : Determine all $m, n \in \mathbb{N}$ such that $gldim(T_{m,n}) < \infty$

The only result existing in the literature is due to Small and Stafford [17]. They proved that $gldim(T_{2,2}) = 5$. In this paper we aim to solve question A in p.i.-degree two (i.e. $n = 2$) using some powerful results of Procesi [15] on the Poincare series of trace rings. The main result will be :

Theorem : $gldim(T_{m,2}) < \infty \Leftrightarrow m = 0, 1, 2$ or 3 .

In proving this theorem we will also show that the trace ring of every prime p.i. ring of p.i.-degree two is a specialization of an iterated Öre extension , solving the embedding problem for quaternionic orders.

In the last section we will show that the rational expression of the Poincaré series of the ring of m generic 2×2 matrices can never be A pure inverse.

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2. Poincare series and finite global dimension

Throughout this paper, k will be a field of characteristic zero and

$$k \langle x_1, \dots, x_m \rangle$$

will be the free k -algebra in m variables. If I_n is the ideal of all identities satisfied by $n \times n$ -matrices in m variables, then $G_{m,n} = k \langle x_1, \dots, x_m \rangle / I_n$ is the ring of $m \times n$ -generic matrices, cfr. e.g. [13],[11].

A more convenient description of this ring is obtained in the following way. Let R be the commutative polynomial ring $k[t_{ij}^l; 1 \leq i \leq j \leq n; 1 \leq l \leq m]$ and consider the matrices $X_l = (t_{ij}^l)_{i,j}$ in $M_n(R)$. Then $G_{m,n}$ is the subring of $M_n(R)$ generated as a k -algebra by the elements $X_l; 1 \leq l \leq m$.

The trace ring of $m \times n$ -generic matrices, $T_{m,n}$, is the subring of $M_n(R)$ generated as a k -algebra by $G_{m,n}$ and all the coefficients of the reduced characteristic polynomials of its elements, cfr. e.g. [1],[4].

$G_{m,n}$ is a positively graded k -algebra if we give every generic matrix degree one. Similarly, $T_{m,n}$ is positively graded if the trace of an element is given the same degree as the element.

In commutative algebraic geometry, the Hilbert polynomial of the homogeneous coordinate ring is important in order to define invariants (e.g. the genus) of the variety [10]. However, if a graded k -algebra is not generated by its elements of degree one, one can no longer prove the existence of such a polynomial. A manageable substitute for it in this case is the Poincaré series, cfr. e.g. [18]. Let us recall the definition :

If $A = \bigoplus_{i=0}^{\infty} A_i$ is a positively graded k -algebra and $M = \bigoplus_{i=0}^{\infty} M_i$ is a graded left A -module, then one defines the Poincaré series of A (resp. of M) to be the power series in $\mathbb{Z}[[t]]$:

$$\mathcal{P}(A, t) = \sum_{i=0}^{\infty} \dim_k(A_i) \cdot t^i$$

$$\mathcal{P}(M, t) = \sum_{i=0}^{\infty} \dim_k(M_i) \cdot t^i$$

If it exists, one is usually interested in a rational expression for these series. Because $T_{m,n}$ is a finite module over its center which is an affine k -algebra [1], it follows from the Hilbert-Serre theorem that the Poincaré series of $T_{m,n}$ is a rational function, cfr. e.g. [8].

Using the representation theory of the general linear and symmetric group, Formanek [8] was able to compute the Poincaré series (in a multi-gradation) of the trace ring $T_{m,n}$. In this paper, we prefer Procesi's approach [15] for the Poincaré series of trace rings of 2×2 - generic matrices. So, from now on we assume $n = 2$ and we will write $T_m = T_{m,2}$. By separating traces, Procesi shows that :

$$T_m \simeq T_m^o[\text{Tr}(X_1), \dots, \text{Tr}(X_m)]$$

where T_m^o is the trace ring of generic 2×2 matrices with trace zero. Therefore,

$$\mathcal{P}(T_m, t) = 1/(1-t)^m \cdot \mathcal{P}(T_m^o, t)$$

Moreover, he gives a k -vectorspace basis for T_m^o .

Recall that a standard Young tableaux is a Young diagram (cfr. e.g. [16]) filled with numbers from 1 to m such that in each row the numbers strictly increase from left to right and in each column the numbers do not decrease from top to bottom.

Theorem [Procesi,15] There is a natural one-to-one correspondence between a k -vectorspace basis of T_m^o and all standard Young tableaux with at most three columns.

Moreover, the degree of an element corresponding to such a standard Young tableau is equal to the number of cells in the corresponding Young diagram.

This result allows us, at least in principle, to compute the Poincaré series of T_m^o :

$$\mathcal{P}(T_m^o, t) = \sum_{a,b,c \in \mathbb{N}} \mathcal{L}_{3^a 2^b 1^c} \cdot t^{3a+2b+1c}$$

where $\mathcal{L}_{3^a 2^b 1^c}$ is the number of standard Young tableaux corresponding to a Young diagram of shape $3^a 2^b 1^c$.

The next result (which is perhaps well known) is included for lack of a convenient reference. Its proof is due to J.T. Stafford :

Lemma 1 : Let A be a left Noetherian, positively graded k -algebra such that $A_0 = k$ and $\text{gldim}(A) < \infty$. If I is a graded left ideal of A , then :

$$\mathcal{P}(A/I, t) = f(t) \cdot \mathcal{P}(A, t)$$

for some polynomial $f(t) \in \mathbb{Z}[t]$.

Proof Using the fact that A is left Noetherian and $\text{gldim}(A) < \infty$ one can show that there exists a resolution :

$$0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \rightarrow A \rightarrow A/I \rightarrow 0$$

of A/I as a left A -module such that F_1, \dots, F_{n-1} are graded free left A -modules of finite rank, F_n is a graded projective left A -module and all morphisms are gradation preserving. Because $A_0 = k$ one can show, cfr. e.g. [6], that F_n is also graded free. Therefore, we have :

$$\mathcal{P}(A/I, t) = \mathcal{P}(A, t) + \sum_{i=1}^n (-1)^i \cdot \mathcal{P}(F_i, t)$$

F_i is graded free with basis, say $f_1^{(i)}, \dots, f_{m_i}^{(i)}$, where $\deg(f_j^{(i)}) = d_j^{(i)}$, then :

$$\mathcal{P}(F_i, t) = \mathcal{P}(A, t) \cdot (t^{d_1^{(i)}} + \dots + t^{d_{m_i}^{(i)}})$$

finishing the proof.

This result gives a necessary condition on the Poincaré series of a graded k -algebra to have finite global dimension :

Corollary 1 : Let A be a left Noetherian, positively graded k -algebra such that $A_0 = k$. If $\text{gldim}(A) < \infty$, then :

$$\mathcal{P}(A, t) = 1/f(t)$$

for some polynomial $f(t) \in \mathbb{Z}[t]$.

Proof Take $I = A_+ = \bigoplus_{i=1}^{\infty} A_i$, then $\mathcal{P}(A/I, t) = \mathcal{P}(k, t) = 1$.

Of course, the main problem is to determine a rational expression for the Poincaré series of T_m^o . We will solve this difficulty by proving a result which is of some independent interest.

3. The embedding theorem in p.i. degree two.

In recent years, several attempts have been made to start off noncommutative algebraic geometry, cfr. e.g. [4],[19]. An essential gap in these theories is, at least to the author, a noncommutative version of the embedding of varieties in affine or projective n -space, cfr. e.g. [10]. Therefore, one would like to answer :

Question B : Let A be an affine prime p.i.-algebra over a field k with trace ring TA , cfr. [1]. Does there exist an iterated Ore extension A over k and an epimorphism $A \rightarrow TA$?

Even if we weaken our hypotheses on A , e.g. A a maximal order having finite global dimension, it is not known to the author whether this question can be answered affirmatively.

In p.i. degree two, such an embedding result exists :

Theorem 1 : The trace ring TA of an affine prime p.i. algebra A of p.i. degree two is an epimorphic image of an iterated Ore extension.

Proof Because A is affine of p.i. degree two, there exists a natural number m and a specialization map :

$$G_{m,2} \quad \rightarrow \quad \Lambda$$

$$\downarrow \qquad \qquad \downarrow$$

$$T_{m,2} \quad \rightarrow \quad T(\Lambda)$$

therefore it suffices to prove the result for the rings $T_m = T_{m,2}$. Consider the iterated Öre extension :

$$\Lambda_m = k[a_{ij}; 1 \leq i < j \leq m][a_1][a_2, \sigma_2, \delta_2] \dots [a_m, \sigma_m, \delta_m]$$

where for every $i < j$ one defines $\sigma_j(a_i) = -a_i$ and $\delta_j(a_i) = a_{ij}$. Of course, one has to verify that every σ_k is an automorphism (which is trivial) and that δ_k is a σ_k -derivation of the subalgebra :

$$\Lambda_m(k) = k[a_{ij}; 1 \leq i < j \leq m][a_1] \dots [a_{k-1}, \sigma_{k-1}, \delta_{k-1}]$$

We have defined δ_k on a generating set , so it is defined on $\Lambda_m(k)$ and we only have to check that it preserves the commutation rules , i.e. we have to verify for $i < j < k$:

$$\delta_k(a_i \cdot a_j + a_j \cdot a_i) = \delta_k(a_{ij}) = 0$$

Now, $\delta_k(a_i \cdot a_j) = a_{ik} \cdot a_j - a_{jk} \cdot a_i$ and $\delta_k(a_j \cdot a_i) = a_{jk} \cdot a_i - a_{ik} \cdot a_j$, done.

Now, we define a map :

$$\phi_m : \Lambda_m \rightarrow T_m^o$$

by sending a_i to $X_i^o = X_i - 1/2 \cdot \text{Tr}(X_i)$ and a_{ij} to $\text{Tr}(X_i^o \cdot X_j^o)$. ϕ_m is an algebra morphism because in T_m^o we have $X_i^o \cdot X_j^o + X_j^o \cdot X_i^o = \text{Tr}(X_i^o \cdot X_j^o)$. This follows from the fact that for any pair of 2×2 matrices A, B one has :

$$A \cdot B + B \cdot A = \text{Tr}(A \cdot B) + \text{Tr}(A) \cdot \text{Tr}(B) - \text{Tr}(A) \cdot B - \text{Tr}(B) \cdot A$$

and $\text{Tr}(X_k^o) = 0$.

Moreover, ϕ_m is epimorphic : it follows from Procesi's description of T_m^o that T_m^o is generated by the elements : $X_i^o, Tr(X_i^o.X_j^o)$ and $Tr(X_i^o.X_j^o.X_k^o)$. Now,

$$Tr(X_i^o.X_j^o.X_k^o) = 1/3.S_3(X_i^o, X_j^o, X_k^o)$$

finishing the proof of our claim.

Finally, we have seen above that $T_m = T_m^o[Tr(X_1), \dots, Tr(X_m)]$ and therefore there exists an epimorphism :

$$\phi_m^{\sim} : \Lambda_m[y_1, \dots, y_m] \rightarrow T_m$$

finishing the proof.

Corollary 2 : Let Λ be a prime affine p.i. algebra of p.i. degree two such that Λ is positively graded as a k -algebra with the gradation induced by the specialization map $G_{m,2} \rightarrow \Lambda$. Then the Poincaré series of $T\Lambda$ has a rational expression of the form :

$$P(T\Lambda, t) = f(t)/(1-t^2)^{m(m-1)/2} \cdot (1-t)^{2.m}$$

for some polynomial $f(t) \in \mathbb{Z}[t]$.

Proof If we define $deg(a_i) = 1$ and $deg(a_{ij}) = 2$, then Λ_m is a positively graded k -algebra which, as a graded vectorspace, looks like the commutative polynomial ring $k[a_i, a_{jk}]$. Therefore, we can compute its Poincaré series :

$$P(\Lambda_m) = 1/(1-t^2)^{m(m-1)/2} \cdot (1-t)^m$$

Now, $\Lambda_m[y_1, \dots, y_m]$ has finite global dimension, its part of degree zero is k , it is positively graded and there exists a gradation preserving epimorphism :

$$\Lambda_m[y_1, \dots, y_m] \rightarrow T_m \rightarrow \Lambda$$

Therefore, we can apply lemma 1 and get :

$$\mathcal{P}(T\Lambda, t) = f(t)/(1-t^2)^{m \cdot (m-1)/2} \cdot (1-t)^{2m}$$

for some polynomial $f(t) \in \mathbb{Z}[t]$.

4. Global dimension of trace rings.

We are now in a position to state and prove the main result of this paper :

Theorem 2 : $gldim(T_m) < \infty \Leftrightarrow m = 0, 1, 2$ or 3 .

Proof : Above we recalled Procesi's expression of the Poincaré series of T_m^o

$$(1) : \mathcal{P}(T_m^o, t) = \sum_{a,b,c \in \mathbb{N}} \mathcal{L}_{3^a 2^b 1^c} \cdot t^{3a+2b+c}$$

where $\mathcal{L}_{3^a 2^b 1^c}$ denotes the number of standard Young tableaux corresponding to a Young diagram of shape $3^a 2^b 1^c$. This number was computed by H. Weyl in case $m \geq 4$:

$$\mathcal{L}_{3^a 2^b 1^c} = (1+b) \cdot (1+c) \cdot (1+b+c/2) \cdot \prod_{j=3}^{m-1} (1+(a+b+c)/j) \cdot \prod_{j=2}^{m-2} (1+(a+b)/j) \cdot \prod_{j=1}^{m-3} (1+a/j)$$

This formula allows us to compute the first terms in (1) :

$$\mathcal{L}_1 = m; \mathcal{L}_2 = m(m-1)/2; \mathcal{L}_1^2 = m(m+1)/2$$

$$\mathcal{L}_3 = m(m-1)(m-2)/6; \mathcal{L}_{21} = m(m+1)(m-1)/3; \mathcal{L}_{1^3} = m(m+1)(m+2)/6$$

$$\mathcal{L}_{31} = (m-1)(m-2)m(m+1)/8; \mathcal{L}_{2^2} = (m-1)(m+1)m^2/12$$

$$\mathcal{L}_{21^2} = (m-1)m(m+1)(m+2)/8; \mathcal{L}_{1^4} = m(m+1)(m+2)(m+3)/24$$

This gives us :

$$(2) : \mathcal{P}(T_m^o, t) = 1 + m \cdot t + m^2 \cdot t^2 + m(2m^2 + 1)/3 \cdot t^3 + m(m+1)(3m^2 - m + 2)/8 \cdot t^4 + \dots$$

On the other hand, in the proof of Theorem 1 we defined the k -algebra epimorphism $\phi_m : \Lambda_m \rightarrow T_m^o$, yielding that :

$$(3) : \mathcal{P}(T_m^o, t) = f(t)/(1-t^2)^{m(m-1)/2} \cdot (1-t)^m$$

for some polynomial $f(t) \in \mathbb{Z}[t]$. Now, suppose that T_m^o (or equivalently T_m) has finite global dimension, then (3) combined with corollary 1 implies that the Poincaré series of T_m^o should have the rational expression :

$$\mathcal{P}(T_m^o, t) = 1/(1+t)^\alpha \cdot (1-t)^\beta$$

for some natural numbers α and β . The power series expansion of this expression is :

$$(4) : 1 + (\beta - \alpha) \cdot t + (\alpha(\alpha + 1)/2 + \beta(\beta + 1)/2 - \alpha \cdot \beta) \cdot t^2 + \dots$$

Comparing (2) with (4), α and β should be solutions of the following set of equations

$$\beta - \alpha = m$$

$$\alpha(\alpha + 1) + \beta(\beta + 1) - 2\alpha\beta = 2m^2$$

Therefore, $\alpha = m(m-1)/2$ and $\beta = m(m+1)/2$ and this entails that $\mathcal{P}(T_m^o, t) = \mathcal{P}(\Lambda_m, t)$ whence ϕ_m should be an isomorphism.

Let us compute the Krull dimension of Λ_m and T_m^o . Clearly, $Kdim(\Lambda_m) = m(m-1)/2 + m$ and Artin [2] and Procesi [12] calculated $Kdim(T_m) = 4m - 3$ whence $Kdim(T_m^o) = 3m - 3$ (at least if $m \geq 2$). Therefore, m has to be a solution of the quadratic equation $m^2 - 5m + 6 = 0$, i.e. $m = 2$ or $m = 3$.

Conversely, if $m = 2$ or $m = 3$, equality of the Krull dimension of Λ and T_m^o implies that ϕ_m is an isomorphism because Λ_m is catenary, finishing the proof.

Remark 1 : The fact that T_3 has finite global dimension has some

independent importance. It is a natural example of a regular order such that its center has infinite global dimension.

More important, Severinho Colier Coutinho [7] deduces out of this result that $K_0(G_{3,2}) \not\cong \mathbb{Z}$.

Remark 2 : We have now a method to compute the rational expression of $\mathcal{P}(T_m^o, t)$. For, the coefficients of the Poincaré series of T_m^o and Λ_m are easy to compute and from this comparison one deduces the coefficients of the polynomial $f_m(t)$ s.t. $\mathcal{P}(T_m^o, t) = f_m(t)/(1-t^2)^{m(m-1)/2} \cdot (1-t)^m$. The first cases are :

$$f_2(t) = 1$$

$$f_3(t) = 1$$

$$f_4(t) = 1 - t^4$$

$$f_5(t) = 1 - 5t^4 + 5t^4 - t^{10}$$

Therefore, one would like to make the following :

Conjecture : The Poincaré series of T_m^o satisfies the following functional equation :

$$\mathcal{P}(T_m^o, 1/t) = \alpha \cdot t^\rho \cdot \mathcal{P}(T_m^o, t)$$

where $\alpha = 1$ if m is odd and $\alpha = -1$ if m is even and $\rho \in \mathbb{Z}$.

Furthermore, $\deg(f_m(t)) = m^2 - 3m$ and $\rho = (m^2 - 7m)/2$.

This conjecture has been verified by the author (using a computer) for $m \leq 15$. Details will appear elsewhere.

5. Poincaré series of generic matrices.

Procesi [15] calculated the Poincaré series of the ring of $m \times 2 \times 2$ generic matrices

$$(5) : \mathcal{P}(G_{m,2}, t) = 1/(1-t)^m \cdot \mathcal{P}(T_m^o, t) - 1/(1-t)^m \cdot ((m \cdot (m-1) \cdot (m-2))/3) \cdot t^3 + 1/(1-t)^{m-1}$$

Theorem 3 : There exists a polynomial $g(t) \in \mathbb{Z}[t]$ such that $\mathcal{P}(G_{m,2}, t) = 1/g(t)$ if and only if $m = 0, 1$.

Proof : We have seen before that :

$$\mathcal{P}(T_m^o, t) = f(t)/(1-t^2)^{m(m-1)/2} \cdot (1-t)^m$$

for some polynomial $f(t) \in \mathbb{Z}[t]$.

If we substitute this in (5) we obtain that whenever $\mathcal{P}(G_{m,2}, t)$ is a pure inverse, then $g(t) = (1+t)^\alpha \cdot (1-t)^\beta$ for some $\alpha, \beta \in \mathbb{N}$.

Computing the first terms of (5) gives us :

$$\mathcal{P}(G_{m,2}, t) = 1 + m \cdot t + m^2 \cdot t^2 + m^3 \cdot t^3 + \dots$$

So we have to solve the same set of equations as in the proof of theorem 2, giving $\alpha = m(m-1)/2$ and $\beta = m(m+1)/2$. Comparing the coefficient of t^3 in (5) with that of the power series expansion of :

$$1/(1+t)^{m(m-1)/2} \cdot (1-t)^{m(m+1)/2}$$

gives us : $m^3 = 1/3.(2.m^3 + m)$ leaving $m = 0$ or $m = 1$ as the only solutions.
The other implication is trivial.

Unfortunately, this result does not imply that $G_{m,2}$ has infinite global dimension because $G_{m,2}$ is not Noetherian. This raises the question whether there exists a (possibly infinite) resolution of k (as left $G_{m,2}$ -module) with graded free modules of finite rank.

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