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CLASSGROUPS OF MAXIMAL ORDERS OVER KRULL DOMAINS

Proefschrift ter verkrijging van de graad van Doctor in de Wetenschappen aan de Universitaire Instelling Antwerpen te verdedigen door Lieven LE BRUYN

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It will be pointed out to me that many important facts and valuable results about local fields can be proved in a fully algebraic context, without any use being made of local capacity and can thus be shown to preserve their validity under far more general conditions. May I be allowed to suggest that I am not unaware of this circumstance, nor of the possibility of similarly extending the scope of even such global results as the theorem of Riemann-Roch? We are dealing here with mathematics, not with theology. Some mathematicians may think that they can gain full insight into God's own way of viewing their favourite topic ; to me , this has always seemed a fruitless and frivolous approach. My intentions in this book are more modest. I have tried to show that, from the point of view which I have adopted, one could give a coherent treatment,logically and aesthetically satisfying, of the topics I was dealing with.

For anyone familiar with the language of 'Galois cohomology' it will be an easy and not unprofitable exercise to translate into it some of the definitions and results in one or two places , this even makes it possible to substitute more satisfactory proofs for ours. For me to develop such an approach systematically would have meant loading a great deal of unnecessary machinery on a ship which seemed well equipped for this particular voyage ; instead of making it more seaworthy , it might have sunk it .

A. Weil , introduction to "Basic Number theory"

CONTENTS

introduction

Whereas the classical theory of maximal orders over Dedekind domains has its roots in algebraic number theory , our main motivation for studying maximal orders over Krull domains comes from a quite different algebraic topic , the theory of rings satisfying a polynomial identity (p.i.-rings for short). It is based on the next two recent results :

The first is Chamarie's generalization of the Mori-Nagata theorem to p.i.-rings : every Noetherian (or affine) p.i.-ring Λ with center R can be embedded in a maximal order $\Lambda^{\sim}$ over $R^{\sim}$, the complete integral closure of R , which is a Krull domain.

The second result is due to Artin and Schofield and states that the trace ring of the ring of generic matrices (over a field of characteristic zero) is a maximal order over a Krull domain.

Beside their ringtheoretical importance (showing that containment in a maximal order might generalize to some extent the commutative theory of normalization of Noetherian domains) , these results show that the spectra of these maximal orders are a feasible noncommutative generalization of affine normal varieties. Therefore, a closer investigation of them might shed some new light on a relatively fresh algebraic topic , namely noncommutative algebraic 'geometry' , cfr. e.g. the work of Artin - Schelter or Van Oystaeyen - Verschoren .

One of the main problems in the theory of maximal orders is to find a suitable generalization of unique factorization domains and , related to this question , to find a proper definition of the classgroup. Several possible definitions were suggested , e.g. a rather obscure K-theoretical class group $W(\Lambda)$ by R. Fossum , the normalizing classgroup $\tilde{Cl}(\Lambda)$ by M. Chamarie and the central classgroup $Cl^c(\Lambda)$, studied by E. Jespers and P. Wauters .

In this thesis we aim to survey some of our results on the normalizing and the central classgroup. The thesis is organized as follows.

In the first part we recall some well-known definitions and results from p.i.-theory and the study of maximal orders.

In the second part we treat the central classgroup. The quiding problem is to determine to what extent $Cl^c(\Lambda) \simeq Cl(R)$ implies that Λ is a (reflexive) Azumaya algebra over R. We were able to prove this result for étale or Zariski tamifiable maximal orders and some counterexamples to the general case are included . This fact enables us to show that certain divisorially graded rings over tamifiable maximal orders are graded (reflexive) Azumaya algebras , providing a new approach to this class of orders. For a more extensive introduction to part II the reader is referred to II.1 .

In the third part we focuss attention to the normalizing class group of a maximal order. This time our approach is of a more geometrical nature. In the study of commutative Krull domains there are some important questions on class groups for which purely ringtheoretical methods seem to be insufficient. To solve them one has to use some geometrical machinery. A typical example of such a situation is presented by some results of V.I. Danilov on the relation between $Cl(R)$ and $Cl(R[[t]])$ for a Noetherian integrally closed domain R. First, the classgroup is expressed in terms of Picard groups of certain open subvarieties of the affine scheme. Then one can use the good functorial and cohomological properties of these Picardgroups to prove the theorems on these open sets and afterwards one can pull the obtained information back to the classgroup.

We try to generalize some of these results to maximal orders over Krull domains. To this end we introduce Weil and Cartier divisors and the corresponding class groups. Since the proofs of the classical theorems on the relation between these invariants do not generalize to the p.i.-case we had to come up with a new approach. These new proofs have the extra advantedge of presenting ringtheoretical interpretations (such as the type number and the genera of a maximal order) for certain cohomology pointed sets.

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A Bibliographical Comment

A rather significant part of this thesis is drawn from the following articles :

Le Bruyn L. ; Universal Bialgebras Associated to Orders ; Communications in Algebra ; 10(5),457-478 (1982)

Le Bruyn L. ; A Note on Maximal Orders over Krull Domains ; Journal of Pure and Applied Algebra ; 28 (1983)

Le Bruyn L. ; On the Jespers - Van Oystaeyen Conjecture ; accepted for publication in Journal of Algebra (1983)

Le Bruyn L. ; Class Groups of Maximal Orders over Krull Domains ; Proceedings E. Noether Days - Antwerp '82 ; North Holland (1983)

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PART I : SOME GENERALITIES

A : SOME DEFINITIONS

For the reader's convenience we briefly recollect the basic definitions and properties concerning maximal orders over Krull domains, a more extensive account of the theory may be found in the monographs by R. Fossum [22] (Krull domains) , I. Herstein [29] (central simple algebras) and I. Reiner [67] (maximal orders) .

1. Krull domains

A commutative domain R with field of fractions K is said to be a Krull domain if the following conditions are satisfied :

(a) : R_p is a discrete valuation ring for all $p \in X^{(1)}(R)$ where $X^{(1)}(R)$ is the set of prime ideals of R of height one;

(b) : $R = \cap\{R_p; p \in X^{(1)}(R)\}$;

(c) : (finite character property) Each nonzero element of R is a unit in all but a finite number of the R_p's , $p \in X^{(1)}(R)$.

An R-module M is said to be divisorial if it is torsion free and in $K \otimes_R M$ we have the equality :

$$M = \cap\{M_p = R_p \otimes_R M; p \in X^{(1)}(R)\}$$

For any R-module M , rk(M) , the rank of M , is defined as the dimension of the K-vectorspace $K \otimes_R M$. An R-module M is said to be an R-lattice if M is torsion free of finite rank and if there exists an R-module F of finite type with $M \subset F \subset K \otimes_R M$. It follows from this that $rk(M) = rk(F)$ and that $r.F \subset M$ for a suitable nonzero element r of R. For any torsion free R-module M the R-module $(M : R)$ is defined by :

$$(M : R) = \{f \in Hom_K(K \otimes_R M, K) : f(M) \subset R\}$$

From [22] we recollect that whenever M is an R-lattice , $\cap\{M_p; p \in X^{(1)}(R)\}$ may be identified in a natural way with $(M : R) : R$ where the latter is viewed as a subset of $K \otimes_R M$ via the canonical isomorphism $K \otimes_R M \simeq (K \otimes_R M)^{**}$ (double upperstar denotes the bidual K-module , i.e. $Hom_K(K \otimes_R M, K), K)$, cfr. e.g. [61]). Furthermore, for any R-lattice M , $(M : R)$ is naturally isomorphic to $Hom_R(M, R) = M^*$. It follows that an R-module M is divisorial iff the canonical homomorphism $M \rightarrow M^{**}$ is an isomorphism, i.e. iff M is a reflexive R-module , cfr. e.g. [22] or [61]. Let M and N be torsion free R-modules and let $V = K \otimes_R M$, $W = K \otimes_R N$. Following an idea of Yuan [92] we introduce the notion of a modified tensorproduct $-\otimes_R' -$. Let $M.N$ denote the image of $M \otimes_R N$ in $V \otimes_K W$. Now,define

$$M \otimes_R' N = \cap\{(M.N)_p; p \in X^{(1)}(R)\}$$

Note that whenever M and N are R-lattices , then there is a natural map from $M \otimes_R N$ to $M \otimes_R' N$ such that $(M \otimes_R N)^{**} \simeq M \otimes_R' N$. Another, more torsion-theoretic, way of interpreting this modified tensorproduct is the following : let σ_p be the kernel functor associated with a prime ideal p of R , cfr. e.g. [74] , and let $\sigma(1) = inf\{\sigma_p; p \in X^{(1)}(R)\}$ then $M \otimes_R' N \simeq Q_{\sigma(1)}(M \otimes_R N)$ for R-lattices M and N , where $Q_\tau(-)$ denotes the localization functor in R-mod associated with the kernel functor τ , cfr. e.g. [74].

A divisorial R-ideal is a divisorial R-sublattice of K. From [22] we retain that the set of all divisorial R-ideals forms a free Abelian group generated by the height one prime ideals of R under the *-multiplication , i.e. $A^*B = (A.B)^{**} = (A.B : R) : R$. We will denote this group by $\mathcal{D}(R)$ and call it the group of divisors of R. By $\mathcal{I}(R)$ we denote the subgroup of $\mathcal{D}(R)$ consisting of the invertible R-submodules of K. $\mathcal{P}(R)$ will be the subgroup of $\mathcal{I}(R)$ consisting of the principal invertible R-submodules of K , i.e. those of the form $R.k$ where $k \in K^* = K - O$.

The classgroup of R , $Cl(R)$, is defined to be the quotient group $\mathcal{D}(R)/\mathcal{P}(R)$. The Picardgroup of R , $Pic(R)$, is defined to be the quotient group $\mathcal{I}(R)/\mathcal{P}(R)$. For more details on these objects and their interrelations , the reader is referred to [22].

2 : Central simple algebras

A not necessarily commutative algebra Σ over a commutative field K is said to be a central simple K-algebra if the following conditions are satisfied :

(a) : Σ is simple , i.e. contains na proper (twosided) ideals ;

(b) : Σ is finite dimensional over K ;

(c) : the center of Σ , $Z(\Sigma)$, equals K .

Weddenburn's theorem asserts that Σ is isomorphic , as a K- algebra to a matrix-ring $M_n(\Delta)$ for some central simple K- skewfield Δ. Moreover , Δ is unique up to isomorphism , i.e. $M_{n_1}(\Delta_1) \simeq M_{n_2}(\Delta_2)$ if and only if $\Delta_1 \simeq \Delta_2$ and $n_1 = n_2$.

The set of all central simple K-algebras is closed under taking tensorproducts over K . Σ is said to be equivalent to Γ iff $M_n(\Sigma) \simeq M_m(\Gamma)$ for some natural numbers n and m. The set of all equivalence classes form an Abelian group under the multiplication rule $[-\otimes_K -]$. This group is the Brauer group $Br(K)$ of K.

If Δ is any central simple K-skewfield , then Δ has a maximal commutative subfield L such that L is a separable field extension of K , $dim_L(\Delta) = dim_K(L) = n$ and $\Delta \otimes_K L \simeq M_n(L)$ (L is said to be a splitting field). This entails in particular that the dimension of any central simple K-algebra is a square. Moreover, for any central simple K-algebra Σ one can find a finite Galois extension L/K such that

$\Sigma \otimes_K L \simeq M_n(L)$ if $dim_K(\Sigma) = n^2$, however L cannot always be taken to be a subfield of Σ , cfr. e.g. [30]. This fact enables us to present a cohomological interpretation for the Brauer group :

$$Br(K) = \varinjlim H^2(Gal(L/K), K^*)$$

where the direct limit is taken over all finite Galois extensions L/K with corresponding Galois group $Gal(L/K)$. For proofs and more details on these results , the reader is referred to [29].

Finally, let us define the reduced trace , $tr(-)$, and the reduced norm , $nr(-)$ of a central simple K-algebra Σ . Take a finite Galois splitting field L and an L-algebra isomorphism :

$$h : \Sigma \otimes_K L \to M_n(L)$$

Now, define $tr(a) = trace(h(a \otimes 1))$ and $nr(a) = det(h(a \otimes 1))$. It is fairly easy to verify , cfr. e.g. [67] , that $tr(-)$ and $nr(-)$ do not depend up[on the choices of L and h . Moreover, $tr(\Sigma) \subset K$ and $nr(\Sigma) \subset K$; $tr(-)$ is an additive map whereas $nr(-)$ is multiplicative.

3 : Maximal orders

Let R be a Krull domain with field of fractions K and let Σ be a central simple K-algebra . An R-order in Σ is a subring Λ of Σ satisfying the following three conditions :

(a) : $R \subset \Lambda$;

(b) : $K.\Lambda = \Sigma$, i.e. Λ contains a K-basis of Σ;

(c) : Λ is integral over R .

By (c) we mean that any element $x \in \Lambda$ satisfies a monic polynomial $x^n + r_1.x^{n-1} + \ldots + r_n = 0$ with all $r_i \in R$. From coroll.1.2 of [21] we retain that any R-order in Σ is actually an R-lattice in Σ. Chamarie [13] showed that Λ is an R-order in Σ iff $Z(\Lambda) = R$ and $K.\Lambda = \Sigma$.

We say that an R-order Λ is maximal if it is not properly contained in another R-order in Σ. If Λ is an R-order , it is easily verified that $\cap\{\Lambda_p; p \in X^{(1)}(R)\}$ is also an R-order containing Λ. Conditions (a) and (b) above are immediate and condition (c) follows from the fact that $R = \cap\{R_p; p \in X^{(1)}(R)\}$. Therefore, we have that Λ is a maximal R-order iff :

(1) : Λ is a divisorial R-lattice in Σ and

(2) : Λ_p is a maximal R_p-order for each $p \in X^{(1)}(R)$.

Another , equivalent condition for an R-order Λ in Σ to be maximal is that $(I :_l I) = (I :_r I) = \Lambda$ for each nonzero twosided ideal I of Λ , cfr. e.g. [13] or [50]. We denote for any two subsets A and B of Σ :

$$(A :_l B) = \{x \in \Sigma : x.A \subset B\}$$

$$(A :_r B) = \{x \in \Sigma : A.x \subset B\}$$

If Λ is a maximal R-order in Σ , a divisorial Λ-ideal A is a divisorial R-lattice in Σ which is a twosided Λ-module. Another, equivalent characterization is : A is a twosided Λ-submodule of Σ such that $A.k \subset \Lambda$ for some $k \in K^*$ and $(A : \Lambda) : \Lambda = A$. Let $D(\Lambda)$ denote the set of all divisorial Λ-ideals , then we define a multiplication on $D(\Lambda)$ by :

$$A^*B = (A.B)^{**} = (A.B : \Lambda) : \Lambda$$

From [21] we retain that $D(\Lambda)$ equipped with this multiplication is isomorphic to the free Abelian group generated by $X^{(1)}(\Lambda)$, the set of all height one prime ideals of Λ. In II.2 we will present another proof of this result relating it to the so called Van Geel - primes and arithmetical pseudo valuations.

With $I(\Lambda)$ we will denote the subgroup of $D(\Lambda)$ consisting of those divisorial Λ-ideals A which are invertible , i.e. there exists a divisorial Λ-ideal B such that $A.B = B.A = \Lambda$.

As in the study of commutative Krull domains , it is possible to introduce class- and Picardgroups of Λ by taking classes modulo a subgroup of "principal" divisorial Λ-ideals. There are two possible candidates :

$\mathcal{P}^c(\Lambda) = \{\Lambda.k; k \in K^*\}$ which leads to the so called central classgroup , $Cl^c(\Lambda)$, and the central Picardgroup , $Pic^c(\Lambda)$.

$\mathcal{P}(\Lambda) = \{\Lambda.n; n \in \Sigma : \Lambda.n = n.\Lambda\}$ which leads to the so called normalizing classgroup , $Cl(\Lambda)$, and the normalizing Picardgroup , $Pic(\Lambda)$.

Studying these objects and their relation to the ringtheoretical structure of Λ will be the main objective of these thesis.

B : NUTSHELL ON P.I.-THEORY

The classical theory of maximal orders over Dedekind domains , cfr. e.g. [67] , has its roots in algebraic number theory. Our main motivation for studying maximal orders over Krull domains comes from a quite different algebraic topic , namely p.i.-theory , cfr. e.g. [30],[62].

1. p.i.-rings

Let us recall that a noncommutative ring Λ with center R satisfies a polynomial identity (p.i.) of degree d if there exists a nontrivial polynomial :

$$p(X_1, \ldots, X_n)$$

of degree d in n noncommuting variables with coefficients in R , such that for every n-tuple $(\lambda_1, \ldots, \lambda_n)$ of elements of Λ we have that :

$$p(\lambda_1, \ldots, \lambda_n) = 0$$

The theorems of Kaplansky and Posner , cfr. e.g. [62] , characterize the prime rings Λ satisfying a polynomial identity (p.i.-rings for short) in the following way

A prime ring Λ is a p.i.-ring if and only if Λ is an order in a central simple K-algebra (where K is the field of fractions of R) Σ of dimension say n^2 , and in this case Λ satisfies the standard identity of degree $2.n$:

$$\sum_{\sigma \in S_{2.n}} \epsilon(\sigma) X_{\sigma(1)} \dots X_{\sigma(2.n)}$$

where $S_{2.n}$ denotes the symmetric group of order $2.n$ and $\epsilon(\sigma)$ is the sign of the permutation σ.

For an extensive account of the properties of p.i.-rings, the reader is referred to [30],[62] or [90].

2. (reflexive) Azumaya algebras

An important topic in p.i.-theory is the study of Azumaya algebras and related to it the study of the Brauer group of a commutative ring. We will recall here briefly the definitions and some results. We restrict ourselves to the case that the center R is a Krull domain. Further generalization can be found in [19].

Let Λ be an R-order in a central simple K-algebra Σ , suppose Λ is a divisorial R-lattice and consider the natural R-algebra morphism :

$$m : \Lambda^e = \Lambda \otimes_R \Lambda^{opp} \to End_R(\Lambda)$$

where Λ^{opp} denotes the opposite ring of Λ (i.e, the abelian group Λ with reversed multiplication) and m is determined by : $m(\sum a_i \otimes b_i)(\lambda) = \sum a_i.\lambda.b_i$. Since Λ is a divisorial R-lattice , so is $End_R(\Lambda)$, cfr. e.g. [61]. This entails that m extends to an R-algebra morphism m' making the diagram below into a commutative one

$$\begin{array}{ccc} \Lambda \otimes_R \Lambda^{opp} & \rightarrow & \Lambda \otimes'_R \Lambda^{opp} \\ & {\scriptstyle m} \searrow \quad \swarrow {\scriptstyle m'} & \\ & End_R(\Lambda) & \end{array}$$

An Azumaya algebra over R is an R-order Λ as above such that m is an R-algebra isomorphism. Actually an Azumaya algebra can be defined more generally as an R-algebra over any commutative ring R such that Λ is a finitely generated projective $\Lambda \otimes_R \Lambda^{opp}$-module. Among the many important properties of Azumaya algebras we recall the following :

(1) : Λ is a finitely generated projective R-module ;

(2) : All twosided ideals of Λ are centrally generated ;

(3) : Λ is an Azumaya algebra over R if and only if Λ/P is a central simple $R/R \cap P$ -algebra for every maximal ideal P of Λ .

For further use , we include here an intrinsic characterization of Azumaya algebras essentially due to M.Artin and C.Procesi , cfr. e.g. [62] or [30].

In studying the center of a prime ring satisfying a polynomial identity it is useful to have 'central-valued' or 'central' polynomials , i.e. polynomials which when evaluated always yield elements of the center. Of course, we are only interested in non-constant polynomials , i.e. taking at least two values. It is clearly sufficient to have central polynomials for matrixrings. The first class of such central polynomials was constructed by E. Formanek. More recently, Razmyslov has discovered multilinear central polynomials. If Λ is a prime p.i.-ring with center R , then the Formanek center of Λ , $F(\Lambda)$, is defined to be the ideal of R generated by the values taken by a central polynomial for Λ . If P is a prime ideal of R such that $F(\Lambda) \not\subset P$, then Λ_P is an Azumaya algebra over R_P and the inverse implication holds also. This entails :

(4) : Λ is an Azumaya algebra over R iff $F(\Lambda) = R$.

Two Azumaya algebras over R , Λ and Γ , are said to be equivalent if there exist finitely generated projective R-modules P and Q such that there is an R-algebra isomorphism :

$$\Lambda \otimes_R End_R(P) \simeq \Gamma \otimes_R End_R(Q)$$

The equivalence classes of Azumaya algebras over R form an Abelian group under the tensorproduct , $Br(R)$, the Brauer group of R . For more details , the reader is referred to [19],[35] and [52] ,a.o. .

The study of Azumaya algebras over R reduces roughly to the determination of the Brauer group of R and to the structure of the trivial Azumaya-algebras , i.e. the study of finitely generated projective R-modules and their endomorphism rings.

Extending an idea of Yuan [92] , M. Orzech defines in [61] a reflexive Azumaya algebra to be a divisorial R-order Λ such that the R-algebra morphism :

$$m' : \Lambda \otimes'_R \Lambda^{opp} \rightarrow End_R(\Lambda)$$

defined above is an isomorphism. Two reflexive Azumaya algebras over R, Λ and Γ , are said to be equivalent if there exist divisorial R-lattices M and N such that there is an R-algebra isomorphism :

$$\Lambda \otimes'_R End_R(M) \simeq \Gamma \otimes'_R End_R(N)$$

The equivalence classes of reflexive Azumaya algebras form an Abelian group under the modified tensor product , $\beta(R)$, the so called reflexive Brauer group of R.

It is easily verified that Λ is a reflexive Azumaya algebra if and only if Λ_p is an Azumaya algebra for every $p \in X^{(1)}(R)$, or equivalently , $F(\Lambda) \not\subset p$ for every $p \in X^{(1)}(R)$ (here Λ is already supposed to be a divisorial R-lattice). The second part of the following proposition extends a result of Riley [68] :

Proposition B.1 :

(1) : If Λ is a (reflexive) Azumaya algebra over a Krull domain R , then Λ is a maximal R-order.

(2) : A reflexive Azumaya algebra Λ over R is an Azumaya algebra if and only if Λ is a flat R-module.

Proof :

(1) : Because Λ is a divisorial R-lattice , it suffices to check that Λ_p is a maximal R_p-order for all $p \in X^{(1)}(R)$. To this end we need to verify that an Azumaya algebra over a Krull domain is a maximal order, i.e. that $(I :_l I) = (I :_r I) = \Lambda$ for every twosided ideal I of Λ. Write $x = c^{-1}.\lambda$ where $c \in R$ anf $\lambda \in \Lambda$ and suppose that $I.x \subset I$. Because I and $\Lambda.\lambda.\Lambda$ are centrally generated , we obtain that $c^{-1}.(\Lambda.\lambda.\Lambda \cap R).(I \cap R).\Lambda \subset (I \cap R).\Lambda$ yielding that : $c^{-1}.(\Lambda.\lambda.\Lambda \cap R).(I \cap R) \subset (I \cap R)$ whence $c^{-1}.(\Lambda.\lambda.\Lambda \cap R) \subset R$ because R is a Krull domain , i.e. $c^{-1}.\Lambda.\lambda.\Lambda \subset \Lambda$. Similarly one proves that $(I :_l I) = \Lambda$.

(2) : In view of the definitions we have to prove that the R-algebra morphism $i : \Lambda \otimes_R \Lambda^{opp} \to \Lambda \otimes'_R \Lambda^{opp}$ is an isomorphism. By flatness of Λ this morphism is clearly monomorphic. To prove surjectivity let $\alpha = \sum \lambda_i \otimes \mu_i / r \in \cap (\Lambda \otimes_R \Lambda^{opp})_p$ where $\lambda_i \in \Lambda$, $\mu_i \in \Lambda^{opp}$ and $r \in R$. Because R is a Krull domain, R satisfies the finite character property , i.e. $I = \{p \in X^{(1)}(R) : r \notin U(R_p)\}$ is a finite set. Now, define : $J = X^{(1)}(R) - I$, $\Gamma = \cap \{\Lambda_p^{opp}; p \in I\}$, $\Gamma' = \cap \{\Lambda_p^{opp}; p \in J\}$. Then, $\alpha \in \Lambda \otimes_R \Gamma'$ and clearly : $\alpha \in \cap \{(\Lambda \otimes_R \Lambda^{opp})_p; p \in I\} = \Lambda \otimes_R \cap \{\Lambda_p^{opp}; p \in I\}$ because I is a finite set and Λ is a flat R-module (thus , tensoring with Λ commutes with finite intersections , cfr. [12]). Therefore :

$$\alpha \in \Lambda \otimes_R \Gamma \cap \Lambda \otimes_R \Gamma' = \Lambda \otimes_R (\Gamma \cap \Gamma') = \Lambda \otimes_R \Lambda^{opp}$$

because Λ is flat and Λ^{opp} is a divisorial R-lattice (as a reflexive Azumaya algebra). The inverse implication is of course trivial since an Azumaya algebra is a finitely generated projective R-module.

We will now give an example which shows that the flatness condition in the foregoing proposition cannot be dropped. First, let us recall the definition of a regular ring.

A ring R is said to have global dimension $\leq n$ (written $gldim(R) \leq n$) if every (left) R-module M admits a projective resolution of lenght $\leq n$, i.e. there is an exact sequence :

$$0 \to P_n \to P_{n-1} \to \dots \to P_1 \to P_0 \to M \to 0$$

with all P_i being projective R-modules. A commutative local ring is said to be regular if it is Noetherian and has finite global dimension. For a commutative Noetherian ring one has $gldim(R) = sup\{gldim(R_p); p \in Spoec(R)\}$ and one may even restrict attention to maximal ideals.

We say that R is regular if and only if R_p is a regular local ring for every prime ideal p of R. One of the most striking properties of regular local domains is given by the Auslander-Buchsbaum theorem , cfr. e.g. [6] , which states that they are unique factorization domains. Another noteworthy property for regular rings is that a reflexive finitely generated module E is projective iff $End_R(E)$ is projective.

Example B.2 : (reflexive Azumaya $\neq$ Azumaya)

Suppose that R is a regular local ring of dimension greater than two , then there exists a non-projective finitely generated R-lattice E such that $E = E^{**}$. Consequently, $End_R(E)$ is a reflexive Azumaya algebra. Now, suppose $End_R(E)$ is an Azumaya algebra , i.e. in particular a finitely generated projective R-module , then E would be projective , a contradiction.

E.g. if R is a regular local ring of dimension three and if E is a finitely generated non-projective reflexive R-lattice , then $rad(F(End_R(E))) = m$ where m is the unique maximal ideal of R , because for a regular local ring of dimension smaller or equal to two all maximal orders in matrixrings are Azumaya algebras.

Let us recall two basic properties of $\beta(R)$, proofs can be found in [92] or [61]

(1) : $\beta(R) = \cap\{Br(R_p); p \in X^{(1)}(R)\}$;

(2) : The following sequence is exact :

$$1 \to Pic(R) \to Cl(R) \to BCl(R) \to Br(R) \to \beta(R)$$

Here, $BCl(R)$ is the so called Brauer-classgroup of R. It is defined by taking the set of R-module isomorphism classes of divisorial R- lattices M such that $End_R(M)$ is a finitely generated projective R-module (and hence an Azumaya algebra by Prop.B.2) and then taking equivalence classes with respect to the relation :

$$M \sim N \Leftrightarrow M \otimes_R' P \simeq N \otimes_R' Q$$

for some finitely generated projective R-modules P and Q. The cokernel of the natural morphism $Br(R) \to \beta(R)$ is not so easy to determine.It may be nontrivial.

Finally, let us mention that there exists a cohomological interpretation of the Brauer group of a ring. Galois cohomology used in the field-case has to be replaced by étale cohomology , cfr. e.g. [52].

Up to some extent one may study (reflexive) Azumaya algebras, by :

(1) : studying the groups $Br(R)$ and $\beta(R)$;

(2) : studying the trivial (reflexive) Azumaya algebras , i.e. Azumaya algebras or maximal orders in matrixrings over K and this study usually reduces to module-theoretic questions about finitely generated projective R-modules and divisorial R-lattices resp. .

PART II :
THE CENTRAL CLASSGROUP OF A MAXIMAL ORDER

1. A PROGRAM FOR STUDYING MAXIMAL ORDERS

a. : introduction

Although maximal orders over Krull domains were already studied in the early sixties by e.g. Auslander and Goldman [7] , Maury [49] , Riley [68] , Ramras [64] , [65] , a.o. , relatively little is known about the structure of a general maximal order. The main purpose of this chapter is to present a unified approach to maximal orders over Krull domains. In this first section we will merely scetch this program whereas the next sections contain the proofs and the technical machinery needed.

The question we will be concerned with primarely is the following : if Λ is an arbitrary maximal order over a Krull domain R , is it possible to embed Λ in a natural way in a maximal order Λ' over a Krull domain R' :

$$\begin{array}{ccc} \Lambda & \rightarrow & \Lambda' \\ \uparrow & & \uparrow \\ R & \rightarrow & R' \end{array}$$

such that :

(a) : Λ' is a 'nice' maximal order , i.e. we can study the structure of Λ' over R' ;

(b) : there is a good connection between Λ and Λ', R and R' which enables us to descent structural results of Λ' over R' to results of Λ over R.

It is not difficult to satisfy condition (a) , e.g. by localization the bad part of Λ can be killed , but then one cannot expect too much for condition (b). The idea we will pursue in this chapter is the following : is it possible to choose Λ' to be a graded maximal order which satisfies (a) and such that there is a good connection between graded properties of Λ' (resp. of R') and ungraded properties of Λ (resp. of R). We will now show that this is possible for 'allmost all' maximal orders and moreover we will give a method for studying the exceptional maximal orders.

Of course, we first have to explain what is meant by a 'nice' maximal order . As is clear from the introductory chapter there are at least two classes of maximal R-orders which are reasonably understood , namely Azumaya algebras and reflexive Azumaya algebras.

b. : the central classgroup

Having clarified what we mean by a 'nice' maximal order , our next aim is to define and study an invariant associated to a general maximal R-order Λ which describes how far Λ is from being 'nice' , i.e. from being a (reflexive) Azumaya algebra.

There xists such an invariant in the literature, namely the different and the reduced discriminant of a maximal order , cfr. e.g. Reiner [67] Let us briefly recall its definition :

If Λ is a maximal R-order in the central simple K-algebra Σ and if $tr : \Sigma \to K$ is the reduced trace map, then we can define the following set :

$$\Lambda^t = \{x \in \Sigma : tr(x.\Lambda) \subset R\}$$

It is clear from the definition that Λ^t is a right Λ- submodule of Σ (because $tr(-)$ is an additive map) and that $\Lambda \subset \Lambda^t$. The different of the maximal order Λ , $diff(\Lambda)$, will then be defined to be :

$$diff(\Lambda) = (\Lambda^t)^{-1} = \{x \in \Sigma : \Lambda^t.x.\Lambda^t \subset \Lambda^t\}$$

It is not hard to verify that $diff(\Lambda)$ is actually a twosided (!) divisorial Λ-ideal contained in Λ. Moreover the definition of the different is left-right symmetric , i.e. one could also define :

$$^t\Lambda = \{x \in \Sigma : tr(\Lambda.x) \subset R\}$$

and $diff'(\Lambda) = ({}^t\Lambda)^{-1}$, but then it is rather easy to verify that $diff(\Lambda) = diff'(\Lambda)$.

Further, one defines the reduced discriminant of Λ to be :

$$disc(\Lambda) = nr(diff(\Lambda))$$

therefore, $disc(\Lambda)$ is an ideal of R. Hence, there are only a finite number of height one prime ideals of R , say $p_1, \ldots, p_n$, containing $disc(\Lambda)$. The main result is that these prime ideals are precisely the height one primes of R such that Λ_p is not an Azumaya algebra over R_p.

Therefore, if $\Lambda = diff(\Lambda)$ then Λ is a reflexive Azumaya algebra and $diff(\Lambda)$ (or $disc(\Lambda)$) measures how far Λ is from being reflexive Azumaya.

However, since these invariants are usually rather hard to compute and because it is even harder to describe their behaviour under ring extensions, we looked for a more manageable invariant. The invariant we propose is related to the central classgroup which has been studied by e.g. E. Jespers and P. Wauters [33]. It is defined to be :

$$Cl^c(\Lambda) = D(\Lambda)/P^c(\Lambda)$$

where $P^c(\Lambda)$ denotes the subgroup of $D(\Lambda)$ consisting of the divisorial Λ-ideals which are generated by one central element. Of course, there is a natural map :

$$\phi : Cl(R) \to Cl^c(\Lambda)$$

induced by the morphism :

$$\Phi : D(R) \to D(\Lambda); \Phi(A) = (\Lambda.A)^{**}$$

and it is clear from the definitions and the fact that there is a one-to-one correspondence between $X^{(1)}(R)$ and $X^{(1)}(\Lambda)$ that ϕ is an injective groupmorphism.

Let us first give some examples :

If Λ is an Azumaya algebra over R (or even more generally : a reflexive Azumaya algebra) , then $Cl(R) \simeq Cl^c(\Lambda)$. This fact may be derived from the fact that there is a natural one-to-one correspondence between $\mathcal{D}(\Lambda)$ and $\mathcal{D}(R)$ given by $A \to (\Lambda.A)^{**}$ and $B \to (B \cap K)^{**}$.

If $\Lambda = \mathbb{C}[X, -]$, then we have seen above that every height one prime ideal $P \neq (X)$ is centrally generated since it does not contain the Formanek center. And for $P = (X)$ we know that P^2 is centrally generated, yielding that $Cl^c(\Lambda) \simeq \mathbb{Z}/2.\mathbb{Z}$.

Guided by these examples one could conjecture :

Jespers – VanOystaeyen conjecture : If Λ is a maximal order over a Krull domain R , then the following two statements are equivalent :

(1) : Λ is a reflexive Azumaya algebra

(2) : $Cl(R) \simeq Cl^c(\Lambda)$, i.e. $Coker(\phi) = 1$

What can we say about $Coker(\phi)$? If $R \to S$ is a ringmorphism between two commutative Krull domains satisfying pas d'éclatement (or no blowing up , pour les anglophonnes),i.e. if $ht(P \cap R) \leq 1$ for any height one prime ideal of S, it is well known that $Coker(Cl(R) \to Cl(S))$ describes the splitting and ramification of height one prime ideals , or equivalently , of the associated discrete valuations. The ringextension we are interested in ,$R \subset \Lambda$, satusfies p.d.é. since $ht(P \cap R) = 1$ for every height one prime ideal P of Λ and moreover , no height one prime ideal of R splits in Λ. So, $Coker(\phi)$ measures in some way how the essential discrete valuations associated to R ramify in Σ. Now, the most manageable noncommutative generalization of valuation theory known to the author is the theory of Van Geel-primes and their associated arithmetical pseudo-valuations , cfr. e.g. [80]. This theory will be generalized in II.2 to maximal orders over Krull domains and we will prove that $Coker(\phi)$ does indeed tells us how the discrete valuations on R ramify in Σ with respect to the corresponding Van Geel primes. Moreover, we will prove :

Theorem 1.1 : If Λ is a maximal order over a Krull domain R then $Coker(\phi)$ is a finite group whose exponent is bounded by the p.i.-degree of Λ (i.e. by n if $dim_K(\Sigma) = n^2$).

Returning to the Jespers-Van Oystaeyen conjecture . It is sufficient to study the local case , i.e. if R is a discrete valuation ring in K such that the associated Van Geel prime in Σ does not ramify (that means , if Λ is a maximal R-order in Σ , then the unique maximal ideal of Λ is generated by the uniformizing parameter of R) , does it follow that any maximal R-order in Σ is an Azumaya algebra ?

The key lemma in our approach to this problem (which will be proved in the next section) is the following :

Lemma 1.2 : If R is a discrete valuation ring and if Λ is a maximal R-order such that its maximal ideal is centrally generated , then either one of the following two situations occur :

(a) : Λ is an Azumaya algebra over R

(b) : $Z(\Lambda/\Lambda.m)$ is a purely inseparable field extension of $R/R.m$

The proof of this lemma comes down to a verification of the fact that there is a one-to-one correspondence between prime ideals of Λ[61] lying over $\Lambda.m$ and the prime ideals of R[61] lying over $R.m$.

An immediate consequence of this result is that the Jespers-Van Oystaeyen conjecture is true for all applications in algebraic number theory ($R/R.m$ is finite and hence perfect) and in algebraic geometry over a basefield of characteristic zero ($char(R/R.m) = 0$ and hence perfect).

In order to tackle the general case , we will however need some extra conditions on the maximal order Λ since D. Saltman [73] has provided the following counterexamples to the conjecture (in another context):

Example 1.3 :

equicharacteristic case : Let F be a field of characteristic p and $K = F((t))$, the field of Laurent sequences over F equipped with the natural discrete valuation and let R be the associated (complete) discrete valuation ring. Let $\{a, b\}$ be contained in a p-basis for F (e.g. over its prime field) and let Δ be the cyclic algebra $[a.t^{-p}, b]$. Choose an element $\alpha \in \Delta$ such that $\alpha^p - \alpha = a.t^{-p}$, then $(\alpha.t)^p - t^{p-1}.(\alpha.t) = a$ whence $K(\alpha)/K$ is a field extension such that the corresponding residue fields are $F(a^{1/p})$ and F. Since $b \notin (F(a^{1/p}))^p$ one can verify easily that b is not a norm of $K(\alpha)/K$ yielding that Δ is a skewfield (for details on cyclic algebras we refer the reader to [30]). Since any discrete valuation on a complete field extends to a finite dimensional skewfield over it, there exists a valuation ring Λ in Δ over R with $Cl^c(\Lambda) \simeq 1$ and one verifies that $\Lambda/\Lambda.t = F(a^{1/p}, b^{1/p})$.

case of changing characteristic : Let K be a field of characteristic zero and residue class field F of characteristic p. Suppose K contains a primitive p^{th}-root of unity, say ω. Again, assume that $\{a, b\}$ is a part of a p-basis for F (over its prime field). Choose preimages $a', b' \in K$ of a and b and let Δ be the cyclic algebra $[a', b']$ defined over K. Again, there exists a valuation ring Λ in Δ such that $\Lambda/\Lambda.t = F(a^{1/p}, b^{1/p})$ and such that $Cl^c(\Lambda) \simeq 1$.

The above explains why we have to introduce Zariski (or ètale) tamifiable maximal orders. An order Λ over a Dedekind domain D is said to be hereditary if every onesided ideal of Λ is a projective Λ-module. Note that Λ is hereditary if and only if the Jacobson-radical of Λ_p is a left and right projective Λ_p-module for every height one prime ideal p in D. A reflexive order Λ over a Krull domain R is said to be a tame order in the sense of R. Fossum if and only if Λ_p is an hereditary order for every $p \in X^{(1)}(R)$.

If Λ is a maximal order over a Krull domain R in some central simple K algebra Σ, then Λ is said to be Zariski tamifiable if and only if the following condition holds: for every $p \in X^{(1)}(R)$ there exists a separable splitting subfield $L(p)$ of Σ such that the integral closure $S(p)$ of R_p in $L(p)$ has the property that $\Lambda_p \otimes_{R_p} S(p)$ is an hereditary order over the discrete valuation ring $S(p)$.

Of course, a special case of such a situation (which explains the terminology) is when $\Lambda \otimes_R S$ is a tame order over S where S is the integral closure of R in some separable splitting subfield of Σ.

Λ is said to be étale tamifiable if and only if the following condition is satisfied : for every $p \in X^{(1)}(R)$, there exists an étale extension $S(p)$ of R_p such that $S(p)$ splits Σ , i.e. $\Sigma \otimes_{R_p} S(p) \simeq M_n(L)$ where L is the field of fractions of $S(p)$ and $n = p.i.d.(\Lambda)$. More details on étale ringextensions can be found in e.g. Raynaud [66] or Milne [52].

The main result which will be proved in the next section is then :

Theorem 1.4 : If Λ is a maximal order over a Krull domain R , then the following two statements are equivalent :

(1) : Λ is a reflexive Azumaya algebra over R ;

(2) : $Cl^c(\Lambda) \simeq Cl(R)$ and Λ is Zariski or étale tamifiable.

This shows that for a rather extensive class of maximal orders , the cokernel of the natural monomorphism :

$$\phi : Cl(R) \to Cl^c(\Lambda)$$

is an invariant which measures how far Λ is from being a reflexive Azumaya algebra.

(c) : generalized Rees rings :

The central classgroup of a maximal R-order Λ turns out to be a useful invariant measuring (for a large class of maximal orders , i.e. the tamifiable ones) how far Λ is from being a reflexive Azumaya algebra.

Our next aim is to develop a method for killing off the 'bad' part of $Cl^c(\Lambda)$, namely $Coker(\phi)$.

We will use the construction of the so called generalized Rees rings, first introduced by F. Van Oystaeyen [88] in the commutative $\mathbb{Z}$- graded case over a base ring which is a Dedekind domain. In II.3 , which is largely the content of a joint paper with F. Van Oystaeyen, [42], we will extend this construction in three directions :

(1) : our construction takes as the grading group any torsion free Abelian group, eventually satisfying the ascending chain condition on cyclic subgroups

(2) : the basering does no longer have to be commutative ;

(3) : we will assume that the basering is a so called relative maximal order, which were introduced by the author in [40] in order to present a unified approach to rings having an arithmetical ideal structure, such as maximal orders, h.n.p.-rings [71] , Bass-orders [71] , tame orders in the sense of R. Fossum [21],... .

Generalized Rees rings over these relative maximal orders provide new examples of orders with an arithmetical ideal structure.

In this preliminary section we will not go into the details of the most general case, but we will restrict attention to generalized Rees rings over maximal orders and how they may be applied to kill off the bad part of the central classgroup.

Throughout, Λ will be a maximal order over a Krull domain R and $\{P_1, ..., P_n\}$ will be the finite number of height one prime ideals of Λ which are not centrally generated and $coker(\phi) = \oplus \mathbb{Z}/n_i\mathbb{Z}$ (cfr. theorem 1.3 above)

We consider the $\mathbb{Z} \oplus ... \oplus \mathbb{Z}$-graded subring $\Lambda(\Phi)$ of $\Sigma[X_1, X_1^{-1}, ..., X_n, X_n^{-1}]$

equipped with the natural gradation i.e. $deg(X_i) = (0,...,1,...,0)$ with 1 on spot i ,which is defined by :

$$\Lambda(\Phi)(m_1,...,m_n) = (P_1^{m_1}...P_n^{m_n})^{**}.X_1^{m_1}...X_n^{m_n}$$

Part (a) of the next theorem is an adaptation of a similar result in II.3 , therefore we will merely present an outline of the proof.

Theorem 1.5 : If Λ is a maximal order over a Krull domain R , then :
(a) : $\Lambda(\Phi)$ is a p.i. maximal order over its center $R(\Phi)$ which is a Krull domain
(b) : $Cl^c(\Lambda(\Phi) \simeq Cl(R(\Phi))$

Proof :
(a) : In view of [13] we have to check the following two facts :
1. For any ideal I of $\Lambda(\Phi)$, $(I :_l I) = (I :_r I) = \Lambda(\Phi)$
2. $\Lambda(\Phi)$ satisfies the ACC on divisorial ideals (i.e. ideals of Λ satisfying $(I : \Lambda(\Phi)) : \Lambda(\Phi) = I$).

(1) : Since $\Lambda(\Phi)$ is a graded p.i. ring , its graded ring of quotients , $Q^g(\Lambda(\Phi)) = \Sigma[X_i, X_i^{-1}]$ is obtained by inverting central homogeneous elements and it is an Azumaya algebra over the Krull domain $K[X_i, X_i^{-1}]$,cfr.[42]. So, $\Sigma[X_i, X_i^{-1}]$ is a maximal order. Now, let I be any ideal of $\Lambda(\Phi)$ and suppose that $I.q \subset I$ for some $q \in Q(\Lambda(\Phi))$. Then, $Q^g(\Lambda(\Phi)).I.q \subset Q^g(\Lambda(\Phi)).I$ and by maximality of $Q^g(\Lambda(\Phi))$ this yields that $q \in Q^g(\Lambda(\Phi))$ .Hence we may decompose q in its homogeneous components , $q = q_{i_1} + ... + q_{i_k}$ with $i_1 \leq .... \leq i_k$ (note that $\mathbb{Z} \oplus .... \oplus \mathbb{Z}$ can be given the structure of an ordered group. We obtain : $C_0(I).q_{i_k} \subset C_{i_k}(I)$ where $C_i(I)$ denotes the set of all leading coefficients of elements of I of degree i. Therefore, $(C_0(I).q_{i_k})^{**} \subset C_{i_k}(I)^{**}$ whence : $q_{i_k} \in (C_0(I)^{**})^{-1} * C_{i_k}(I)^{**}$. By [42] , this means that $q_{i_k} \in \Lambda(\Phi)(i_k)$. Replacing q by $q - q_{i_k}$ and repeating the foregoing argumentation one finally arrives at $q \in \Lambda(\Phi)$, finishing the proof of (1).

(2) : If $\{I_n; n \in \mathbb{N}\}$ is an ascending chain of divisorial $\Lambda(\Phi)$ ideals , then the ascending chain $\{(Q^g.I_n)^{**}; n \in \mathbb{N}\}$ becomes stationary , i.e. there is an $n' \in \mathbb{N}$

such that $(Q^g.I_m)^{**} = (Q^g.I_{n'})^{**}$ for every $m \geq n'$. On the other hand , because Λ is a maximal order , there exists an $n'' \in \mathbb{N}$ such that : $C_0(I_m)^{**} = C_0(I_{n''})^{**}$ for every $m \geq n''$. Let $N = sup(n', n'')$, then $I_m = I_N$ for every $m \geq N$,cfr. [42].

(b) : The graded central classgroup of $\Lambda(\Phi)$, $Cl_g^c(\Lambda(\Phi))$ is defined to be :

$$Cl_g^c(\Lambda(\Phi)) = D_g(\Lambda(\Phi))/P_g^c(\Lambda(\Phi))$$

where $D_g(\Lambda(\Phi))$ is the subgroup of $D(\Lambda(\Phi))$ of the $\mathbb{Z}\oplus...\oplus\mathbb{Z}$-graded divisorial ideals of $\Lambda(\Phi)$ and $P_g^c(\Lambda(\Phi)) = \{\Lambda(\Phi).c \mid c \in k[X_1, X_1^{-1}, ..., X_n, X_n^{-1}]$ and homogeneous $\}$. By [33,Th.3.2] the following sequence is exact :

$$1 \to Cl_g^c(\Lambda(\Phi)) \to Cl^c(\Lambda(\Phi)) \to Cl^c(\Sigma[X_1, X_1^{-1}, ..., X_n, X_n^{-1}] \to 1$$

Now, $\Sigma[X_1, X_1^{-1}, ..., X_n, X_n^{-1}]$ being an Azumaya-algebra over a factorial domain, $Cl^c(\Sigma[X_1, X_1^{-1}, ..., X_n, X_n^{-1}] = 1$ whence : $Cl_g^c(\Lambda(\Phi) \simeq Cl^c(\Lambda(\Phi))$.

Furthermore, it is easy to verify that the sequence below is exact :

$$1 \to < [P_1], ..., [P_n] > \to Cl^c(\Lambda) \to Cl_g^c(\Lambda(\Phi)) \to 1$$

Similarly, $Cl_g(R(\Phi)) \simeq Cl(R(\Phi))$ and :

$$1 \to < [p_1], ..., [p_n] > \to Cl(R) \to Cl_g(R(\Phi)) \to 1$$

whence one finally obtains the exact diagram :

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & < [p_i] > & \to & Cl(R) & \to & Cl(R(\Phi)) & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & < [P_i] > & \to & Cl^c(\Lambda) & \to & Cl^c(\Lambda(\Phi)) & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \oplus\mathbb{Z}/n_i.\mathbb{Z} & & \oplus\mathbb{Z}/n_i.\mathbb{Z} & & & &
\end{array}$$

finishing the proof.

Our next aim will be to prove that $\Lambda(\Phi)$ is tamifiable if Λ is. In order to do this, we need some more information about the center of $\Lambda(\Phi)$, which we will denote by $R(\Phi)$. By direct calculation it is clear that all rings $R(\Phi)$ occuring in this way are of the following type :

Let R be a Krull domain , then for any finite set of height one prime ideals $\{p_1, ..., p_n\}$ of R and for any set of natural numbers $\{m_1, ..., m_n\}$ one can define the so called lepidopterous Rees ring (in the terminology of F. Van Oystaeyen) $R(p_i, m_i)$ to be the $\mathbb{Z} \oplus ... \oplus \mathbb{Z}$- graded subring of $K[X_1, X_1^{-1}, ..., X_n, X_n^{-1}]$ defined by :

$$R(p_i, m_i)(i_1, ..., i_n) = (p_1^{[[i_1/m_1]]} * ... * p_n^{[[i_n/m_n]]}.X_1^{i_1}...X_n^{i_n}$$

where $[[a/b]] = sign(a/b).[| a/b |]$ (as usual, $[-]$ denotes the integral part of -). These lepidopterous Rees rings are readily seen to be Krull domains , cfr. e.g. [87]. They form a directed system in the following way :

$$R(p_i, m_i) \leq R(p'_j, m'_j)$$

iff $\{p_i\} \subset \{p'_j\}$ and $m_i \mid m'_j$ for corresponding values of i and j.

In our case, $R(\Phi) = R(p_i, m_i)$ where $\{p_1, ..., p_n\}$ are the ramified prime ideals in Λ and $m_i = e_{p_i}$, the ramification indices. We are now in a position to prove :

Proposition 1.6 :

(1) : If Λ is a Zariski tamifiable maximal order over R , then $\Lambda(\Phi)$ is Zariski tamifiable over $R(\Phi)$.

(2) : If Λ is an étale tamifiable maximal order over R , then $\Lambda(\Phi)$ is étale tamifiable over $R(\Phi)$.

Proof :

(i) : Let $p \in X^{(1)}(R)$ and suppose that L is a separable splitting subfield of Σ such that the integral closure $S(p)$ of R_p in L satisfies : $\Lambda_p \otimes S(p)$ is hereditary. Of course, $L(X_1, ..., X_n)$ is a separable splitting subfield of $\Sigma(X_1, ..., X_n)$. Let $S(\Phi)$ be the integral closure of $R(\Phi)$ in $L(X_1, ..., X_n)$. Because $R(\Phi)$ is a graded Krull domain , so is $S(\Phi)$ by an argument similar to [81]. Now, let P be any height one prime ideal of $S(\Phi)$, then either P is a graded prime ideal or P_g (the set of homogeneous elements) $= 0$, by a standard argument cfr. e.g. [nasf]

Suppose first that $P_g = 0$. Then the localization of $\Lambda(\Phi) \otimes S(\Phi)$ at P is a localization of $(\Sigma \otimes_K L)[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$. Therefore, it will be an Azumaya algebra over the Krull domain $S(\Phi)_P$, hence a maximal and certainly a tame order.

Next, suppose that P is a graded prime ideal and that $P \cap R = p$. If $p \notin \{p_1, \ldots, p_n\}$, then the localization of $\Lambda(\Phi) \otimes S(\Phi)$ at P is a localization of $(\Lambda_p \otimes S(p))[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ and therefore it is a tame order because the class of tame orders is closed under polynomial extensions and central localizations.

Finally, suppose $p = p_1$, then $(\Lambda(\Phi) \otimes S(\Phi))_P = (\Lambda(\Theta) \otimes S(\Theta))_q[X_2, X_2^{-1}, \ldots, X_n, X_n^{-1}]$ where $\Lambda(\theta)(n) = P_1^n X_1^n$, $S(\Theta)$ is the integral closure of $R(\Phi)$, the center of $\Lambda(\Theta)$ in $L(X_1)$ and $p = P \cap S(\Theta)$. Now, $\Lambda(\Theta) \otimes S(\Theta)$ is readily checked to be an overring of $(\Lambda \otimes S)(\Theta)$ in $(\Sigma \otimes L)(X_1)$. Furthermore, $(\Lambda_p \otimes S(p))(\Theta)$ is a tame order by [42] or [48] and therefore so is $(\Lambda(\Theta) \otimes S(\Theta))_q$, finishing the proof.

(2) is rather trivial since étale extensions are preserved under tensorproduct and localization.

The foregoing results complete the proposed program for tamifiable maximal orders. In section II.3 we will prove that there is a natural one-to-one correspondence between divisorial Λ-ideals and graded divisorial (onesided) $\Lambda(\Phi)$-ideals. The connection between graded divisorial ideals of $R(\Phi)$ and divisorial ideals of R is not that easy to determine. The reader is referred to a fothcomming article of M. Vanden Bergh for an exposition of this relation in the $\mathbb{Z}$- graded case over a Dedekind domain R. It would be extremely interesting to generalize his methods to the $\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$- graded case and with R being a Krull domain.

In order to study the (reflexive) Azumaya algebras $\Lambda(\Phi)$ over the graded Krull domain $R(\Phi)$ it might be interesting to study the graded (reflexive) Brauer group of $R(\Phi)$ introduced by F. Van Oystaeyen in the $\mathbb{Z}$-graded case [86] and consequently generalized in [91] to arbitrary grading groups G. We will here merely present the definitions and some of the problems in the theory .

A graded order Λ over the graded Krull domain R is said to be a graded Azumaya (resp. reflexive Azumaya) algebra if the natural morphism :

$$\Lambda \otimes_R \Lambda^{opp} \to End_R(\Lambda) \text{ resp. } \Lambda \otimes'_R \Lambda^{opp} \to END_R(\Lambda)$$

is a degree preserving isomorphism. It is easy to check that a graded (reflexive) Azumaya algebra is just a (reflexive) Azumaya algebra admitting a gradation extending the gradation of the center. Two graded (reflexive) Azumaya algebras Λ and Γ are said to be gr-equivalent if there exist graded finitely generated projective (resp. reflexive R-lattices) P and Q (resp. M and N) such that there exists a degree preserving isomorphism :

$$\Lambda \otimes_R END_R(P) \simeq \Gamma \otimes_R END_R(Q)$$

$$\Lambda \otimes'_R END_R(M) \simeq \Gamma \otimes'_R END_R(N)$$

where the rings $END_R(-)$ and tensorproducts are equiped with the natural gradation, cfr. e.g. [58]. The set of gr-equivalence classes of graded (reflexive) Azumaya algebras forms a group with respect to the tensorproduct , $Br^g(R)$ (resp. $\beta^g(R)$). These groups are called the graded (reflexive) Brauer group of R. These groups were extensively studied in the $\mathbb{Z}$-graded case by F. Van Oystaeyen, A. Verschoren and S. Caenepeel [86],[91]. One of the key problems in the theory is the determination of the kernel of the natural morphism $Br^g(R) \to Br(R)$.

Another problem is whether $Br(R)$ may be reconstructed from $Br^g(R)$'s obtained by changing the gradation. That this is indeed possible in certain cases has been demonstrated by F. Van Oystaeyen.

As in the ungraded case , the study of graded (reflexive) Azumaya algebras shows two different faces :

(a) : determination of the groups $\beta^g(R)$ (resp. $Br^g(R)$),

(b) : the study of graded reflexive (resp. projective) R-modules or equivalently the study of graded maximal orders in $M_n(K^g)$.

It is an interesting problem to investigate whether $\beta^g(R(\Phi))$ may be generated by the classes of reflexive Azumaya algebras of the form $\Lambda(\Phi)$ where Λ is a maximal R-order. A natural idea would be to try to prove that the part of degree $(0,...,0)$ of a graded reflexive Azumaya algebra is a maximal R-order. However, this need not be the case as the following counterexamples due to M. Vanden Bergh shows

Example 1.7 :

Let Λ be an hereditary,not maximal order over a discrete valuation ring R. If we form the $\mathbb{Z}$-graded generalized Rees ring over Λ starting from the Jacobson radical of Λ then it is not difficult to see that this is an Azumaya algebra over its center. However, its part of degree 0 is Λ.

Another example of such a situation is the following : let Λ be a maximal order with prime discriminant over a Dedekind domain R in a quaternionalgebra Σ over its field of fractions K . Form the generalized Rees ring over Λ starting from this ramified prime and suppose that it is an Azumaya algebra (is always the case if Λ is tamifiable, e.g. for quaternions over $\mathbb{Q}$). Now, split this Azumaya algebra with the integral closure of the center in a separable splitting subfield , then we find a graded Azumaya algebra with part of degree 0 not a maximal order.

For every lepidopterous Rees ring $R(p_i, m_i)$ there exists a natural morphism

$$\theta(p_i, m_i) : \beta^g(R(p_i, m_i)) \to Br(K)$$

which is defined to be the composition of the morphism

$$\beta^g(R(p_i, m_i)) \to Br^g(K[X_1, X_1^{-1}, ..., X_n, X_n^{-1}]$$

and the natural isomorphism $Br^g(K[X_1, X_1^{-1}, ..., X_n, X_n^{-1}]) \to Br(K)$ which exists because $K[X_1, X_1^{-1}, ..., X_n, X_n^{-1}]$ is strongly graded and hence there is a natural one-to-one correspondence between graded Azumaya algebras over the graded field $K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ and Azumaya algebras over K.

Hence, one can define a morphism :

$$\theta : \varinjlim \beta^g(R(p_i, m_i)) \to Br(K)$$

and we claim that $Coker(\theta)$ gives a measure for the aboundancy of non-tamifiable maximal orders over R , for :

Proposition 1.8 :

If R is a Krull domain such that every maximal R-order is tamifiable , then θ is surjective.

Proof :

The map θ is defined by sending a class of $\beta(R(p_i, m_i))$ say $[\Lambda]$ to $[(\Lambda \otimes K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}])_0]$ So take a representant Σ of a class in $Br(K)$ and let Λ be a maximal R-order in Σ. Because Λ is tamifiable , $\Lambda(\Phi)$ is a reflexive Azumaya algebra over $R(\Phi)$ and it is easy to see that $\Lambda(\Phi) \otimes K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] = \Sigma$, finishing the proof.

d. : universal measuring bialgebras

Finally, let us return briefly to the study of non-tamifiable maximal orders. In II.4 we will develop a method which might help to describe their structure. The motivation behind it is Saltman's approach in [73] for his class of non-tamifiable maximal orders. over a discrete valuation ring. He is only concerned with the case that $R/R.m \to \Lambda/\Lambda.m$ is a commutative (!) , purely inseparable field extension of exponent one and degree p^r. It turns out that the structure of Λ is entirely described by the R-derivations of Λ.

Inspired by this special case, one might expect that for more general non-tamifiable maximal orders, higher derivations of Λ will come into the picture as well as the extension $Z(\Lambda/\Lambda.m) \rightsquigarrow \Lambda/\Lambda.m$ which depends on the structure of the group of R-automorphisms of Λ .

Therefore, it would be usefull to have some invariant associated to Λ which describes in a unified way both the (higher) derivations and the automorphisms of Λ over R.

If A is an algebra over a field K , the universal measuring bialgebra $M_K(A, A)$ and more in particular its maximal cocommutative pointed subbialgebra $H_K(A, A)$, both introduced and studied by M.E. Sweedler [76], [77] satisfies the requirements of the proposed invariant we are looking for.

For general definitions and more details on coalgebras, bialgebras etc. , the reader is referred to Sweedler's monograph [76] or the more compact book of Abe [[1]]. Roughly, coalgebras are the dual objects of algebras (reverse all the arrows in the diagrams defining an algebra) and bialgebras are both algebras and coalgebras such that the defining maps are compatible with one another. Standard examples of bialgebras are groupalgebras and universal enveloping algebras of Liealgebras of derivations, cfr. [76]. A bialgebra M is said to measure an algebra A to itself if there exists a K-linear map :

$$\psi : M \otimes_K A \to A$$

such that the following conditions are satisfied :

(1) : $\psi(m \otimes a.a') = \sum \psi(m_{(1)} \otimes a).\psi(m_{(2)} \otimes a')$

(2) : $\psi(m \otimes 1) = \epsilon(m).1_A$

for all $a, a' \in A$ and $m \in M$; where $\delta(m) = \sum m_{(1)} \otimes m_{(2)}$ is the comultiplication map of M and $\epsilon : M \to K$ is the counit map. If m is a group-like element of M , i.e. $\Delta(m) = m \otimes m$ and $\epsilon(m) = 1$, then it is easy to see that $\psi(m \otimes -) : A \to A$ is a K-algebra morphism of A. If m is a primitive element of M i.e. $\Delta(m) = m \otimes 1 + 1 \otimes m$ and $\epsilon(m) = 0$ then $\psi(m \otimes -) : A \to A$ is a K-derivation of A. This shows that measuring bialgebras present a concept which unifies actions of both automorphisms and derivations. Sweedler [76] has shown that there is a universal object $M_K(A, A)$ in the category of bialgebras measuring A to A. That is, if $\psi : M \otimes_K A \to A$ is a measuring bialgebra then there exists a unique bialgebra map $F : M_K(A, A) \to M$ such that the diagram below is a commutative one :

$$\begin{array}{ccc} M_K(A,A) \otimes_K A & \rightarrow & A \\ \downarrow & \nearrow & \\ M \otimes_K A & & \end{array}$$

Therefore, in order to study measurings from A to A , it will be sufficient to study subbialgebras of $M_K(A, A)$ and their induced measurings. However, in most cases this $M_K(A, A)$ is far too large to admit a manageable description. Therefore, one restricts attention to the maximal cocommutative pointed subbialgebra $H_K(A, A)$ of $M_K(A, A)$. Note that this $H_K(A, A)$ suffices for our purposes since automorphisms and derivations can be described by measurings with cocommutative pointed bialgebras. If $char(K) = 0$, then a beautiful theorem due to Kostant describes $H_K(A, A)$, namely :

$$H_K(A, A) \simeq K.Gal_K(A) \# U(Der_K(A))$$

where $.\#.$ denotes the smashed product , cfr. [76]. These objects can be used in several fields , e.g. to obtain a Galois theory for commutative field extensions , cfr. [77].

Of course, one can define algebras,coalgebras,bialgebras,measurings etc. in the obvious way over an arbitrary ring R but then it is no longer possible to construct a universal measuring bialgebra in general. This forced S. Chase and M.E. Sweedler in [15] to introduce the notion of a so called Galois object in order to get a more or less satisfactory Galois theory. However, the condition of being a Galois object restricts the ringextensions in this Galois theory rather severely.

Therefore, it would be interesting to find a nice class of commutative rings (e.g. the Dedekind or Krull domains) for which it is possible to extend the construction of Sweedler. In II.5 we show that this is indeed possible. The reason why this generalization holds for Dedekind domains is that there are enough projective modules (every f.g. torsion free module is projective) whereas for Krull domains it holds because they may be viewed as a global sheaftheoretic version of a discrete valuation ring.

We will show that for every order Λ over a Dedekind domain D in a central simple K-algebra Σ there exists a universal measuring bialgebra $M_D(\Lambda, \Lambda)$. Further it is rather easy to show that $M_D(\Lambda, \Lambda) \otimes_D K$ is a subbialgebra of $M_K(\Sigma, \Sigma)$. Now, if we define :

$$H_D(\Lambda, \Lambda) = \{x \in M_D(\Lambda, \Lambda) : i(x \otimes 1) \in H_K(\Sigma, \Sigma)\}$$

where i is the canonical inclusion map. This $H_D(\Lambda, \Lambda)$ is shown to be a maximal pointed cocommutative subbialgebra of $M_D(\Lambda, \Lambda)$. Similarly , one may define the pointed irreducible component of the unit element :

$$H_D^1(\Lambda, \Lambda) = \{x \in H_D(\Lambda, \Lambda) : i(x \otimes 1) \in H_K^1(\Sigma, \Sigma)\}$$

and it turns out that this $H_D^1(\Lambda, \Lambda)$ is a not necessarely finitely generated D-order in $H_K^1(\Sigma, \Sigma)$, i.e. :

$$H_D^1(\Lambda, \Lambda) \otimes_D K \simeq H_K^1(\Sigma, \Sigma)$$

However , $H_D(\Lambda, \Lambda)$ need not be an order in $H_K(\Sigma, \Sigma)$. E.g. if D is a discrete valuation ring and if Λ is a maximal order then $H_D(\Lambda, \Lambda)$ is a D-order in $H_K(\Sigma, \Sigma)$ if and only if Λ is a discrete noncommutative valuation ring. In general, for a maximal D-order Λ , $G(H_K(\Sigma, \Sigma))/G(H_D(\Lambda, \Lambda))$ measures the conjugated maximal D-orders.

Furthermore, we present a method in order to lift most results from Dedekind domains to Krull domains.

2. THE CENTRAL CLASSGROUP OF A MAXIMAL ORDER.

a. primes, pseudo-valuations and maximal orders.

In this section we aim to develop a machinery which resembles the theory of essential valuations associated to a commutative Krull domain, cfr. e.g. [22].

First, we aim to establish that the set of all divisorial Λ- ideals , $\mathcal{D}(\Lambda)$, is an Abelian group. To this end we use the axiomatic construction of the group of E. Artin. The construction we present here runs along the lines of G. Maury and J. Raynaud in [50]. We denote by $\mathcal{F}(\Lambda)$ the set of all nonzero fractional Λ- ideals, i.e. R-lattices in Σ which are twosided Λ- modules , or equivalently , twosided Λ-submodules A of Σ such that $A.k \subset \Lambda$ for some suitable $k \in K^*$. Clearly, if A and B are in $\mathcal{F}(\Lambda)$, then their product $A.B$ is also a fractional Λ-ideal. $\mathcal{F}(\Lambda)$ equipped with this multiplication and ordered by inclusion satisfies the following properties :

(1) : $\mathcal{F}(\Lambda)$ is an associative semigroup with identity element Λ ;

(2) : $\mathcal{F}(\Lambda)$ is a lattice ; $A+B = sup(A,B)$ and $A \cap B = inf(A,B)$;

(3) : If $A, B, C \in \mathcal{F}(\Lambda)$, then $A \leq B$ implies that $A.C \leq B.C$ and moreover $A.(B+C) = A.B + A.C$, $(B+C).A = B.A + C.A$;

(4) : If $\{A_i; i \in I\}$ is a nonempty set of elements of $\mathcal{F}(\Lambda)$ such that $\sum A_i \in \mathcal{F}(\Lambda)$, then for any $C \in \mathcal{F}(\Lambda)$, $sup(C.A_i)$ and $sup(A_i.C)$ exist in $\mathcal{F}(\Lambda)$ and $sup(C.A_i) = \sum C.A_i = C.\sum A_i = C.sup(A_i)$ and similarly $sup(A_i.C) = sup(A_i).C$;

(5) : For every $A \in \mathcal{F}(\Lambda)$, $(A : \Lambda) \in \mathcal{F}(\Lambda)$ (note that because Λ is a maximal R-order , $(A :_l \Lambda) = (A :_r \Lambda) = (A : \Lambda)$,cfr. e.g. [k]) and $A.(A : \Lambda).A \leq A$ moreover if $A.X.A \leq A$ for some $X \in \mathcal{F}(\Lambda)$ then this implies that $X \leq (A : \Lambda)$;

(6) : $\Lambda^2 \leq \Lambda$, moreover if $\Lambda \leq \Gamma$, $\Gamma^2 \leq \Gamma$ and $\Gamma \in \mathcal{F}(\Lambda)$, then $\Lambda = \Gamma$ because

Λ is a maximal R-order.

An abstract ordered semigroup satisfying these requirements is said to be an Artin-setting. Now, we define an equivalence relation on $\mathcal{F}(\Lambda)$ by saying that $A \sim B$ iff $(A : \Lambda) : \Lambda = (B : \Lambda) : \Lambda$ iff $A^{**} = B^{**}$. The set of equivalence classes , $\mathcal{F}(\Lambda)/\sim$,is isomorphic to the set of divisorial Λ-ideals , $\mathcal{D}(\Lambda)$, which becomes an associative semigroup by defining the $*$-multiplication rule : $A * B = (A.B : \Lambda) : \Lambda = (A.B)^{**}$. Proposition 1.4 of [50] now yields that $\mathcal{D}(\Lambda)$ with this multiplication is an Abelian group.

Now, we reverse the ordering on $\mathcal{D}(\Lambda)$, i.e. $A \leq' B$ iff $A \supset B$. It is easy to verify , cfr. e.g. [50] , that every nonempty subset of $\mathcal{D}(\Lambda)$ has a supremum for this new ordering $A_1 \cap \ldots \cap A_n$ and an infimum $(A_1 + \ldots + A_n : \Lambda) : \Lambda$.

Proposition 2.1 : $\mathcal{D}(\Lambda) \simeq \mathbb{Z}^{(I)}$ for a certain index set I and this isomorphism is order preserving.

Proof :

We have seen that $\mathcal{D}(\Lambda)$ is an ordered Abelian group such that any two elements have a supremum and an infimum. Because Λ is a maximal order over a Krull domain R , Λ satisfies the ascending chain condition on divisorial ideals contained in Λ . This entails that every nonempty subset of positive elements of $\mathcal{D}(\Lambda)$, i.e. of divisorial Λ-ideals contained in Λ , has a minimal element. A well-known theorem on ordered Abelian groups satisfying these requirements , cfr. [12] , yields that $\mathcal{D}(\Lambda) \simeq \mathbb{Z}^{(I)}$ for some index set I and that this isomorphism is order-preserving.

Here, the order relation on $\mathbb{Z}^{(I)}$ is of course defined by : $(\alpha_i)_i \leq (\beta_i)_i$ iff $\alpha_i \leq \beta_i$ for all $i \in I$. Let $\psi : \mathcal{D}(\Lambda) \to \mathbb{Z}^{(I)}$ be an order preserving isomorphism.

Put $e_i = (\delta_{ij})_j$ where δ_{ij} is the Kronecker-delta and let $P_i = \psi^{-1}(e_i)$. Thus, any element $A \in \mathcal{D}(\Lambda)$ can be written uniquely as :

$$A = P_1^{n_1} * ... * P_k^{n_k}; n_i \in \mathbb{Z}$$

In order to avoid heavy notation, we will sometimes use the abbreviation : $A^d = (A : \Lambda) : \Lambda = A^{**}$.

Lemma 2.2 : P_i is a prime ideal of Λ.

Proof :

Let $x, y \in \Lambda$ be such that $x.\Lambda.y \subset P_i$.It is straightforward to check that $(\Lambda.x.\Lambda)^d * (\Lambda.y.\Lambda)^d = (\Lambda.x.\Lambda.y.\Lambda)^d \subset P_i$. Further, $\psi(\Lambda.x.\Lambda^d) = \sum n_j.e_j$ and $\psi(\Lambda.y.\Lambda^d) = \sum m_j.e_j$ where $n_j, m_j \geq 0$. In particular, $\psi(\Lambda.x.\Lambda^d) + \psi(\Lambda.y.\Lambda^d) = \sum(n_j + m_j).e_j \geq \psi(P_i) = e_i$.Therefore, $n_i \geq 1$ or $m_j \geq 1$ yielding that either $x \in P_i$ or $y \in P_i$, finishing the proof.

Lemma 2.3 : Let P be any nonzero prime ideal of Λ , then P contains some P_i.

Proof :

Since Λ is a prime p.i. ring P contains a nonzero central element , say r. Since $\Lambda.r \in \mathcal{D}(\Lambda)$ we may write :

$$P \supset \Lambda.r = P_1^{n_1} * .... * P_k^{n_k} \supset P_1^{n_1}...P_k^{n_k}$$

and all $n_i \geq 0$.Therefore $P \supset P_i$ for some $i \in I$.

Proposition 2.4 : $\mathcal{D}(\Lambda)$ is generated by the height one prime ideals of Λ.

Proof :

Suppose P is a height one prime ideal of Λ , then by the foregoing lemma $P = P_i$ for some $i \in I$. Conversely, let P be a prime generator of $\mathcal{D}(\Lambda)$. If P is not a height one prime ideal, then $0 \subsetneq Q \subsetneq P$ for some prime ideal Q of Λ.Again by the foregoing lemma this entails that $P_i \subsetneq P$ for some $i \in I$. Therefore, $\psi(P) < \psi(P_i) = e_i$ which is a contradiction because $\psi(P) > 0$.

In the commutative case, valuation theory is a powerfull tool in order to study Krull domains. A useful noncommutative generalization of valuation theory is the theory of the so called Van Geel - primes (cfr.[57] , [80]). We aim to relate pseudovaluation functions on the set of divisorial Λ-ideals to Van Geel-primes in Σ.Let us start by recalling some definitions :

Following J. Van Geel we will call a pair $(M, \mathcal{O})$ a prime in the central simple K-algebra Σ if and only if it satisfies the following properties :

(P1) : $\mathcal{O}$ is a subring of Σ;

(P2) : M is a prime ideal of $\mathcal{O}$;

(P3) : For all $x, y \in \Sigma$, $x.\mathcal{O}.y \subset M$ implies that either $x \in M$ or $y \in M$.

If $(M, \mathcal{O})$ is a prime in Σ , then so is (M, Σ^M) where we denote by Σ^M the set $\{x \in \Sigma : x.M \subset M \text{ and } M.x \subset M\}$.

Primes are natural generalizations of commutative valuation rings, for, if $\Sigma = K$ is a commutative field , then (M, K^M) is a prime in K if and only if K^M is a valuation ring in K and M is its unique maximal ideal.

Extending the terminology of [57] to the case of a maximal order Λ over a Krull domain R in a central simple K-algebra Σ , we define :

Definition 2.5 : An arithmetical pseudovaluation v on $\mathcal{D}(\Lambda)$ is a function $v : \mathcal{D}(\Lambda) \to \Gamma \cup \{\infty\}$ where Γ is a totally ordered group satisfying :

(V1) : $\forall A, B \in \mathcal{D}(\Lambda) : v(A * B) = v(A) + v(B)$;

(V2) : $\forall A, B \in \mathcal{D}(\Lambda) : v((A + B)^{**}) \geq \min(v(A), v(B))$;

(V3) : $\forall A, B \in \mathcal{D}(\Lambda)$: if $A \subset B$ then $v(A) \geq V(B)$;

(V4) : $v(\Lambda) = 0$ and $v(0) = \infty$.

For any element $x \in \Sigma$ we will denote by $C_x = (\Lambda.x.\Lambda)^{**} \in D(\Lambda)$. The next result is a generalization of a similar result in [57] :

Proposition 2.6 :

(1) : To any arithmetical pseudo valuation v on $D(\Lambda)$ one may associate a prime in Σ.

(2) : To any prime (P, Σ^P) in Σ such that $P = \cap\Lambda_p.P.\Lambda_p$ and $\Lambda \subset \Sigma^P$ we may associate an arithmetical pseudo valuation on $D(\Lambda)$.

Proof :

(1) : Let v be an arithmetical pseudo valuation on $D(\Lambda)$. Define $P = \{x \in \Sigma \mid v(C_x) > 0\}$. By definition of v , P is clearly a multiplicatively closed additive subgroup of Σ_+ , the Abelian group Σ ,yielding that P is an ideal of Σ^P. If $x, y \in \Sigma$ such that $x.\Sigma^P.y \subset P$, then $x.\Lambda.y \subset P$ because $\Lambda \subset \Sigma^P$. Therefore, $0 < v(\cap\Lambda_p.x.\Lambda_p.y.\Lambda_p) = v(\cap\Lambda_p.(\cap\Lambda_p.x.\Lambda).(\cap\Lambda.y.\Lambda_p).\Lambda_p) = v(C_x * C_y) = v(C_x) + v(C_y)$ and thus either $v(C_x) > 0$ or $v(C_y) > 0$ yielding that (P, Σ^P) is a prime in Σ.

(2) : If (P, Σ^P) is a prime in Σ such that $\cap\Lambda_p.P.\Lambda_p = P$ and $\Lambda \subset \Sigma^P$, define for any divisorial Λ-ideal I :

$$v(I) = \{x \in \Sigma \mid C_x * I \subset P\}$$

Now, let Γ be the set $\{v(I) \mid I \in D(\Lambda)\}$, then Γ is totally ordered by inclusion. To show this , suppose that $I, J \in D(\Lambda)$ are such that both $v(I) \not\subset v(J)$ and $v(J) \not\subset v(I)$. Therefore, there exist elements $x, y \in \Sigma$ such that $C_x * I \subset P$, $C_x * J \not\subset P$, $C_y * I \subset P$ and $C_y * I \subset P$. Because (P, Σ^P) is a prime , we then obtain :

$$(C_x * J).\Sigma^P.(C_y * I) \not\subset P$$

yielding that for some $z \in \Sigma^P$: $C_x * J * C_z * C_y * I \not\subset P$. But, $D(\Lambda)$ is an abelian group whence $C_x * I * C_z * C_y * I \not\subset P$ and because for any $z \in \Sigma^P$ we have trhat $C_z * P \subset P$ and $P * C_z \subset P$ this is a contradiction, finishing the proof of our claim.

We claim that $v(I)+v(J) = v(I*J)$ is a well defined addition on Γ which turns Γ into an ordered group with unit element $v(\Lambda)$. For, if $v(I) = v(I')$ and $v(J) = v(J')$ for i, I', J and $J' \in \mathcal{D}(\Lambda)$ then we have for any $x \in v(I * J) : C_x * I * J \subset P$ whence $C_x * I \subset v(J) = v(J')$ hence $C_x * I * J' = C_x * J' * I \subset P$ and thus finally since $v(I) = V(I')$, $C_x * J' * I' = C_x * I' * J' \subset P$ follows , i.e. $x \in v(I' * J')$. The fact that $v(\Lambda)$ is a unit element is obvious.

$v(I) \leq v(J)$ yields $v(I) + v(H) \leq v(J) + v(H)$ for any $H \in \mathcal{D}(\Lambda)$, for , if $x \in v(I * H)$ then $C_x * H * I = C_x * I * H \subset P$, i.e. $C_x * H \subset v(I) \subset v(J)$ whence $C_x * J * H \subset P$. The required properties for v to be an arithmetical pseudo valuation follows directly from the definition of v.

Let us define for all i :

$$v_i : \mathcal{D}(\Lambda) \to \mathbb{Z} : A = P_1^{n_1} * ... * P_k^{n_k} \to n_i$$

Proposition 2.7 : v_i is an arithmetical pseudovaluation on $\mathcal{D}(\Lambda)$.

Proof :

It is easy to check that $A = P_1^{n_1} * ... * P_k^{n_k} = (\Lambda : (\Lambda : P_1^{n_1}...P_k^{n_k}))$. Since $\mathcal{D}(\Lambda)$ is a commutative group, we see that for all $I, J \in \mathcal{D}(\Lambda)$ we have $v_i(I * J) = v_i(I) + v_j(J)$. Now, let $I \subset J$ and let $\psi(J) = \sum m_j.e_j$, $\psi(I) = \sum n_j.e_j$. Since $\psi(J) \leq \psi(I)$ we have that $v_i(J) \leq v_i(I)$.

Next, we have to establish that $v_i((I+J)^d) \geq min(v_i(I), v_i(J))$. Suppose first that both I and J are in Λ. Then $I = (\Lambda : (\Lambda : P_1^{n_1}...P_k^{n_k})) \subset (\Lambda : (\Lambda : P_i^{n_i})$ because all $n_i \geq 0$. Similarly, $J = (\Lambda : (\Lambda : P_1^{m_1}...P_k^{m_k})) \subset (\Lambda : (\Lambda : P_i^{m_i}))$ whence $(I + J) \subset (\Lambda : (\Lambda : P_i^{n_i})) + (\Lambda : (\Lambda : P^{m_i})) = (\Lambda : (\Lambda : P_i^{k_i}))$ where $k_i = min(n_i, m_i)$ yielding that $v_i((I + J)^d) \geq min(v_i(I), v_i(J))$. If $I \not\subset \Lambda$ or $J \not\subset \Lambda$, there exists an element $c \in R$ such that $c.I \subset \Lambda$ and $c.J \subset \Lambda$. Hence, $v_i(((c.\Lambda).I + (c.\Lambda).J)^d) \geq min(v_i((c.\Lambda).I), v_i((c.\Lambda).J)) = min(v_i(c.\Lambda) + v_i(I), v_i(c.\Lambda) + v_i(J))$. Therefore, $v_i(c.\Lambda) + v_i((I+J)^d) \geq v_i(c.\Lambda) + min(v_i(I), v_i(J))$ completing the proof.

Lemma 2.8 : Let v be an arithmetical pseudovaluation on $\mathcal{D}(\Lambda)$ and let $\{I_j\}$ be an arbitrary set of divisorial ideals such that $\sum I_j \in \mathcal{F}(\Lambda)$, then $v((\sum I_j)^d) = inf(v(I_j))$.

Proof :

One inequality is obvious since $v((\sum I_j)^d) \leq v(I_j)$ for every j. The proof of the converse inequality is similar to the proof of proposition 2.7.

Corollary 2.9 : If v is an arithmetical pseudovaluation on $\mathcal{D}(\Lambda)$ and if $I \in \mathcal{D}(\Lambda)$, then $v(I) = inf\{v((\Lambda.x.\Lambda)^d) \mid x \in I\}$.

Now, let us consider again the arithmetical pseudovaluation :

$$v_i : \mathcal{D}(\Lambda) \to \mathbb{Z} : A = P_1^{n_1} * \dots * P_k^{n_k} \to n_i$$

and let us assume for the sake of simplicity that $i = 1$. Denote $Q_1 = \{x \in \Sigma \mid v_1(C_x) > 0\}$ and $\Lambda_1 = \{x \in \Sigma \mid v.Q_1 \subset Q_1$ and $Q_1.x \subset \Lambda_1\}$. It is straightforward to check that $Q_1 \cap \Lambda = P_1$.

Proposition 2.10 : $\Lambda_1 = \{x \in \Sigma \mid v_1(C_x) \geq 0\} = \{x \in \Sigma \mid x.I \subset \Lambda$ and $I.x \subset \Lambda$ for some ideal I of Λ not contained in $P_1\}$.

Proof :

(1) : Suppose $x \in \Sigma, v_1(C_x) \geq 0$ and $y \in Q_1$, i.e. $v_1(C_x) > 0$. We immediatly have that $\Lambda.x.y.\Lambda \subset \Lambda.x.\Lambda.\Lambda.y.\Lambda \subset C_x * C_y$, hence $C_{x.y} \subset C_x * C_y$. This yields $v_1(C_{x.y}) \geq v_1(C_x) + v_1(C_y) > 0$. Therefore, $x.Q_1 \subset Q_1$ and similarly $Q_1.x \subset Q_1$. Conversely, suppose $x \in \Lambda_1$, so in particular $x.P_1 \subset Q_1$. Corollary 2.8 above yields that there is an element $y \in P_1$ such that $v_1(C_y) = 1$. Hence , $\Lambda.x.\Lambda.y.\Lambda \subset Q_1$. We claim that $v_1((\Lambda.x.\Lambda.y.\Lambda)^d) > 0$. If $\Lambda.x.\Lambda.y.\Lambda \subset \Lambda$ we may write $(\Lambda.x.\Lambda.y.\Lambda)^d = (C_{x.r_1.y} + \dots + C_{x.r_n.y})^d$ (because the divisorial ideals contained

in Λ satisfy the ascending chain condition). If $\Lambda.x.\Lambda.y.\Lambda \not\subset \Lambda$, then it may be multiplied by a central element such that the image is in Λ and then the argument used before may be repeated. Since $x.r_j.y \in Q_1$ for all j and lemma 2.7 yields that $v_1((\Lambda.x.\Lambda.y.\Lambda)^d) > 0$. Therefore, $v_1((\Lambda.x.\Lambda.y.\Lambda)^d) = v_1(C_x) + v_1(C_y) \geq 1$ and $v_1(C_y) = 1$ hence $v_1(C_x) \geq 0$.

(2) : Let $x \in \Lambda_1$, that means $v_1(C_x) \geq 0$. Write $C_x = P_1^{n_1} * \ldots * P_k^{n_k}$ and $n_1 \geq 0$. Multiply this equality by those $P_i^{n_i}$ with $n_i < 0$. Then $I * C_x = P_1^{n'_1} * \ldots * P_k^{n'_k}$ where $I \in D(\Lambda)$ and on the right side of this equality all n'_i are positive. Hence we have that $I * C_x \subset \Lambda$. Now I is the product in $D(\Lambda)$ of positive powers of P_i and $i \neq 1$. Hence $I \not\subset P_1$. Conversely, suppose $I.x \subset \Lambda$ and $I \not\subset P_1$. Take $y \in I - P_1$. Then $y.\Lambda.x \subset \Lambda \subset \Lambda_1$. Similarly, as in the first part of the proof we have $v_1((\Lambda.x.\Lambda.y.\Lambda)^d) = v_1(C_y) + v_1(C_x) \geq 0$. But, $v_1(C_y) = 0$ since $y \in \Lambda - P_1$ and therefore $v_1(C_x) \geq 0$ and thus finally $x \in \Lambda_1$.

It is easy to derive from the foregoing proposition and from the fact that there is a one-to-one correspondence between the height one prime ideals of R and those of Λ that $\Lambda_1 \simeq \Lambda_{p_1}$ where $p_1 = P_1 \cap \Lambda$. This yields that Λ_1 is a left and right principal ideal ring and that it is a maximal order over a discrete valuation ring. Thus, in particular, any twosided ideal of Λ_1 is some power of its Jacobson radical $J(\Lambda_1)$.

Let R_v be the center of Λ_1 and let m be the uniformizing parameter of the associated discrete valuation in K, then by the foregoing remarks we know that $\Lambda_1.m = J(\Lambda_1)^e$. This e is called the ramification index of the valuation v in Σ.

b : the central classgroup :

In this section we will introduce and study an invariant which describes how the arithmetic of the maximal order differs from that of its center : the central classgroup.

This central classgroup of a maximal R-order Σ , $Cl^c(\Lambda)$, is defined to be the quotient group of $\mathcal{D}(\Lambda)$ by $\mathcal{P}^c(\Lambda)$, where $\mathcal{P}^c(\Lambda)$ is the subgroup of $\mathcal{D}(\Lambda)$ consisting of those divisorial Λ-ideals of the form $\Lambda.k$ for some $k \in K^*$. Alternatively, one could view $Cl^c(\Lambda)$ to be the twosided Λ- module isomorphism classes of divisorial Λ-ideals , for if $f : \Lambda \simeq A$ as a twosided Λ-module , then there exists an element x such that $f(\lambda) = \lambda.x$. Because f is right linear we obtain that $f(\lambda.\lambda') = f(\lambda').\lambda$ and therefore $\lambda'.\lambda.x = \lambda'.x.\lambda$ for all $\lambda, \lambda' \in \Lambda$ yielding that $\lambda.x = x.\lambda$, i.e. $x \in K^*$.

If R is a discrete valuation ring in K and if Λ is a maximal R-order in some central simple K-algebra Σ and if $M = J(\Lambda)$ is the unique maximal ideal of Λ , then $M^e = \Lambda.m$ where m is the uniformizing parameter of R and e is the ramification index of this valuation in Σ , cfr. the previous section. Since all maximal orders over a discrete valuationring are conjugated , it follows that this ramification index does not depend upon the particular choice of Λ. If $e \geq 2$ we say that the valuation ramifies in Σ , otherwise it is said to be unramified.

Our first objective is to study the relation between the central classgroup of a maximal order over a Krull domain and the classgroup of the center.

Proposition 2.11 : If Λ is a maximal order over a Krull domain R in some central simple K-algebra Σ , then the following sequence of groups is exact :

$$1 \to Cl(R) \to Cl^c(\Lambda) \to \bigoplus \mathbb{Z}/e_p\mathbb{Z} \to 1$$

where $\bigoplus \mathbb{Z}/e_p\mathbb{Z}$ is a finite group which describes the ramification- indices of the essential valuations associated to R in Σ and moreover its exponent is bounded by the p.i.-degree of Λ.

Proof :

Consider the natural map $\phi : \mathcal{D}(R) \to \mathcal{D}(\Lambda)$ which is defined by $\phi(A) = (\Lambda.A)^{**}$. Because there is a natural one- to-one correspondence between $X^{(1)}(R)$ and $X^{(1)}(\Lambda)$, this map is readily checked to be a groupmonomorphism and for every $p \in X^{(1)}(R)$, there exists a unique natural number e_p and a unique prime ideal $P \in X^{(1)}(\Lambda)$ such that $(\Lambda.p)^{**} = (P^{e_p})^{**}$. It is straightforward to check that e_p is the ramification index of the discrete valuation associated to p in Σ.

Furthermore , $\phi(\mathcal{P}(R)) = \mathcal{P}^c(\Lambda)$ so we obtain the following exact diagram :

$$\begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \to & \mathcal{P}(R) & \to & \mathcal{D}(R) & \to & Cl(R) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathcal{P}^c(\Lambda) & \to & \mathcal{D}(\Lambda) & \to & Cl^c(\Lambda) & \to & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & \bigoplus \mathbb{Z}/e_p\mathbb{Z} & & & & \end{array}$$

where $\phi : Cl(R) \to Cl^c(\Lambda)$ is the induced morphism. Applying the snake lemma entails that the sequence below is exact :

$$1 \to Cl(R) \to Cl^c(\Lambda) \to \bigoplus \mathbb{Z}/e_p\mathbb{Z} \to 1$$

Therefore , we are left to prove that $Coker(\phi)$ is a finite group whose exponent is bounded by the p.i.-degree of Λ. Let c be an arbitrary nonzero element in the Formanek center $F(\Lambda)$ of Λ , then there are only a finite number of height one prime ideals p in R such that $c \in p$ by the finite character property of R. Because the localization of Λ at any of the other height one prime ideals of R is an Azumaya algebra , it follows that $e_p = 1$ for allmost all $p \in X^{(1)}(R)$ showing finiteness of $Coker(\phi)$.

As for the last claim :

Let P be an height one prime ideal of Λ with ramification index e_p , then $(nr(P))^{**} = (p^f)^{**}$ for some natural number f. Then , $(\Lambda.p^f)^{**} = (P^{e_p.f})^{**}$ and therefore taking reduced norms on both sides yields : $(p^{f.n})^{**} = (p^{f^2.e_p})^{**}$ whence finally $e_p \leq n$, finishing the proof.

Example 2.12 : Let $\Lambda = \mathbb{C}[X,-]$, then Λ is a maximal order over $R = \mathbb{R}[X^2]$. If $P \neq (X) \in X^{(1)}(\Lambda)$ then $pid(\Lambda/P) = 2$ and $pid(\Lambda/(X)) = pid(\mathbb{C}) = 1$ whence $e_{P\cap R} = m1$ for all $P \neq (X)$. Furthermore, $e_{(X^2)} = 2$ and therefore (since $Cl(\mathbb{R}[X^2]) = 1$), $Cl^c(\mathbb{C}[X,-]) = \mathbb{Z}/2\mathbb{Z}$.

Proposition 2.13 : If Λ is a reflexive Azumaya algebra over the Krull domain R , then $Cl(R) \simeq Cl^c(\Lambda)$.

Proof :

If $F(\Lambda)$ denotes the Formanek center of Λ , then $X^{(1)}(R) \subset X(R.F(\Lambda)) = \{p \in Spec(R) : R.F(\Lambda) \not\subset p\}$ because Λ is a reflexive Azumaya algebra. This entails that Λ_p is an Azumaya algebra over R_p for every $p \in X^{(1)}(R)$, hence $e_p = 1$ and proposition b.1 now finishes the proof.

The main problem in the rest of this section will be to determine to what extent the inverse implication of the foregoing proposition holds , i.e. for which maximal orders are the ramified height one prime ideals exactly those lying outside the open set of $Spec(R)$ determined by the Formanek-center of Λ.

Since $Coker(\phi)$ depends only upon the ramified essential valuations of R in Σ we have :

Proposition 2.14 : If Λ and Γ are two maximal R-orders in the same central simple K-algebra Σ , then :

$$Cl^c(\Lambda) \simeq Cl^c(\Gamma)$$

From [33] we retain that this central classgroup has nice functorial properties with respect to ringextensions in the sense of C. Procesi. So, if $\Lambda \subset \Gamma$ is an extension of maximal orders which satisfies the pas d'éclatement condition , i.e. $ht(P \cap \Lambda) \leq 1$ for every $P \in X^{(1)}(\Gamma)$. Note that $\Lambda \subset \Gamma$ satisfies pas d'éclatement if and only if $Z(\Lambda) \subset Z(\Gamma)$ satisfies pas d'éclatement.

In [33] it is shown that whenever $\Lambda \to \Gamma$ is an extension which satisfies pas d'éclatement , then the natural map :

$$\mathcal{D}(\Lambda) \to \mathcal{D}(\Gamma)$$

defined by sending A to $(\Gamma.A)^{**}$ is a groupmorphism and it induces a morphism :

$$Cl^c(\Lambda) \to Cl^c(\Gamma)$$

c : the Jespers-Van Oystaeyen conjecture for tamifiable maximal orders

F. Van Oystaeyen and later E. Jespers [34] asked whether for a maximal order Λ over a Dedekind domain R , the vanishing of the central classgroup implies that Λ is an Azumaya algebra over R. This conjecture is equivalent to the following :

(Jespers-Van Oystaeyen conjecture) If Λ is a maximal order over a Krull domain R then the following statements are equivalent:
(1) : Λ is a reflexive Azumaya algebra in the sense of M. Orzech [61].
(2) : $Cl^c(\Lambda) \simeq Cl(R)$.

In this section we aim to prove this conjecture for a large class of maximal orders. Let us start by recalling some definitions :

If Λ is an order over a Dedekind domain R , then Λ is said to be an hereditary order (or an h.n.p.-ring in the terminology of Robson [69]) if every one sided ideal of Λ is a projective Λ-module. Note that , unlike in the commutative case , an hereditary order need not be maximal. A standard counterexample is :

$$\Lambda \simeq \begin{pmatrix} R & M \\ R & R \end{pmatrix}$$

where M is an ideal of the Dedekind domain R. An order Λ over a Krull domain R is said to be a tame order in the sense of R. Fossum [21] if Λ_p is an hereditary order over R_p for every $p \in X^{(1)}(R)$.

(a) : the local case

In this first section we will restrict attention to the case where Λ is a maximal order over a discrete valuation ring R such that $Cl^c(\Lambda) = 1$. This means that the unique maximal ideal M of Λ is of the form Λ.m where m is the uniformizing parameter of R . In order to check that Λ is an Azumaya algebra over R , it is sufficient to check that $\Lambda/\Lambda.m$ is a separable algebra over R/Rm ,[19].

The condition which appears in the literature , cfr. e.g. [67] , is that $Z(\Lambda/\Lambda.m)$ is a separable field extension of R/Rm (for, $\Lambda/\Lambda.m$ is a simple p.i.-ring whence separable over its center). Our first aim is to improve some results of Reiner [67] and Riley [68] and to reduce the study of the Jespers-Van Oystaeyen conjecture to two special cases. The proof of the next proposition relies heavily upon some results of J. Mc Connell [51] and M. Chamarie [13] on the localizability of prime ideals in maximal orders.

Proposition 2.15 : If Λ is a maximal order over a discrete valuation ring R with $Cl^c(\Lambda) = 1$, then one of the following situations occurs :

(a) : $Z(\Lambda/\Lambda.m) = R/R.m$ in which case Λ is an Azumaya algebra

(b) : $Z(\Lambda/\Lambda.m)$ is a purely inseparable field extension of R/R.m

Proof : The proof will be split up in several steps :

step 1 : First, we claim that it is sufficient to check that prime ideals of the polynomial ring $\Lambda[t]$ which lie over $\Lambda.m$ satisfy the unique-lying-over property with respect to the center R[t]. For, it is rather easy to see that this set of prime ideals corresponds bijectively to $\mathrm{Spec}(\Lambda/\Lambda.m[t])$. Now, $\Lambda/\Lambda.m$ is a simple p.i.-algebra, whence there is a one-to-one correspondence between $\mathrm{Spec}(\Lambda/\Lambda.m[t])$ and $\mathrm{Spec}(Z(\Lambda/\Lambda.m)[t])$. If the claimed condition is satisfied, this entails that there is a one-to-one correspondence between $\mathrm{Spec}(Z(\Lambda/\Lambda.m)[t])$ and $\mathrm{Spec}(R/R.m[t])$,i.e. there are no irreducible polynomials over R/Rm which decompose over $Z(\Lambda/\Lambda.m)$ in distinct irreducible polynomials. Because $Z(\Lambda/\Lambda.m)$ is a finite field extension of R/Rm this entails that $Z(\Lambda/\Lambda.m)$ cannot contain separable elements over R/Rm not belonging to R/Rm , finishing the proof of our claim.

step 2 : In [13] , M. Chamarie proved that a prime ideal P of a maximal order over a Krull domain satisfies the unique-lying-over property with respect to its center if and only if $C(P)$, the multiplicatively closed set of elements which are regular modulo P, satisfies the left and right Öre - conditions. Let us first verify that every P $\in$ Spec $\Lambda[t]$ such that $P \cap \Lambda = \Lambda.m$ satisfies the AR-property. By [51] 2.7 it is sufficient that P has a centralizing set of generators. Now, m $\in$ P and $P/\Lambda.m[t] = \Lambda/\Lambda.m[t].c'$ for some c' in $Z(\Lambda/\Lambda.m)[t]$, because every ideal in a polynomial ring over a simple ring is generated by a central element . So, (m,c) is a centralizing set of generators of P. Using [51] Th.6 and Coroll.7 , it will now be sufficient to check that every ideal of $\Lambda[t]$ has a centralizing set of generators. In fact, the proof of [51] Th.6 uses only the fact that certain ideals H_n have a centralizing set of generators, so we just have to check this property for ideals intersecting Λ nontrivially.

step 3 : Let I be any ideal of $\Lambda[t]$ such that $I \cap \Lambda \neq 0$, then $I \cap \Lambda = \Lambda.m^n$ for some natural number n. Let $I_1 = \mu_1(I)$ where $\mu_1 : \Lambda[t] \to \Lambda[t]/(m^n)$ is the canonical epimorphism and let $c_1 \in I$ be of minimal degree such that $\mu_1(c_1) \neq 0$. If m_1 is the leading coefficient of c_1 , then clearly $\mu_1(m_1) \neq 0$ and $\Lambda.m_1.\Lambda = \Lambda.m^{l_1}$ where $l_1 < n$, for , otherwise one could lower the degree of c_1. So, we may suppose that the leading coefficient of c_1 equals m^{l_1}. Because $m^{l_1} \in R$ and the degree of c_1 is minimal , $c_1\lambda - \lambda c_1 \in (m^n)$ for every $\lambda \in \Lambda$ yielding that $\mu_1(c_1) \in Z(\Lambda[t]/(m^n))$. If $l_1 = 0$ (i.e. $m_1 = 1$) then $\mu_1(I) = \Lambda[t]/(m^n).\mu_1(c_1)$, finishing the proof. If $0 < l_1 < n$ and if $I \neq (m^n, c_1)$, choose $c_2 \in I$ of minimal degree such that $\mu_2(c_2) \neq 0$ where $\mu_2 : \Lambda[t] \to \Lambda[t]/(m^n, c_1)$ is the canonical epimorphism. Clearly, by a minimal degree argument as before we may assume that the leading coefficient of c_2 equals m^{l_2} for some $l_2 < l_1$ and that $c_2\lambda - \lambda c_2 \in (m^n, c_1)$ for every $\lambda \in \Lambda$ whence $\mu_2(c_2) \in Z(\Lambda[t]/(m^n, c_1))$. Continuing in this way leads after a finite number of steps to an element c_m such that either $= (m^n, c_1, ..., c_m)$ or the leading coefficient of c_{m+1} is 1 yielding that $I = (m^n, \;, ..., c_{m+1})$, finishing the proof.

Remark 2.16 :

The condition : $Z(\Lambda/\Lambda.m)$ is not a purely inseparable field extension of $R/R.m$, is always satisfied in the cases under consideration in algebraic number theory and algebraic geometry. For, in these cases , Σ is a central simple algebra over a global field or over a functionfield of a variety over a basefield of characteristic zero , yielding that $R/R.m$ is a perfect field.

This vast amount of good examples may account for the manifest lack of interest of order-theorists in the question whether there exist maximal orders satisfying the condition of Prop.2.15.b .

Proposition 2.17 : If Λ is a maximal order over a discrete valuation ring R in a central simple K-algebra Σ and if L is a separable splitting subfield of Σ and let S be the integral closure of R in L. Then, the following two statements are equivalent :

(1) : Λ is an Azumaya algebra over R ;

(2) : $Cl^c(\Lambda) \simeq 1$ and $\Lambda \otimes_R S$ is an hereditary S-order.

Proof :

The implication (1) $\Rightarrow$ (2) is trivial since Azumaya algebras over discrete valuation rings are maximal orders and hence in particular hereditary.

Conversely, we have to check that the situation (b) of Prop.a.1 cannot occur. Again, we devide the proof in three steps :

step 1 : Suppose that $Z(\Lambda/\Lambda.m)$ is a proper purely inseparable field extension of $R/R.m$. By a result of [35] we know that the natural map between the Brauer-groups :

$$[-\otimes Z(\Lambda/\Lambda.m)] : Br(R/R.m) \to Br(Z(\Lambda/\Lambda.m))$$

is an epimorphism. So, there exists a central simple algebra A over $R/R.m$ such that $M_k(A) \otimes Z(\Lambda/\Lambda) \simeq M_l(\Lambda/\Lambda.m)$.

Replacing Λ by $M_l(\Lambda)$, Σ by $M_l(\Sigma)$ A by $M_k(A)$ etc. we may therefore assume that $\Lambda/\Lambda.m$ contains a simple algebra A over $R/R.m$ such that $\Lambda/\Lambda.m \simeq A \otimes Z(\Lambda/\Lambda.m)$. Now, if $\mu : \Lambda \to \Lambda/\Lambda.m$ denotes the natural epimorphism we will denote by $\dot{A}_1 = \mu^{-1}(\dot{A})$. Because $\dot{A}_1$ and $\dot{\Lambda}$ share the common twosided ideal $\dot{\Lambda}.m$, $\dot{A}_1$ is an order in Σ and the center of A_1 equals R. Furthermore, $\Lambda.m$ is the unique nonzero prime ideal of A_1 and $Z(A_1/\Lambda.m) = R/R.m$. Actually, A_1 is a Bäckström - order [70] with associated hereditary order Λ .

step 2 : Let L be a separable splitting field for Σ contained in Σ. Further, let S be the integral closure of R in L. It is easy to check that $S = L \cap \Lambda$ is a discrete valuation ring with uniformizing parameter m as is well known. Now, $\Lambda \otimes S$ is by assumption an hereditary order in $M_n(L)$ which is not maximal because otherwise Λ would be Azumaya (cfr. [70] Th.VI.2.8 or an easy descent argument). Now, by results of Harada or Artin [5] one can describe $\Lambda \otimes S$ in the following way

$$\Lambda \otimes_R S \simeq \begin{pmatrix} M_{n_1}(S) & S_{n_1 \times n_2} & \cdot & \cdot & S_{n_1 \times n_j} \\ m.S_{n_2 \times n_1} & M_{n_2}(S) & \cdot & \cdot & S_{n_2 \times n_j} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ m.S_{n_j \times n_1} & m.S_{n_j \times n_2} & \cdot & \cdot & M_{n_j}(S) \end{pmatrix}$$

with $n_1 + n_2 + ... + n_j = n$ and $j \geq 2$. Clearly, $A_1 \otimes S$ has an ideal n common with $\Lambda \otimes S$ namely $(\Lambda \otimes S).m$, therefore :

$$(\Lambda \otimes S).m \subset A_1 \otimes S \subset \Lambda \otimes S$$

This implies that there are at least j prime ideals of $A_1 \otimes S$ lying over mS . The proof will be complete if we can show that this is not possible.

step 3 : Because $\xi : A_1 \to A_1 \otimes S$ is a central extension, prime ideals intersect A_1 in prime ideals, so we have to calculate the fiber of ξ in $\Lambda.m$. A_1 being a finite module over its center and $\Lambda.m$ satisfying the unique- lying-over property with respect to the center, $\Lambda.m$ is localizable whence there is a one-to-one correspondence between this fiber and Spec$(A \otimes S)$ = Spec$(S/S.m)$ because A is a simple algebra with center $R/R.m$ and since $S/S.m$ is a field the proof is finished.

We now present an étale approach to our problem. For more details on étale extensions the reader is referred to Raynaud [66] or Milne [52].

Proposition 2.18 : If Λ is a maximal order over a discrete valuation ring R in a central simple K-algebra Σ , then the following two statements are equivalent

(1) : Λ is an Azumaya algebra over R ;

(2) : $Cl^c(\Lambda) \simeq 1$ and there exists an étale extension $R \subset S$ such that S splits Σ , i.e. $\Sigma \otimes_R S \simeq M_n(L)$ where L is the field of fractions of S and $n = p.i.d.(\Lambda)$.

Proof :

(1) $\Rightarrow$ (2) : is easy since any Azumaya algebra may be split by an étale extension , cfr. e.g. [35].

(2) $\Rightarrow$ (1) : First, we aim to prove that $S \otimes_R \Lambda$ is an hereditary order over S. First we will assume that S is a Galois extension of R and thus there exists an element $u = \sum x_i \otimes y_i$ in $S \otimes_R S$ such that $\sum x_i.y_i = 1$ and $(s \otimes 1).u = (1 \otimes s).u$ for any element $s \in S$.

Now, let $J = J(S \otimes_R \Lambda)$ be the Jacobson radical of $S \otimes_R \Lambda$. Because Λ is hereditary and J is a f.g. Λ-module ($R \subset S$ is finite) , J considered as a left Λ-module (denoted by J_Λ) is finitely generated and projective.We now define a map :

$$J \to S \otimes_R J_\Lambda$$

by sending an element j to $\sum x_i \otimes y_i.j$. It is easily verified that this is a $S \otimes_R \Lambda$-map which splits the $S \otimes_R \Lambda$-map :

$$S \otimes_R J_\Lambda \to J$$

defined by sending $s \otimes j$ to $s.j$. Finally, because $S \otimes_R J_\Lambda$ is a finitely generated $S \otimes_R \Lambda$- module , so is $J = J(S \otimes_R \Lambda)$ entailing that $S \otimes_R \Lambda$ is hereditary.

As for the general case , we may always assume (after localization) that S is a discrete valuation ring with uniformizing parameter m such that $R/R.m \subset S/S.m$ is a separable field extension. We now aim to prove that $S \otimes_R rad(\Lambda) \simeq rad(S \otimes_R \Lambda)$. One inclusion being trivial, it is sufficient to prove that $S \otimes_R \Lambda / S \otimes_R rad(\Lambda)$ is semisimple. Now, $S \otimes_R \Lambda / S \otimes_R rad(\Lambda) \simeq S \otimes_R \Lambda / rad(\Lambda) \simeq S/S.m \otimes_{R/R.m} \Lambda/rad(\Lambda)$ and is therefore semisimple Artinean because $R \to S$ is unramified.

Now, by our assumptions , S splits Σ and therefore $S \otimes_R \Lambda$ is of the form as described in the proof of proposition a.3 . Therefore, it will be sufficient to check that the fiber of the natural map :

$$S \to S \otimes_R \Lambda$$

at $S.m$ consists of one element (by a descent argument as in proposition 2.17). This fiber is in a one-to-one correspondence with $Spec(S/S.m \otimes \Lambda/\Lambda.m) \simeq Spec(S/S.m \otimes Z(\Lambda/\Lambda.m))$. By proposition 2.15 we know that $Z(\Lambda/\Lambda.m)$ is a purely inseparable field extension over $R/R.m$ and $S/S.m$ is separable over $R/R.m$, whence this fiber consists of precisely one element as desired.

(b) : the global case

Now, let Λ be a maximal order over a Krull domain R in a central simple K-algebra Σ. Λ is said to be Zariski-tamifiable if for every $p \in X^{(1)}(R)$ one can find a separable splitting subfield L of Σ such that $\Lambda \otimes_{R_p} S(p)$ is an hereditary order over $S(p)$ if $S(p)$ denotes the integral closue of R_p in L.

A special case of such a situation (which explains the terminology) is the following : assume that S is the integral closure of R in some separable splitting subfield L of Σ and that $\Lambda \otimes_R S$ is a tame order in the sense of R. Fossum , then Λ is Zariski-tamifiable.

Theorem 2.19 : With notations as before , we have :

(1) : Λ is a reflexive Azumaya algebra over R if and only if $Cl(R) \simeq Cl^c(\Lambda)$ and Λ is Zariski-tamifiable.

(2) : Λ is an Azumaya algebra over R if and only if $Cl(R) \simeq Cl^c(\Lambda)$, Λ is Zariski-tamifiable and Λ is a flat R-module.

Proof :

In view of II.1.a , we have only to prove part (1). Now, it is readily verified that $Cl(R) \simeq Cl^c(\Lambda)$ entails that for every $p \in X^{(1)}(R)$ we have that Λ_p is a maximal R_p-order with $Cl^c(\Lambda_p) \simeq 1$. Because Λ is supposed to be Zariski-tamifiable , Prop.2.15 entails that Λ_p is an Azumaya algebra over R_p. Therefore, $X^{(1)}(R) \subset X(R.F(\Lambda)$ yields that Λ is a reflexive Azumaya algebra, finishing the proof.

A maximal R-order Λ in the central simple algebra Σ is said to be étale tamifiable if and only if the following condition is satisfied : for every $p \in X^{(1)}(R)$, there exists an étale extension $R_p \subset S(p)$ which splits Σ.

Theorem 2.20 : With notations as before , we have :

(1) : Λ is a reflexive Azumaya algebra over R if and only if $Cl(R) \simeq Cl^c(\Lambda)$ and Λ is étale tamifiable.

(2) : Λ is an Azumaya algebra over R if and only if $Cl(R) \simeq Cl^c(\Lambda)$, Λ is étale tamifiable and Λ is a flat R-module.

Proof :

As before, we only have to prove part (1) . Again, $Cl(R) \simeq Cl^c(\Lambda)$ entails that for every $p \in X^{(1)}(R)$ we have that $Cl^c(\Lambda_p) \simeq 1$. Proposition 2.18 then entails that Λ_p is an Azumaya algebra , whence $X^{(1)}(R) \subset X(R.F(\Lambda))$, finishing the proof.

3. : NEW EXAMPLES OF ORDERS

a. : introduction

The contruction of generalized Rees rings , first introduced by F. Van Oystaeyen [88] in the commutative case , has been generalized in [42]. Roughly, the general situation may be described as follows. Let Σ be a (rather arbitrary) ring and let G be an arbitrary group , consider the crossed product :

$$\Sigma[X_\tau, \psi, c]$$

where X_τ is a symbol for each $\tau \in G$, $\psi : G \to Aut(\Sigma)$ is a groupmorphism and $c : G \times G \to U(Z(\Sigma))$ is a 2-cocycle , i.e. c represents an element of $H^2(G, U(Z(\Sigma)))$ where $U(Z(\Sigma))$ is the group of units of the center of Σ. The ringstructure of $\Sigma[X_\tau, \psi, c]$ is determined by the rules : $X_\tau.x = \psi(\tau)(x).X_\tau$ for all $x \in \Sigma, \tau \in G$ and $X_\tau.X_\gamma = c(\tau, \gamma).X_{\tau.\gamma}$ for all $\tau, \gamma \in G$.

A generalized Rees ring is then a subring of $\Sigma[X_\tau, \psi, c]$ of the form $\bigoplus \Lambda_\tau.X_\tau$, where $\Lambda_e = \Lambda$ is a subring of Σ and Λ_τ , $\tau \in G$ is a twosided Λ-module in Σ such that $\Lambda_\tau.\Lambda_\gamma^{\psi(\tau)} \subset \Lambda_{\tau.\gamma}$.

Taking $G = \mathbb{Z}$, Λ a commutative Dedekind domain and Σ its field of fractions and if we have moreover that $\Lambda_\tau.\Lambda_\gamma^{\psi(\tau)} = \Lambda_{\tau.\gamma}$, then we obtain the commutative generalized Rees rings sudied in [88]. Another commutative example may be obtained by taking G to be an Abelian, torsion free group satisfying the ascending chain condition on cyclic subgroups , taking Λ to be a commutative Krull domain with field of fractions Σ and if we impose $(\Lambda_\tau.\Lambda_\gamma)^{\psi(\tau)})^{**} = \Lambda_{\tau.\gamma}$ for all $\tau, \gamma \in G$, where $(-)^{**}$ denotes the bidual module of $(-)$. These rings were defined in [42].

Now, in the noncommutative case it is plausible that Σ is going to be a central simple algebra whereas Λ is some subring having nice arithmetical properties , e.g. a maximal order over a Dedekind or Krull domain , an H.N.P.-ring or a tame order in the sense of R.Fossum [21].

The author introduced relative maximal orders [40] in order to present a unified approach to rings having an arithmetical ideal structure and indeed all examples mentioned before reduce to special cases of relative maximal orders. This motivates the introduction of generalized Rees rings over general relative maximal orders.

The ringtheoretical tools used in this section are : general techniques in the theory of G-graded rings , properties of Picard and relative Picard groups of orders and the arithmetical features of relative maximal orders. For a more extensive account of these topics the reader is referred to [23],[91],[41],[40] and [58].

Although further generalization is certainly possible we restrict attention to the case of prime rings satisfying a nontrivial polynomial identity , and usually even to orders over Krull domains.

b. : relative maximal orders and Picard groups

Throughout Λ will be a prime p.i. ring with classical ring of quotients Σ which is a central simple K-algebra , K the field of fractions of $R = Z(\Lambda)$, $\mathcal{L}$ will be the multiplicatively closed filter of all nonzero (twosided) ideals of Λ and $\mathcal{L}(\rho)$ will be a multiplicatively closed subfilter such that the generated left (resp. right) ideal filters $\mathcal{L}^l(\rho)$ (resp. $\mathcal{L}^r(\rho)$) are idempotent in the sense of [74] or [83].

We say that Λ is a $\rho-$maximal order in Σ if there exists no intermediate ring $\Lambda \subset \Gamma \subset \Sigma$ such that $I.\Gamma.J \in \mathcal{L}(\rho)$ for some $I, J \in \mathcal{L}(\rho)$. This concept provides a relativation with respect to $\mathcal{L}(\rho)$ of the classical notrion of a maximal order as introduced before. It is fairly easy to verify [40] that Λ is a $\rho-$maximal order in Σ if and only if $(I :_l I) = (I :_r I) = \Lambda$ for all ideals $I \in \mathcal{L}(\rho)$. Unless mentioned otherwise Λ will be a $\rho-$maximal order in Σ.

A fractional ideal A of Λ is a twosided Λ-submodule of Σ such that $I.A$ and

$A.I$ both belong to $\mathcal{L}(\rho)$ for some ideal $I \in \mathcal{L}(\rho)$. The set of all fractional Λ-ideals , $\mathcal{F}_\rho(\Lambda)$, is closed under multipliation. Therefore, $\mathcal{F}_\rho(\Lambda)$ is an ordered set (with respect to inclusion) equipped with a multiplication. The construction of the E. Artin group (cfr. previous section) associated to such a set comes down to taking equivalence classes of elements of $\mathcal{F}_\rho(\Lambda)$ with respect to the relation :

$$A \sim B \Leftrightarrow (A : \Lambda) = (B : \Lambda)$$

where it should be noted that for every $A \in \mathcal{F}_\rho(\Lambda)$ we have $(A :_l \Lambda) = (A :_r \Lambda)$, cfr. [40]. The E. Artin group associated with $\mathcal{F}_\rho(\Lambda)$ will be denoted by $\mathcal{D}_\rho(\Lambda)$ and it is obvious from the construction that $\mathcal{D}_\rho(\Lambda)$ may be identified with the set of all divisorial Λ-ideals (i.e. fractional Λ- ideals such that $(A : \Lambda) : \Lambda = A$) equipped with the $*$-multiplication : $A * B = (A.B : \Lambda) : \Lambda$. It follows that $\mathcal{D}_\rho(\Lambda)$ is an Abelian group , cfr. [40].

From now on we will also assume that Λ satisfies the ascending chain condition on divisorial Λ-ideals contained in Λ. Note that this is a relativation of the notion of a maximal order over a Krull domain. Considering $\mathcal{D}_\rho(\Lambda)$ with the reversed ordering one obtains that every finite set of elements of $\mathcal{D}_\rho(\Lambda)$ admits a supremum and an infimum and moreover any nonempty set of divisorial Λ-ideals contained in Λ has a minimal element. As in the previous section one deduces from these facts that $\mathcal{D}_\rho(\Lambda) \simeq \mathbb{Z}^{(S)}$ for some index set S. The divisorial Λ-ideals contained in Λ and which are maximal (with respect to inclusion) as such form a set of free generators for $\mathcal{D}_\rho(\Lambda)$. One easily verifies that these generators are actualy prime ideals of Λ and conversely that all divisorial prime ideals are generators,cfr. [40]. Let $\mathcal{P}(\rho)$ be the set of free generators of $\mathcal{D}_\rho(\Lambda)$. To any prime ideal P of Λ we associate the filter of ideals :

$$\mathcal{L}(\Lambda - P) = \{I \leq_2 \Lambda; I \not\subset P\}$$

and we define

$$\mathcal{L}(\sigma) = \cap\{\mathcal{L}(\Lambda - P) \cap \mathcal{L}(\rho); P \in \mathcal{P}(\rho)\}$$

We will always assume that $\mathcal{L}^l(\sigma)$ and $\mathcal{L}^r(\sigma)$ are idempotent filters, i.e. σ may be regarded as a kernel functor in Λ-mod as well as in mod-Λ. All imposed conditions will be satisfied in the examples we will encounter, e.g. for orders over Krull domains.

With $Q_\rho(\Lambda)$ we will denote $\{x \in \Sigma : I.x \subset \Lambda \text{ and } x.I \subset \Lambda \text{ for some } I \in \mathcal{L}(\rho)\}$. Clearly, all fractional Λ-ideals are contained in $Q_\rho(\Lambda)$.

Lemma 3.1 : Let Λ be a $\rho-$maximal order in Σ. If $A \in \mathcal{D}_\rho(\Lambda)$ and $I \in \mathcal{F}_\rho(\Lambda)$, then $(I :_l A)$ and $(I :_r A)$ are divisorial Λ-ideals and

$$(I :_l A) = (I^d :_l A) = A * I^{-1} = I^{-1} * A = (I^d :_r A) = (I :_r A)$$

where $I^d = (I : \Lambda) : \Lambda$ and $I^{-1} = \{x \in Q_\rho(\Lambda) : I.x.I \subset I\}$.

Proof :

It is clear from the definitions above that $(A^{-1}.I)^{-1}$ and $I^{-1} * A$ are divisorial and that $(A^{-1}.I)^{-1} = I^{-1} * A$. If $x \in Q_\rho(\Lambda)$ is such that $I.x \subset A$, then $A^{-1}.I.x \subset A^{-1}.A \subset \Lambda$ hence $x \in (A^{-1}.I)^{-1}$. Conversely, $x \in I^{-1} * A$ entails that $I.x \subset I.(I^{-1}.A)^d \subset I^d.(I^{-1}.A)^d \subset A$. Therefore, $(I :_r A) = I^{-1} * A$ and similarly one proves $A * I^{-1} = (I :_l A)$. The fact that $\mathcal{D}_\rho(\Lambda)$ is an Abelian group combined with $(I^d)^{-1} = I^{-1}$ now finishes the proof.

Proposition 3.2 : For $A, B \in \mathcal{D}_\rho(\Lambda) : Q^l_\sigma(AB) = A * B = Q^r_\sigma(AB)$.

Proof :

If $x \in Q^r_\sigma(AB)$ then $x.I \subset A.B \subset (A.B)^d$ for some $I \in \mathcal{L}(\sigma)$. Since $I \in \mathcal{F}_\rho(\Lambda)$ and I is not contained in any $P \in \mathcal{P}(\rho)$ it follows that $I^d = \Lambda$. Now it follows from the foregoing lemma that $x.I^d = x.\Lambda \subset (A.B)^d$. Conversely, $x \in (A.B)^d$ yields $x.(\dot{A}.\ddot{B} : \dot{\Lambda}) \subset \dot{\Lambda}$ and thus $x.(\dot{A}.\ddot{B} : \dot{\Lambda}).\dot{A}.\ddot{B} \subset \dot{A}.\ddot{B}$. Because $(\dot{A}.\ddot{B} : \dot{\Lambda}).\dot{A}.\ddot{B}$ is an ideal of Λ not contained in any $P \in \mathcal{P}(\rho)$ and since $(A.B : \Lambda).A.B \in \mathcal{L}(\rho)$ it follows that $(A.B)^d \subset Q^r_\sigma(AB)$. In a similar manner one can prove that $Q^l_\sigma(AB) = (A.B)^d$, finishing the proof.

Remark 3.3 : More generally, if a subfilter $\mathcal{L}(\rho')$ of $\mathcal{L}(\rho)$ is such that the generated filters of left (resp. of right) ideals are idempotent , then for any divisorial Λ-ideal A we have : $Q^l_{\rho'}(A) = Q^r_{\rho'}(A) = \{x \in Q_\rho(\Lambda) : I.x \subset A$ and $x.I \subset A$ for some $I \in \mathcal{L}(\rho')\}$.

Picard groups of orders have been studied by A. Fröhlich in [23] but explicit results are only obtained for orders over Dedekind domains. In order to study the more general situation of orders over Krull domains we do not only have to study the Picard group $Pic(\Lambda)$ but also the relative Picard group $Pic(\Lambda, \sigma)$ for some suitable kernel functor σ. This group arose in the study of the Picard group of a Grothendieck category in [91']. E.g. if Λ is a commutative Krull domain then $Pic(\Lambda, \sigma)$ is nothing but the classgroup in case $\sigma = inf\{\sigma_{\Lambda-P}; P \in X^{(1)}(\Lambda)\}$. Let us first recall some generalities :

Let σ be any idempotent kernel functor in Λ-mod. A twosided Λ-module P is said to be σ-flat if for every exact sequence :

$$0 \to K \to M \to N \to 0$$

in Λ-mod with $\sigma(K) = K$ (i.e. K is σ-torsion) we have that the kernel of the extended morphism $P \otimes_\Lambda M \to P \otimes_\Lambda N$ is a σ-torsion module too. A twosided Λ-module P is said to be σ-invertible if $P \otimes_\Lambda -$ maps σ-torsion modules to σ-torsion modules and moreover there exists a twosided Λ-module Q having the same property and such that $Q_\sigma(P \otimes_\Lambda Q) \simeq Q_\sigma(\Lambda) \simeq Q_\sigma(Q \otimes_\Lambda P)$, where all isomorphisms are twosided Λ- module isomorphisms. It is known that σ-invertible modules are σ-flat , cfr. [41]. The isomorphism classes of σ-invertible modules form a group under the modified tensor product $Q_\sigma(-\otimes_\Lambda -)$. This group is called the relative Picard group of Λ with respect to σ and it will be denoted by $Pic(\Lambda, \sigma)$. One can relate $Pic(\Lambda,\sigma)$ to the Picard group of the localized Grothendieck category (Λ, σ)-mod , cfr. [91]. For further use we need two basis lemmas about $Pic(\Lambda, \sigma)$ in the noncommutative case :

Lemma 3.4 : Let Λ be a $\rho-$maximal order , let σ be defined as before and consider $P \in Pic(\Lambda,\sigma)$, then :

(a) : $End_{\Lambda}({}_{\Lambda}P) \simeq \Lambda^{opp}$;

(b) : $End_{\Lambda-\Lambda}(P) \simeq Z(\Lambda)$, where $End_{\Lambda-\Lambda}(-)$ stands for the ring of twosided Λ-module endomorphisms ;

(c) : If $P \simeq \Lambda$ in Λ-mod , then there exists an $\alpha \in Aut(\Lambda)$ such that $P \simeq_1 \Lambda_\alpha$ as twosided Λ-modules .

Proof :

(a) : Define $\theta : \Lambda^{opp} \to End_{\Lambda}({}_{\Lambda}P)$ by $\theta(\lambda)(p) = p.\lambda$ for every $p \in P$. If $\theta(\lambda) = 0$ then $P.\lambda = 0$ entails that $(Q \otimes_{\Lambda} P).\lambda = 0$. Because any Λ-linear morphism : $Q \otimes_{\Lambda} P \to Q \otimes_{\Lambda} P$ extends in a unique way to $Q^l_\sigma(Q \otimes_{\Lambda} P) \to Q^l_\sigma(Q \otimes_{\Lambda} P)$ we may take $Q \in Pic(\Lambda,\sigma)$ such that $Q^l_\sigma(Q \otimes_{\Lambda} P) \simeq Q^l_\sigma(P \otimes_{\Lambda} Q) \simeq Q^l_\sigma(\Lambda)$ and deduce from this that multiplication by λ extends to the zero map $Q^l_\sigma(\Lambda) \to Q^l_\sigma(\Lambda)$, i.e. $\lambda = 0$ because $1 \in Q^l_\sigma(\Lambda)$ and therefore θ is injective. It follows from lemma a.1 above that $Q^l_\sigma(\Lambda) = \Lambda$. Note also that the definition of $Pic(\Lambda,\sigma)$ implies that we may take P to be σ-closed , i.e. $Q^l_\sigma(P) = P$ and also that for every $M \in \Lambda$-mod , $Q^l_\sigma(P \otimes_{\Lambda} M) = 0$ if and only if $\sigma(M) = M$. Now consider $f \in End_{\Lambda}({}_{\Lambda}P)$,then $1_Q \otimes f$ extends in a unique way to a left Λ-linear morphism $g = Q^l_\sigma(1_Q \otimes f)$: $\Lambda \to \Lambda$ which makes the diagram below into a commutative one :

$$\begin{array}{ccccc} & Q \otimes_{\Lambda} P & \to & \Lambda & \\ 1_Q \otimes f & \downarrow & & \downarrow & g \\ & Q \otimes_{\Lambda} P & \to & \Lambda & \end{array}$$

If $g(1) = \lambda$, then g is determined by right multiplication by λ. Define $h : P \to P$ by $h(p) = p.\lambda$, then we have $Q^l_\sigma(1_Q \otimes_{\Lambda} f) = Q^l_\sigma(1_Q \otimes_{\Lambda} h)$. Now, $Im(f - h) \subset P$ and is therefore σ-torsion free while on the other hand $Q^l_\sigma(Q \otimes_{\Lambda} Im(f - h)) = 0$, yielding that $f = h$,finishing the proof.

(b) : If f is a twosided Λ-module endomorphism of P then by part (a) it is given as right multiplication by some $\lambda \in \Lambda$. Right linearity of f then implies that $P(\lambda.x - x.\lambda) = 0$ for all $x \in \Lambda$. By an argument as in part (a) it follows that $\lambda.x - x.\lambda = 0$ for all $\lambda \in \Lambda$, i.e. $\lambda \in Z(\Lambda)$.

(c) : Recall first that for $\beta, \alpha \in Aut(\Lambda)$, the twosided Λ-module ${}_\beta\Lambda_\alpha$ is defined to be the Abelian group $\Lambda_{(+)}$ with left Λ-action defined by $\lambda.x = \beta(\lambda)x$ and right Λ-action given by $x.\lambda = x\alpha(\lambda)$ for all $\lambda \in \Lambda$, $x \in \Lambda_{(+)}$.

Let $f : P \to \Lambda$ be an isomorphism in Λ-mod and let $\lambda \in \Lambda$. Define $g : P \to P$ by $g(p) = f^{-1}(f(p)\lambda)$. Because $\Lambda^{opp} \simeq End_\Lambda({}_\Lambda P)$, there exists an $\alpha(\lambda) \in \Lambda$ such that $g(p) = p\alpha(\lambda)$, i.e. $f(p\alpha(\lambda) = f(p)\lambda$ for all $p \in P$.Evidently, $\alpha \in Aut(\Lambda)$ and then f can be considered as a twosided Λ-module isomorphism ${}_1P_\alpha \to \Lambda$.

Lemma 3.5 : With notations as before, there exists a canonical groupmorphism $\alpha : Pic(\Lambda, \sigma) \to Aut(Z(\Lambda))$.

Proof :

For $c \in Z(\Lambda)$, $P \in Pic(\Lambda, \sigma)$, define $f : P \to P$ by $f(p) = c.p$. By part (a) of the foregoing lemma , there exists a unique element $\alpha_P(c) \in \Lambda$ such that $c.p = p.\alpha_P(c)$. Now, on one hand : $c.p.\lambda = p.\alpha_P(c).\lambda$ but on the other hand $c.(p.\lambda) = p.\lambda.\alpha_P(c)$ whence $P.(\lambda.\alpha_P(c) - \alpha_P(c).\lambda) = 0$,i.e. $\lambda.\alpha_P(c) = \alpha_P(c).\lambda = 0$ for all $\lambda \in \Lambda$, i.e. $\alpha_P(c) \in Z(\Lambda)$. Injectivity of $\alpha_P(-)$ is clear. Moreover, multiplication by $c \in Z(\Lambda)$ on the left extends to left multiplication by c , $Q^l_\sigma(P \otimes_\Lambda Q) \to Q^l_\sigma(P \otimes_\Lambda Q)$. Furthermore, $\alpha_{P\otimes_\Lambda Q} = \alpha_Q \circ \alpha_P$ so, if Q represents $[P]^{-1}$ in $Pic(\Lambda, \sigma)$, then $\alpha_Q \circ \alpha_P = \alpha_{Q\otimes_\Lambda P} = \alpha_{P\otimes_\Lambda Q} = \alpha_P \circ \alpha_Q = 1_{Z(\Lambda)}$ showing that $\alpha_P(-)$ is epimorphic and that $\alpha : Pic(\Lambda, \sigma) \to Aut(Z(\Lambda))$ is a groupmorphism.

The next proposition clarifies the relationship between relative invertible modules and divisorial Λ-ideals.

Proposition 3.6 : If Λ is a ρ -- maximal order and if $A \in D_\rho(\Lambda)$, then A is σ-invertible.

Proof :

Suppose that $M \in \Lambda$-mod is such that $\sigma(M) = M$ and consider $A \otimes_\Lambda M$. If $y = \sum a_j \otimes m_j \in A \otimes_\Lambda M - \sigma(A \otimes_\Lambda M)$ then pick an ideal $I \in \mathcal{L}(\sigma)$ such that $I.m_j = 0$ for all m_j. Because $\mathcal{D}_\rho(\Lambda)$ is an Abelian group $Q_\sigma(A.I) = Q_\sigma(I.A)$ and thus for every $i \in I$ there exists an ideal $J_{(i)} \in \mathcal{L}(\sigma)$ such that $J_{(i)}.i.a_j \in A.I$. Therefore, $J_{(i)}.i.\sum a_j \otimes m_j \subset \sum A.I \otimes m_j = 0$. Consequently, $I.(\sum a_j \otimes m_j) \subset \sigma(A \otimes_\Lambda M)$ and idempotency of σ entails that $y \in \sigma(A \otimes_\Lambda M)$, a contradiction ; i.e. $\sigma(A \otimes_\Lambda M) = A \otimes_\Lambda M$.

The second property for σ-invertible modules will follow from the more general result :

Proposition 3.7 : If $A, B \in \mathcal{D}_\rho(\Lambda)$, then $Q_\sigma(A \otimes_\Lambda B) \simeq Q_\sigma(A.B)$ as twosided Λ-modules.

Proof :

Multiplication in Σ defines a surjective twosided Λ- module morphism θ : $A \otimes_\Lambda B \to A.B$. If $\sum a_j \otimes b_j \in Ker(\theta)$, then : $A.A^{-1}.\sum a_j \otimes b_j \subset \sum A \otimes A^{-1}.a_j.b_j = 0$. From $Q_\sigma(A.A^{-1}) = A * A^{-1} = \Lambda$ it follows that $A.A^{-1} \in \mathcal{L}(\sigma)$ and thus $\sum a_j \otimes b_j \in \sigma(A \otimes B)$. Localizing θ at σ yields :

$$Q_\sigma(A \otimes_\Lambda B) \simeq Q_\sigma(A.B) = A * B$$

and all isomorphisms are twosided.

Remark 3.8 : The relative version of the Picent-group in [23] is defined in the following way : let $Picent(\Lambda, \sigma)$ be the subgroup of $Pic(\Lambda, \sigma)$ consisting of the isomorphism classes of σ-invertible twosided Λ-modules P such that $c.P = P.c$ for all $c \in \mathcal{Z}(\Lambda)$; i.e. such that the canonical α_P of lemma 3.5 is the identity. In other words the following sequence is exact :

$$1 \to Picent(\Lambda, \sigma) \to Pic(\Lambda, \sigma) \to Aut(Z(\Lambda))$$

c. : divisorially graded rings

In this section we will consider rings graded by an arbitrary group G For generalities on graded rings, the reader is referred to [58] or, in particular, to [58'] where rings graded by arbitrary groups are being studied extensively.

A graded ring $\Lambda = \bigoplus_{\tau \in G} \Lambda_\tau$ is said to be divisorially graded if the following two conditions are satisfied :

(dg.1) : Λ_e is a ρ – maximal order , where e is the neutral element of G ;

(dg.2) : For all $\tau \in G$ we have that $Q_\sigma(\Lambda.\Lambda_\tau) = \Lambda$ where σ is the kernel functor in $\Lambda_e - mod$ derived from $\mathcal{P}(\rho)$ as in the foregoing section.

Lemma 3.9 : If Λ is divisorially graded by G , then :

(a) : For every $\tau \in G : Q_\sigma(\Lambda_\tau.\Lambda_{\tau^{-1}}) = \Lambda_e$;

(b) : For every $\tau \in G : Q_\sigma(\Lambda_\tau) = \Lambda_\tau$;

(c) : For every graded left ideal L of Λ we have : $Q_\sigma(L) = Q_\sigma(\Lambda.L_e)$.

Proof :

First note that Λ_τ is σ-torsion free for every $\tau \in G$ since $\Lambda_\tau \subset \Lambda = Q_\sigma(\Lambda.\Lambda_\tau)$.

(a) : If $x \in \Lambda_e$ then it follows from (dg.2) that for every $\tau \in G$, there exists an ideal $I \in \mathcal{L}(\sigma)$ such that $I.x \subset \Lambda.\Lambda_\tau^{-1}$. Hence, $I.x \subset (\Lambda.\Lambda_\tau^{-1})_e = \Lambda_\tau.\Lambda_\tau^{-1}$ and therefore : $Q_\sigma(\Lambda_\tau.\Lambda_\tau^{-1}) = \Lambda_e$.

(b) : The inclusion $\Lambda_\tau \subset \Lambda.\Lambda_\tau$ entails that $Q_\sigma(\Lambda_\tau) \subset Q_\sigma(\Lambda.\Lambda_\tau) = \Lambda$. Because σ is a localization functor in degree e , it follows immediatly from this that $Q_\sigma(\Lambda_\tau) \subset \Lambda_\tau$.

(c) : If $\gamma \in G$ and $x \in \Lambda_\gamma$, then $\Lambda_\gamma.\Lambda_{\gamma^{-1}}.x \subset \Lambda.L_e$ and it follows from part (a) that $x \in Q_\sigma(\Lambda.L_e)$. Consequently, $Q_\sigma(L) \subset Q_\sigma(\Lambda.L_e)$, the converse implication is of course trivial.

Lemma 3.10 : Let Λ be divisorially graded by G. Let $M \in \Lambda - gr$ be such that $Q_\sigma(M) = M$ in Λ_e-mod , then there is a canonical graded isomorphism of degree e in $\Lambda - gr$: $Q_\sigma(\Lambda \otimes_{\Lambda_e} M_e) \simeq M$.

Proof :

Consider the exact sequence :

$$0 \to K \to \Lambda \otimes_{\Lambda_e} M_e \to M$$

where the right morphism is the canonical (multiplication) morphism , which is clearly graded by degree e , hence K is a graded Λ- module. Since $\sigma(M) = 0$, $\sigma(\Lambda \otimes M) \subset K$. Conversely, if $x \in K_\tau$ for some $\tau \in G$, then write $x = \sum \lambda_\tau^{(i)} \otimes m_e^{(i)}$ with $\lambda_\tau^{(i)} \in \Lambda_\tau$ for all i.

$f(x) = 0$ entails that $\Lambda_\tau.\Lambda_\tau^{-1}.x \subset \Lambda_\tau \otimes \Lambda_\tau^{-1}(\sum \lambda_\tau^{(1)}.m_e^{(i)} = 0$ whence $x \in \sigma(\Lambda \otimes M_e)$ because $\Lambda_\tau.\Lambda_\tau^{-1} \in \mathcal{L}(\sigma)$, so $K = \sigma(\Lambda \otimes M_e)$. It follows from the inclusion

$$(\Lambda \otimes M_e)/\sigma(\Lambda \otimes M_e) \to M$$

that $Q_\sigma(\Lambda \otimes M_e) \to Q_\sigma(M) = M$ is monomorphic.

Because $Q_\sigma(\Lambda \otimes M_e)$ is σ-closed and since it contains M , for , if $m \in M_\tau$ then $\Lambda_\tau.\Lambda_\tau^{-1}.m \subset \Lambda_\tau.M_e$ it follows that $Q_\sigma(\Lambda \otimes M_e) \simeq M$ in Λ-mod. Finally , taking into account that M_τ is σ-torsion over $\Lambda_\tau.M_e$, it follows that $Q_\sigma(\Lambda_\tau \otimes M_e) \simeq M_\tau$, therefore , the isomorphism $Q_\sigma(\Lambda \otimes M_e) \simeq M$ is a graded isomorphism of degree e.

Remark 3.11 : Let M be a graded Λ-module such that $Q_\sigma(M) = M$, then it is clear that $Q_\sigma(M(\tau) = M(\tau)$ holds for all $\tau \in G$. Here, $M(\tau)$ is the graded Λ-module such that $M(\tau)_\gamma = M_{\tau.\gamma}$. Moreover , $Q_\sigma(\Lambda \otimes_{\Lambda_e} M(\tau)) \simeq M(\tau)$ in Λ-gr.

Let us now turn to the structure theorem for divisorially graded rings.First, note that for any σ there is a canonical embedding of $Pic(\Lambda)$ (in the sense of

Fröhlich [23]) into $Pic(\Lambda, \sigma)$. This allows us to restrict the constructions we care about to describe , to $Pic(\Lambda)$ if one wants to.

Let Λ be any ρ – maximal order and let G be any group and consider an arbitrary grouphomomorphism :

$$\Phi : G \to Pic(\Lambda, \sigma)$$

We write $\Phi(\tau) = [P_\tau]$. A factorset f associated to Φ is a set $f = f_{\gamma,\tau}; \gamma, \tau \in G$ of twosided Λ-module isomorphisms :

$$f_{\gamma,\tau} : Q_\sigma(P_\gamma \otimes_\Lambda P_\tau) \to P_{\gamma.\tau}$$

$$\alpha : P_e \to \Lambda$$

such that the following diagrams commute for all $\gamma, \tau, \theta \in G$:

$$\begin{array}{ccc} Q_\sigma(P_\gamma \otimes P_\tau \otimes P_\theta) & \to & Q_\sigma(P_\gamma \otimes P_{\tau.\theta}) \\ f_{\gamma,\tau} \otimes 1_\theta \downarrow & & \downarrow f_{\gamma,\tau.\theta} \\ Q_\sigma(P_{\gamma.\tau} \otimes P_\theta) & \to & P_{\gamma.\tau.\theta} \end{array}$$

$$\begin{array}{ccc} P_\gamma \otimes P_e & \to & P_\gamma \otimes \Lambda \\ & \searrow & \cong \\ & & P_\gamma \end{array}$$

$$\begin{array}{ccc} P_e \otimes P_\gamma & \to & \Lambda \otimes P_\gamma \\ & \searrow & \cong \\ & & P_\gamma \end{array}$$

In writing down these diagrams we implicitly used the fact that for any $P \in Pic(\Lambda, \sigma)$ we have :

$Q_\sigma(P \otimes_\Lambda M) \simeq Q_\sigma(P \otimes_\Lambda Q_\sigma(M))$ for any $M \in \Lambda$-mod;

$Q_\sigma(\dot{N} \otimes_\Lambda \ddot{P}) \simeq \ddot{Q}_\sigma(\dot{Q}_\sigma(\dot{N}) \ddot{\otimes}_\Lambda \ddot{P})$ for any twosided Λ-module $\dot{N}$.

Extending the notation introduced in [88] to this case , we write $F_s(\Phi)$ for the set of factorsets associated to Φ. We make the additive group $\bigoplus P_\tau$ into a ring by defining the multiplication rule as follows : for $x \in P_\gamma$, $y \in P_\tau$ define $x.y =$

$f_{\gamma,\tau}(x \otimes y)$. The ring defined in this manner will be denoted by $\Lambda < f, \theta, G >$. With these notations we have :

Theorem 3.12 : The ring $\Lambda < f, \theta, G >$ is a divisorially graded ring containing a subring isomorphic to Λ. Conversely, if Λ is divisorially graded by G, then there exists a groupmorphism $\Phi : G \to Pic(\Lambda_e, \sigma)$ and a factorset $f \in F_s(\Phi)$ such that $\Lambda \simeq \Lambda_e < f, \Phi, G >$.

Proof :

Because P_e is a twosided Λ-module isomorphic to Λ it follows that $P_e \otimes_\Lambda P_e \simeq \Lambda$ whence $P_e \otimes_\Lambda P_e \simeq Q_\sigma(P_e \otimes_\Lambda P_e)$. It is clear that $\Lambda < f, \theta, G >$ is a ring and that P_e is a subring since $f_{e,e} : P_e \otimes_\Lambda P_e \to P_e$ is an isomorphism. Again, it follows from $P_e \simeq \Lambda$ as twosided Λ-modules that $P_e = \Lambda.p = p.\Lambda$ for some $p \in P$ such that $\lambda.p = p.\lambda$ for all $\lambda \in \Lambda$. Consequently, $P_e \otimes_\Lambda P_e = \Lambda(p \otimes p) = (p \otimes p).\Lambda$. Since $f_{e,e}$ is a twosided Λ-module isomorphism we have that $f_{e,e}(p \otimes p) = \mu.p$ forome $\mu \in \Lambda$. Bilinearity of $f_{e,e}$ then entails that $\mu \in Z(\Lambda)$ and even that $\mu \in U(Z(\Lambda))$,the group of units of $Z(\Lambda)$. Put $p_1 = \mu^{-1}.p$, then $f_{e,e}(p_1 \otimes p_1) = p_1$ and $\Lambda \to P_e$ defined by sending λ to $\lambda.p_1$ is a ringisomorphism. It remains to show that p_1 is the identity element for $\Lambda < f, \theta, G >$. Since $P_e \simeq \Lambda$, i.e. $P_e \otimes_\Lambda P_\gamma \simeq P_\gamma \simeq Q_\sigma(P_\gamma)$ for all $\gamma \in G$ we have that $P_e \otimes_\Lambda P_\gamma \simeq Q_\sigma(P_e \otimes_\Lambda P_\gamma)$ for all $\gamma \in G$. Therefore , if $x \in P_\gamma$ then there exists an element $y \in P_\gamma$ such that $x = f_{e,\gamma}(p_1 \otimes y)$. Then we calculate :

$f_{e,\gamma}(p_1 \otimes x) = f_{e,\gamma}(p_1 \otimes f_{e,\gamma}(p_1 \otimes y)) = f_{e,\gamma}(f_{e,e}(p_1 \otimes p_1) \otimes y) = f_{e,\gamma}(p_1 \otimes y) = x$ and this establishes that $p_1.x = x$ for all $x \in \Lambda < f, \theta, G >$ and similarly one can show that $x.p_1 = x$.

Conversely, let Λ be divisorially graded by G, then by lemma c.3 , the canonical Λ_e-bimodule morphism $\Lambda_\gamma \otimes_\Lambda \Lambda_{\gamma^{-1}} \to \Lambda_e$ extends to an isomorphism $\tilde{Q}_\sigma(\Lambda_\gamma \tilde{\otimes}_\Lambda \Lambda_{\gamma^{-1}}) \to \Lambda_e$. Similarly, $\tilde{Q}_\sigma(\Lambda_{\gamma^{-1}} \tilde{\otimes}_\Lambda \Lambda_\gamma) \simeq \Lambda_e$ as twosided Λ_e-modules. To conclude from this that Λ_γ is σ-invertible we have to verify that $\Lambda_\gamma \otimes_\Lambda T$ is σ-torsion if $T \in \Lambda_e$-mod is σ-torsion.

Take $x = \sum \lambda_i \otimes t_i \in \Lambda_\gamma \otimes_\Lambda T$ and let $I \in \mathcal{L}(\sigma)$ be such that $I.t_i = 0$

for every i. Because $\Lambda_{\gamma^{-1}}.\Lambda_\gamma \subset \Lambda_e$ we have that $\Lambda_\gamma.I.\Lambda_{\gamma^{-1}}.\Lambda_\gamma \subset \Lambda_\gamma.I$ whence $I'.x \subset \sum \Lambda_\gamma \otimes I.t_i = 0$ if $I' = \Lambda_\gamma.I.\Lambda_{\gamma^{-1}} \in \mathcal{L}(\sigma)$. Therefore, $\Lambda_\gamma \otimes T$ is σ-torsion.

Hence, $\Lambda_\gamma \in Pic(\Lambda_e, \sigma)$ for every $\gamma \in G$. Because $Q_\sigma(\Lambda_\tau \otimes \Lambda_\gamma) \simeq \Lambda_{\tau.\gamma}$ as twosided Λ_e-modules, there is a groupmorphism $\Phi : G \to Pic(\Lambda_e, \sigma)$ given by $\Phi(\gamma) = [\Lambda_\gamma]$ forall $\gamma \in G$. Extending the multiplication map $f_{\gamma,\tau} : \Lambda_\gamma \otimes \Lambda_\tau \to \Lambda_{\gamma.\tau}$ to a twosided Λ_e- module isomorphism $f_{\gamma,\tau} : Q_\sigma(\Lambda_\gamma \otimes \Lambda_\tau) \to \Lambda_{\gamma.\tau} = Q_\sigma(\Lambda_{\gamma.\tau})$ we obtain a factorset f associated to Φ and it is clear that $\Lambda \simeq \Lambda_e < f, \Phi, G >$, finishing the proof.

Remark 3.13 :

(1) : For any $\tau \in G$, take P_τ' such that $[P_\tau] = [P_\tau']$ in $Pic(\Lambda_e, \sigma)$ and let $\alpha_\tau : P_\tau \to P_\tau$' be a twosided Λ_e- module isomorphism. Then one can easily check that up to replacing $f_{\gamma,\tau}$ by $f_{\gamma,\tau}$' which is defined by the commutative diagram :

$$\begin{array}{ccc} P_\gamma \otimes P_\tau & \to & P_{\gamma.\tau} \\ \downarrow & & \downarrow \\ P'_\gamma \otimes P'_\tau & \to & P'_{\gamma.\tau} \end{array}$$

then $P_e < f, \Phi, G >$ and $P_e' < f', \Phi, G >$ are graded isomorphic.

(2) : The composition morphism :

$$G \to Pic(\Lambda_e, \sigma) \to Aut(Z(\Lambda_e))$$

allows us to give a cohomological classification of all divisorially G-graded rings over Λ_e in the sense that graded isomorphism classes of divisorially G-graded rings Λ with Λ_e as its part of degree e , correspond bijectively to $H^2(G, U(Z(\Lambda_e)))$.

Proposition 3.14 : Let Λ be a $\rho-$maximal order and let Γ be divisorially graded by G such that $\Gamma_e = \Lambda$. If $Pic(\Lambda, \sigma) = [\Lambda]$, then $\Gamma \simeq \Lambda[X_\sigma, \phi, c]$ is a crossed product of Λ and G described by a groupmorphism $\phi : G \to Aut(\Lambda)$ and a 2-cocycle $c : G \times G \to U(Z(\Lambda))$.

Proof :

It follows from Th.3.12 that $\Gamma = \bigoplus P_\tau$ where $P_\tau \in Pic(\Lambda, \sigma)$. By the assumptions we have that $P_\tau \simeq \Lambda$ as twosided Λ-modules , hence by lemma b.5 we know that $P_\tau \simeq_1 \Lambda_{\phi_\tau}$ for some $\phi_\tau \in Aut(\Lambda)$. Put $X_\tau = \theta_\tau(1)$ where θ_τ denotes the bimodule isomorphism :

$$\theta_\tau :_1 \Lambda_{\phi_\tau} \to P_\tau$$

Then $P_\tau = P_e.X_\tau = X_\tau.P_e$ follows and from $X_\tau.P_e.X_{\tau^{-1}} = P_e, X_{\tau^{-1}}.P_e.X_\tau = P_e$ it follows that there exist elements $\lambda, \mu \in P_e = \Lambda$ such that $1 = X_\tau.\lambda.X_{\tau^{-1}}$ and $1 = X_{\tau^{-1}}.\mu.X_\tau$, i.e. each X_τ is invertible in Γ. Because $X_\tau.X_\gamma.X_{\tau.\gamma}^{-1}$ commutes with $P_e = \Lambda$ is follows that $X_\tau.X_\gamma = c_{\tau,\gamma}.X_{\tau.\gamma}$ for some unit $c_{\tau,\gamma}$ in $Z(\Lambda)$. Obviously , $\phi : G \to Aut(\Lambda), \tau \to \phi_\tau$ and $c : G \times G \to U(Z(\Lambda)), c(\tau,\gamma) = c_{\tau,\gamma}$ define a crossed product structure such that $\Gamma \simeq \Lambda[X_\tau, \phi, c]$.

Lemma 3.15 : Let Λ be a $\rho-$maximal order such that both σ and ρ are G-invariant central localizations then $\mathcal{L}(\rho)$ is invariant under α_P , the automorphism of $Z(\Lambda)$ induced by $P \in Pic(\Lambda, \sigma)$, for every $P \in Pic(\Lambda, \sigma)$.

Proof :

Put $G = Pic(\Lambda, \sigma)$ and let Γ be the divisorially graded ring $\Lambda < f, \theta, G >$ where θ is the identity map on $Pic(\Lambda, \sigma)$ and f is the trivial factorset associated to θ. This allows us to prove the lemma in the terminology of G-graded rings which simplifies matters a lot. Take $J \in \mathcal{L}(\rho)$ then for every $\gamma \in G$, $J' = \Gamma_\tau.J.\Gamma_{\tau^{-1}} \in \mathcal{L}(\rho)$.

Since ρ is a central kernel functor we may assume that $J = \Lambda.J_c$ where $J_c = J \cap Z(\Lambda)$. By definition of $\alpha : Pic(\Lambda, \sigma) \to Aut(Z(\Lambda))$ we have that $J.\Gamma_\gamma = \Gamma_\gamma.\alpha_\gamma(J)$. Consequently, we obtain : $\Gamma_\gamma.J.\Gamma_{\gamma^{-1}} = \Gamma_\gamma.\Gamma_{\gamma^{-1}}.\alpha_{\gamma^{-1}}(J) \in \mathcal{L}(\rho)$. Since $\Gamma_\gamma.\Gamma_{\gamma^{-1}} \in \mathcal{L}(\sigma)$, $\alpha_{\gamma^{-1}}(J) \in \mathcal{L}(\rho)$ follows for every $\gamma \in G$.

Lemma 3.16 : Let Λ be a ρ – maximal order such that σ and ρ are as before and moreover ρ is a perfect localization , then $Q_\rho(P) \in Pic(Q_\rho(\Lambda))$ for every $P \in Pic(\Lambda, \sigma)$.

Proof :

Because $Q^l_\rho(\Lambda) = Q^r_\rho(\Lambda)$ and $Q^l_\rho(P) \simeq Q^l_\rho(\Lambda) \otimes_\Lambda P$, $Q^r_\rho(P) \simeq P \otimes_\Lambda Q^r_\rho(\Lambda)$ it will be sufficient to establish that $P \otimes_\Lambda Q_\rho(\Lambda) \simeq Q_\rho(\Lambda) \otimes_\Lambda P$. Note that ρ^l-torsion and ρ^r torsion of a twosided Λ- module is the same because $p.I = 0$ for some $I \in \mathcal{L}(\rho)$ is equivalent to $\alpha_P(I).p = 0$ and we know from the foregoing lemma that $\alpha_P(I) \in \mathcal{L}(\rho)$. Now, if $x \in Q^r_\rho(P)$ then we can write $x = \sum p_i \otimes q_i$ where $p_i \in P$, $q_i \in Q^r_\rho(\Lambda)$. There exists an ideal $J \in \mathcal{L}(\rho)$ such that $J.q_i \subset \Lambda$ for all q_i. Let Q be a representant of $[P]^{-1}$ in $Pic(\Lambda, \sigma)$ then $\alpha_Q(J).x \subset p_i.J \otimes q_i \subset \sum p_i \otimes J.q_i \subset P$ where we use the canonical identifications $\Lambda \otimes_\Lambda P = P = P \otimes_\Lambda \Lambda$. Consequently , $x \in Q_\rho(\Lambda) \otimes_\Lambda P \simeq Q^l_\rho(P)$.

Conversely, if $y \in Q^l_\rho(P)$ then $y \in P \otimes_\Lambda Q_\rho(\Lambda)$ (with identifications as before) follows by means of a similar argument. Therefore, from $Q_\sigma(P \otimes_\Lambda Q) \simeq Q_\sigma(Q \otimes_\Lambda P) \simeq \Lambda$ it follows that $Q_\rho(P \otimes_\Lambda Q) \simeq Q_\rho(\Lambda) \otimes_\Lambda P \otimes_\Lambda Q = Q_\rho(P) \otimes_\Lambda Q_\rho(Q) = Q_\rho \otimes_{Q_\rho(\Lambda)} Q_\rho(Q)$ where we use the fact that $Q_\rho(\Lambda) \otimes_\Lambda Q_\rho(\Lambda) \simeq Q_\rho(\Lambda)$ because $\Lambda \to Q_\rho(\Lambda)$ is an epimorphism in the category of rings. Furthermore, the fact that $Q_\rho(P) \otimes_\Lambda Q_\rho(Q)$ is ρ- torsion free entails : $Q_\rho(P) \otimes_\Lambda Q_\rho(Q) \simeq Q_\rho(P) \otimes_{Q_\rho(\Lambda)} Q_\rho(Q)$. Therefore, we obtain at last : $Q_\rho(\Lambda) \simeq Q_\rho(P \otimes_\Lambda Q) \simeq Q_\rho \otimes_{Q_\rho(\Lambda)} Q_\rho(Q)$ and similarly : $Q_\rho(\Lambda) \simeq Q_\rho(Q \otimes_\Lambda P) \simeq Q_\rho(Q) \otimes_{Q_\rho(\Lambda)} Q_\rho(P)$. Thus, $Q_\rho(P) \in Pic(Q_\rho(\Lambda)$.

Proposition 3.17 : Let Λ be a maximal order over a Krull domain R , let G be a group and $\Phi : G \to Pic(\Lambda, \sigma)$ a groupmorphism and f a factorset associated to Φ. Consider the divisorially graded ring $\Gamma = \Lambda < \Phi, f, G >$. Then $Q^g(\Gamma)$ is a crossed product algebra of the form $\Sigma[X_\tau, \phi, c]$ where Σ is the classical ring of quotients of Λ , $\phi : G \to Aut(\Sigma)$ is a grouphomomorphism and $c : G \times G \to U(K)$ is a 2-cocycle.

Proof :

In the case that Λ is a maximal order over a Krull domain R, $\mathcal{L}(\rho)$ is the filter of all twoisided ideals of Λ and by Posner's theorem : $Q_\rho(\Lambda) = \Sigma$. Because Σ is a central simple K-algebra , $Pic(\Sigma) = [\Sigma]$. Applying lemma 3.15 and the prop 3.14 the result follows.

Remark 3.18 :

(1) : We leave to the reader the verification of the fact that in all examples given in section b. the imposed conditions on σ and ρ are indeed satisfied.

(2) : Note that $\Gamma \subset Q_\rho(\Gamma)$. Indeed , if $J.x = 0$ for some $x \in \Gamma_\tau$ and $J \in \mathcal{L}(\rho)$ then $J.\Gamma.x.\Gamma = 0$. Hence, $J.(\Gamma.x.\Gamma)_e = 0$ yielding that $(\Gamma.x.\Gamma)_e = 0$ because Λ is a prime ring and therefore $Q_\sigma(\Gamma.x.\Gamma) = 0$ which is impossible since Γ_τ is σ-torsion free. Thereforee we may write $\Gamma = \sum I_\tau.X_\tau \subset Q_\rho[X_\tau, \phi, c]$ where for each $\tau \in G$, I_τ is a twosided Λ- submodule of $Q_\rho(\Lambda)$.

(3) : We may replace $Pic(\Lambda, \sigma)$ by $Picent(\Lambda, \sigma)$. In that case, X_τ commutes with the center of $Q_\rho(\Lambda)$ hence ϕ_τ is inner , say $\phi_\tau(\lambda) = a_\tau.\lambda.a_\tau^{-1}$ for some $a_\tau \in Q_\rho(\Lambda)$. Change of variables : $X_\tau \to a_\tau^{-1}$ and correspondingly changing $\phi \to 1_{Q_\rho(\Lambda)}$ and $c \to c$' with c' equivalent to c , we see that in the case of $Picent(\Lambda, \sigma)$ we may assume that X_τ commutes with $Q_\rho(\Lambda)$ and then $\Gamma = \sum I_\tau.X_\tau$ where $\tau \to I_\tau$ defines a grouphomomorphism $G \to D_\rho(\Lambda)$. It is this situation that we will generalize in the following definition :

Definition 3.19 : Let Λ be a ρ − maximal order and G an arbitrary group. Let there be given a grouphomomorphism $\phi : G \to Aut(\Lambda)$ such that $\mathcal{L}(\rho)$ is invariant under each $\phi(\tau)$ and a 2-cocycle $c : G \times G \to U(Z(\Lambda))$. We construct the crossed product $Q_\rho(\Lambda)[X_\tau, \phi, c]$. Now, consider a map $\Phi : G \to D_\rho(\Lambda)$ and look at the Abelian group :

$$\sum \Phi(\tau).X_\tau \subset Q_\rho(\Lambda)[X_\tau, \phi, c]$$

The condition that this set will be a subring of the crossed product can be expressed by : $\Phi(\tau).\phi_\tau(\Phi(\gamma)) \subset \Phi(\tau.\gamma)$. A further condition, assuring that the set will be a divisorially G-graded ring is that : $Q_\sigma(\Phi(\tau).\phi_\tau(\Phi(\gamma))) = \Phi(\tau.\gamma)$.

If these conditions are satisfied then we write $\Lambda(\Phi, \phi, c) = \sum \Phi(\tau).X_\tau$ with ring structure induced from the crossed product $Q_\rho(\Lambda)[X_\tau, \phi, c]$, and we call $\Lambda(\Phi, \phi, c)$ the generalized Rees ring associated to Φ, ϕ, c over Λ.

Proposition 3.20 : In the situation of Prop. 3.17 where we assume moreover that Γ is a p.i.-ring , then : $\Gamma = \sum I_\tau.X_\tau \subset \Sigma[X_\tau, \phi, c]$ with $I_\tau \in \mathcal{D}(\Lambda)$ for each $\tau \in G$.

Proof :

First repeat remark 3.18.(2). Then consider $\tau \in G$ and look at the subring generated by X_τ and $X_{\tau^{-1}}$ over Σ in $\Sigma[X_\tau, \phi, c]$. We claim that there exist a natural number $n \in \mathbb{N}$ such that ϕ_τ^n is inner in Σ , say $\phi_\tau^n(x) = a_\tau.x.a_\tau^{-1}$ for some $a_\tau \in \Sigma$.

If ϕ_τ is in the torsion part of $Aut(\Sigma)$ then our claim is true. If ϕ_τ is not a torsion element then τ is not a torsion element of G since ϕ is a homomorphism. Let $c(\tau)$ be the restriction of $c : G \times G \to U(Z(\Lambda))$ to $< \tau > \times < \tau > \to U(Z(\Lambda))$. Since $< \tau > \simeq \mathbb{Z}$ it follows that $c(\tau) \sim 1$. Therefore, the ring generated by X_τ and $X_{\tau^{-1}}$ over Σ is isomorphic to $\Sigma[X_\tau, X_\tau^{-1}, \phi_\tau]$. The latter is by assumption a p.i.-ring and therefore (cfr. [d5]) ϕ_τ has finite order in $Aut(\Sigma)/Inn(\Sigma)$, finishing the proof of our claim.

Define for each $\gamma, \tau \in G$:

$$I_\gamma^{(\tau)} = I_\gamma \cap \phi_\tau(I_\gamma) \cap \dots \cap \phi_\tau^{n(\tau)} \cap K$$

It is obvious from the definition that $I_\gamma^{(\tau)}$ is ψ_τ-invariant.

Therefore, $I_\gamma.(I_{\gamma^{-1}}^{(\gamma)} \cap \Lambda) \subset \Lambda$. Indeed , from $I_\gamma.X_\gamma.I_{\gamma^{-1}}.X_{\gamma^{-1}} = I_\gamma.\phi_\gamma(I_{\gamma^{-1}}.c_{\gamma,\gamma^{-1}}.X_e$, it follows that $I_\gamma.\phi_\gamma(I_{\gamma^{-1}}) \subset \Lambda = \Gamma_e$. However, $I_{\gamma^{-1}}^{(\gamma)} \cap \Lambda \subset Z(\Lambda)$, i.e. $I_{\gamma^{-1}}^{(\gamma)} \cap \Lambda$ commutes with I_γ and thus we obtain : $J.I_\gamma = I_\gamma.J \subset \Lambda$ with $J = \Lambda.(I_{\gamma^{-1}}^{(\gamma)} \cap \Lambda)$.

Note that $\dot{\phi}_\gamma^{\bar{r}}(\dot{I}_{\gamma^{-1}}) \cap \dot{\Lambda} \neq \ddot{0}$ and hence $\dot{J}$ is a nonzero ideal of $\dot{\Lambda}$, so it follows from the fact that $J.I_\gamma$ is a nonzero ideal of Λ that $I_\gamma \in \mathcal{F}(\Lambda)$. If $Q_\sigma(I_\gamma) \supset I_\gamma$, say $y \in Q_\sigma(I_\gamma) - I_\gamma$ and $y \in Q_\rho(\Lambda)$, then $y \in Q_\sigma(\Lambda(\Phi))$ in $Q_\rho(\Lambda)[X_\tau, \phi, c] = Q_\rho(\Lambda(\Phi))$. Since $Q_\sigma(\Lambda(\Phi)) = \Lambda(\Phi)$ it then follows that $Q_\sigma(I_\gamma) = I_\gamma$ and thus $I_\gamma \in \mathcal{D}_\rho(\Lambda)$.

Proposition 3.21 : If $\Gamma = \Lambda < \Phi, f, G >$ as in Prop.3.17 is a p.i.-ring, then Γ is a generalized Rees ring $\Lambda(\Phi, \phi, c)$ if G is mapped into $\mathcal{D}(\Lambda)$ bu Φ.

Proof :

Follows directly from Prop.3.17 , Prop.3.20 and def.3.19 . In this case Φ is even a grouphomomorphism and this allows us to use the following

Proposition 3.22 : If $\Lambda(\Phi, \phi, c)$ is a generalized Rees ring then Φ is a groupmorphism if and only if $\phi_\tau(I_\gamma) \subset I_\gamma$ for all $\tau, \gamma \in G$.

Proof :

From $Q_\sigma(I_\tau.X_\tau.I_\gamma.X_\gamma) = I_{\tau.\gamma}.X_{\tau.\gamma}$ it follows that :

$$Q_\sigma(I_\tau.\phi_\tau(I_\gamma).c_{\tau,\gamma}.X_{\tau.\gamma}) = Q_\sigma(I_\tau.\phi_\tau(I_\gamma)).X_{\tau.\gamma}$$

If Φ is a groupmorphism then the foregoing reduces to :

$$\phi_\tau(I_\gamma) = I_\tau^{-1} * I_{\tau.\gamma} = I_{\tau^{-1}} * I_{\tau.\gamma} = I_\gamma$$

Conversely, if $\phi_\tau(I_\gamma) \subset I_\gamma$ then we find : $Q_\sigma(I_\tau.I_\gamma) = I_{\tau.\gamma}$ i.e. $I_\tau * I_\gamma = I_{\tau.\gamma}$, proving that Φ is a group homomorphism.

d. : normalizing Rees rings

In this section we investigate a special type of generalized Rees rings, the so called normalizing Rees rings. Let us fix notations and hypotheses for this section as follows.

Let G be a torsion free Abelian group. Let $\Phi : G \to \mathcal{D}_\rho(\Lambda)$ and $\phi : G \to Aut(\Lambda)$ be group homomorphisms , where Λ is a $\rho-$maximal order, and let $c : G \times G \to U(Z(\Lambda))$ be a 2-cocycle. Now we assume that for every $\tau \in G$,$\phi(\tau) = \phi_\tau$ is defined by $\phi_\tau(x) = a_\tau.x.a_\tau^{-1}$ for all $x \in \Lambda$ where $a_\tau \in N_\rho(\Lambda) = \{x \in Q_\rho(\Lambda) : \Lambda.x = x.\Lambda \in \mathcal{F}_\rho(\Lambda)\}$.

Under these conditions the generalized Rees ring $\Lambda(\Phi, \phi, c)$ is said to be a normalizing Rees ring. If no ambiguity is possible then we write this ring as $\Lambda(\Phi)$ and $Q_\rho(\Lambda(\Phi)) = Q_\rho(\Lambda)[X_\tau, \phi, c]$. It follows from proposition 3.22 that for a normalizing Rees ring we always have that $I_\tau = \phi(\tau)$ is ϕ_γ-invariant for every $\tau, \gamma \in G$.

In this section we aim to investigate whether $\Lambda(\Phi)$ is a relative maximal order (with eventually the ascending chain condition on divisorial ideals contained in $\Lambda(\Phi)$) with respect to some suitable $\rho(\Phi)$ whenever Λ is a $\rho-$maximal order (with eventually the ascending chain condition on divisorial ideals contained in Λ).

Throughout, we will assume that $\Lambda(\Phi)$ is a p.i.-ring , this may be documented by giving necessary and sufficient conditions for this to happen but we only hint at the problem in the special case of normalizing Rees rings, here.

Recall that a torsion free Abelian group may be ordered. By this we mean that there exists a subset S of G (called the set of positive elements) such that $e \notin S$, S is multiplicatively closed and for every $\alpha \in G$ either $\alpha = e$, $\alpha \in S$ or $\alpha^{-1} \in S$. The linear ordering of G is then defined by $\alpha < \beta$ if and only if $\alpha^{-1}.\beta \in S$. If I is an ideal of $\Lambda(\Phi)$ and $\gamma \in G$; then we define :

$$C_\gamma(I) = \{x \in Q_\rho(\Lambda) : x^\sim = r.X_\gamma \text{ for some } i \in I\}$$

where for every element $x \in \Lambda(\Phi)$, $x^\sim$ is the homogeneous component of highest degree in the homogeneous decomposition of x. Since we have assumed that $\phi(\tau) \in Aut(\Lambda)$ for all $\tau \in G$, it follows that $C_\gamma(I)$ is a twosided Λ-submodule of $Q_\rho(\Lambda)$.

Lemma 3.23 : If I is an ideal of $\Lambda(\Phi)$, then the following statements are equivalent :

(1) : For all $\tau \in G$, $C_\tau(I) \in \mathcal{F}_\rho(\Lambda)$;

(2) : $C_e(I) \in \mathcal{L}(\rho)$.

Proof :

(1) $\Rightarrow$ (2) is obvious .

(2) $\Rightarrow$ (1) : for each $\gamma \in G$ we have that : $C_e(I).X_e.I_\gamma.X_\gamma = C_e(I).\phi_e(I_\gamma).c_{e,\gamma}.X_\gamma$. Since ϕ_e leaves elements of $\mathcal{D}_\rho(\Lambda)$ invariant and since $c_{e,\gamma} \in U(Z(\Lambda))$, we deduce from the foregoing that : $C_e(I).I_\gamma \subset C_\gamma(I) \subset I_\gamma$. Since both extremes are in $\mathcal{F}_\rho(\Lambda)$, so is $C_\gamma(I)$.

Corollary 3.24 : $\mathcal{L}(\rho(\Phi)) = \{$ ideals I of $\Lambda(\Phi)$ such that for all $\gamma \in G$, $C_\gamma(I) \in \mathcal{F}_\rho(\Lambda)\}$ is a multiplicatively closed filter.

Lemma 3.25 : For every ideal I of $\Lambda(\Phi)$ such that $C_e(I) \in \mathcal{L}(\rho)$: we have : $Q_\sigma(C_\gamma(I)) = Q_\sigma(C_e(I).I_\gamma)$.

Proof :

From $I_{\gamma^{-1}}.I_\gamma.C_e(I) \subset I_{\gamma^{-1}}.C_\gamma(I) \subset C_e(I)$ it follows that : $Q_\sigma(I_{\gamma^{-1}}.I_\gamma.C_e(I)) = Q_\sigma(C_e(I)) \subset Q_\sigma(I_{\gamma^{-1}}.C_\gamma(I)) \subset Q_\sigma(C_e(I))$. Therefore, $Q_\sigma(C_e(I)) = I_{\gamma^{-1}} * Q_\sigma(C_\gamma(I))$. Note that we have used the fact that the I_γ are ϕ_τ-invariant in the foregoing argument. Now, commutativety of $\mathcal{D}_\rho(\Lambda)$ entails finally that : $Q_\sigma(C_e(I)) * I_\gamma = Q_\sigma(C_\gamma(I))$, i.e. , $Q_\sigma(C_e(I).I_\gamma) = Q_\sigma(C_\gamma(I))$.

We will now recall the following lemma :

Lemma d.4 : If Λ is a graded ring of type G satisfying the identities of n by n matrices such that its center $Z(\Lambda)$ is a graded field (i.e. every homogeneous element is invertible), then Λ is an Azumaya algebra.

Proof :

The multilinear Razmyslov polynomial cannot vanish for every homogeneous substitution for the variables (since otherwise it would vanish on Λ). The hypothesis on the center of Λ then entails that the Formanek center of Λ equals $Z(\Lambda)$ and therefore Λ is an Azumaya algebra.

Theorem 3.27 : Let Λ be a ρ – maximal order and suppose that $\Lambda(\Phi)$ satisfies a polynomial identity , then $\Lambda(\Phi)$ is a $\rho(\Phi)$-maximal order , where $\rho(\Phi)$ is defined as in Corollary 3.24 .

Proof :

Because $\Lambda(\Phi)$ is a graded p.i.-ring , its graded ring of quotients, $Q^g(\Lambda(\Phi))$ is obtained by inverting central homogeneous elements and it is a G-graded simple Artinian ring in the sense of [58],[58']. The graded version of Weddenburn's theorem (cfr. [58] Th.1.5.8) yields that $Q^g(\Lambda(\Phi)) \simeq M_n(\Delta)(\Xi)$ for some G-graded skewfield Δ and $\Xi \in G^n$ defining the gradation by

$$((M_n(\Delta)(\Xi))_\gamma)_{i,j} = \Delta_{\tau_i . \gamma . \tau_j^{-1}}$$

where $\Xi = (\tau_1, ..., \tau_n)$ From lemma 4.2 of [58] it follows that $Z(Q^g(\Lambda(\Phi))) = Z(\Delta)$ is a graded field graded by the group $H = \{\gamma \in G : \Delta_\gamma \cap Z(\Delta) \neq 0\}$. The foregoing lemma then entails that $Q^g(\Lambda(\Phi))$ is an Azumaya algebra over $Z(\Delta)$.Because H is again a torsion free Abelian group , it follows from Prop.3.2 of [2] that $Z(\Delta)$ is a completely integrally closed domain yielding that $Q^g(\Lambda(\Phi))$ is a maximal order.

Now, consider $J \in \mathcal{L}(\rho(\Phi))$ and suppose that $J.q \subset J$ for some $q \in Q(\Lambda(\Phi))$. Then, $Q^g(\Lambda(\Phi)).J.q \subset Q^g(\Lambda(\Phi)).J$ and $Q^g(\Lambda(\Phi))$ is a twosided ideal og $Q^g(\Lambda(\Phi))$ yielding that $q \in Q^g(\Lambda(\Phi))$. Hence we can find a decomposition of q into homogeneous components say $q = q_{\tau_1}.X_{\tau_1} + ... + q_{\tau_n}.X_{\tau_n}$ with $\tau_1 < ... < \tau_n$. The relation $J.q \subset J$ then yields :

$$(*) : C_e(J).\phi_e(q_{\tau_n}) \subset C_{\tau_n}(J)$$

By the definition of $\mathcal{L}(\rho(\Phi))$ it follows that $C_\gamma(J) \in \mathcal{F}_\rho(\Lambda)$ for all $\gamma \in G$, so we may deduce from $(*)$ above that : $Q_\sigma(C_e(J)).\phi_e(q_{\tau_n}) \subset Q_\sigma(C_{\tau_n})$, or , equivalently , $\phi_e(q_{\tau_n}) \in Q_\sigma(C_e(J))^{-1} * Q_\sigma(C_{\tau_n}(J))$.

By Lemma 3.25 this means that $\phi_e(q_{\tau_n}) \in I_{\tau_n}$ and by proposition 3.22 $q_{\tau_n} \in I_{\tau_n}$. Now, replacing q by $q - q_{\tau_n}.X_{\tau_n}$ and repeating the foregoing argumentation one finally arrives at $q \in \Lambda(\Phi)$ and therefore $(J :_r J) = \Lambda(\Phi)$. The equality $(J :_l J) = \Lambda(\Phi)$ can be established in a formally similar way.

Corollary 3.28 : If Λ is a maximal order and if $\Lambda(\Phi)$ is a normalizing Rees ring satisfying a polynomial identity , then $\Lambda(\Phi)$ is again a maximal order.

Proof :

It is clear that Λ is a maximal order in Σ if and only if Λ is a ρ−maximal order with respect to $\mathcal{L}(\rho)$ the filter of all nonzero ideals of Λ. Clearly , $\mathcal{L}(\rho(\Phi))$ is in this case the filter of all nonzero ideals of $\Lambda(\Phi)$ and this finishes the proof.

Theorem 3.29 : Let Λ be a $\rho-$ maximal order such that Λ satisfies the ascending chain condition on divisorial ideals contained in Λ and let $\Lambda(\Phi)$ be a normalizing Rees ring satisfying a polynomial identity. If G satisfies the ascending chain condition on cyclic subgroups then $\Lambda(\Phi)$ is a $\rho(\Phi)$-maximal order satisfying the ascending chain condition on divisorial ideals contained in $\Lambda(\Phi)$.

Proof : (adapting notation as in the proof of theorem 4.5)

$Q^g(\Lambda(\Phi))$ is an Azumaya algebra over its center $Z(\Delta)$ which is a graded field graded by a subgroup H of G. Since H also satisfies the ascending chain condition on cyclic subgroups we know that $Z(\Lambda)$ is a factorial domain (by corollary 3.4 of

[2]). Since $Q^g(\Lambda(\Phi))$ is an Azumaya algebra , divisorial ideals correspond bijectively to divisorial ideals of the center and consequently every divisorial ideal of $Q^g(\Lambda(\Phi))$ is generated by a central element.

Now, if $\{A_n; n \in \mathbb{N}\}$ is an ascending chain of divisorial $\Lambda(\Phi)$-ideals contained in $\Lambda(\Phi)$ then the ascending chain $\{Q_\sigma(Q^g(\Lambda(\Phi)).A_n); n \in \mathbb{N}\}$ becomes stationary in $Q^g(\Lambda(\Phi))$, i.e. there exists a natural number $n' \in \mathbb{N}$ such that : $Q_\sigma(Q^g(\Lambda(\Phi)).A_{n'}) = Q_\sigma(Q^g(\Lambda(\Phi)).A_m)$ for every $m \geq n'$. On the other hand, the fact that Λ satisfies the ascending chain condition on divisorial Λ- ideals contained in Λ entails that there exists a natural number $n'' \in \mathbb{N}$ such that : $Q_\sigma(C_e(A_{n''})) = Q_\sigma(C_e(A_m))$ Let N be $sup(n', n'')$ and take $k \geq N$. Since $Q_\sigma(-)$ is a localization in degree e , $Q_\sigma(Q^g(\Lambda(\Phi)).-) = Q_\sigma(Q^g(\Lambda(\Phi)).-)$. Therefore, if $q.A_N \subset A_k$ then $q.A_N.Q^g(\Lambda(\Phi)) \subset A_k.Q^g(\Lambda(\Phi)) = A_N.Q^g(\Lambda(\Phi))$ follows. But $A_N.Q^g(\Lambda(\Phi)) = Q^g(\Lambda(\Phi)).c$ for some $c \in Z(\Delta)$, and $Q^g(\Lambda(\Phi))$ is a maximal order in $Q(\Lambda(\Phi))$, hence $q \in Q^g(\Lambda(\Phi))$ follows.

Write $q = q_{\tau_1}.X_{\tau_1} + \dots + q_{\tau_n}.X_{\tau_n}$ with $\tau_1 < \dots < \tau_n$. The relation $q.A_N \subset A_k$ then yields :

$$q_{\tau_n}.\phi_{\tau_n}(C_e(A_N)) \subset C_{\tau_n}(A_k)$$

Note that we have assumed that Λ was a ρ – maximal order wich satisfies a polynomial identity. Therefore ρ is a central kernel functor and since the ϕ_τ are induced by normalizing elements they leave elements of the center of Λ invariant yielding that ρ is ϕ_τ invariant. Note also that $A_n \in \mathcal{D}_{\rho(\Phi)}(\Lambda(\Phi))$ implies that $A_n \in \mathcal{L}(\rho(\Phi))$ i.e. that $C_e(A_n) \in \mathcal{L}(\rho)$. Because of these remarks (*) yields :

$$(**) : q_{\tau_n}.Q_\sigma(C_e(A_N)) \subset Q_\sigma(C_{\tau_n}(A_k))$$

By assumption on N , $Q_\sigma(C_e(A_N)) = Q_\sigma(C_e(A_k))$ and applying lemma 3.25 to this equality then yields : $Q_\sigma(C_{\tau_n}(A_k)) = Q_\sigma(C_{\tau_n}(A_N))$.

From (**) and again using lemma 3.25 it follows that $q_{\tau_n} \in I_{\tau_n}$. Replacing q by $q - q_{\tau_n}.X_{\tau_n}$ and repeating the same argumentation , we obtain that $q \in \Lambda(\Phi)$.

Therefore, $(A_N :_l A_k) \in \Lambda(\Phi)$ whence $A_N^{-1} * A_k \in \Lambda(\Phi)$ and thus also $A_k \in A_N$ follows. But then $A_k = A_N$ and this finishes the proof.

Corollary 3.30 : If Λ is a maximal order over a Krull domain and if $\Lambda(\Phi)$

is a normalizing Rees ring which satisfies a polynomial identity , then $\Lambda(\Phi)$ is also a maximal order over a Krull domain.

Proof :

A maximal order over a Krull domain is a relative maximal order with respect to the filter of all nonzero ideals such that it satisfies the ascending chain condition on divisorial ideals contained in it. The result now follows immediatly from the foregoing theorem.

Remark 3.31 : A normalizing Rees ring $\Lambda(\Phi) \subset Q_\rho(\Lambda)[X_\tau, \phi, c]$ will be a p.i.-ring if c has the following property :

$$\forall \tau \in G \ , \exists n_\gamma \in \mathbb{N} : c_{\tau,\gamma^n.\gamma} = c_{\gamma^n.\gamma,\tau}$$

Up to changing the variables in $Q_\rho(\Lambda)[X_\tau, \phi, c]$, replacing X_τ by $X_\tau' = a_\tau^{-1}.X_\tau$ where a_τ is the normalizing element of $Q_\rho(\Lambda)$ inducing ϕ_τ in $Q_\rho(\Lambda)$; we get an imbedding of $\Lambda(\Phi)$ in $Q_\rho(\Lambda)[X_\tau', 1, c']$ where :

$$c'_{\gamma,\tau} = a_\gamma^{-1}.\phi_\gamma(a_\tau^{-1}).a_{\gamma.\tau}.c_{\gamma,\tau}$$

with $a_\gamma^{-1}.\phi_\gamma(a_\tau^{-1}) = a_\tau^{-1}.a_\gamma^{-1}$. The statement of the remark becomes now rather straightforward to verify.

In the non-normalizing case one has to add the condition on ϕ_τ that some power of ϕ_τ is inner in $Q_\rho(\Lambda)$, for every $\tau \in G$. This can be proved by using the restriction technique as in proposition 3.20 in order to get rid of the cocycle.

d. : some remarks on class groups.

Throughout this section Λ will be a $\rho-$maximal order satisfying the ascending chain condition on divisorial Λ-ideals contained in Λ ; σ and ρ will be central kernel functors which are G-invariant. Notations will be the same as in the preceding section . The set of $\rho(\Phi)$-divisorial ideals of $\Lambda(\Phi)$ which are graded submodules of $Q^g(\Lambda(\Phi))$ will be denoted by $\mathcal{D}^g_{\rho(\Phi)}(\Lambda(\Phi))$.

Proposition 3.32 : The map $\chi : \mathcal{D}_\rho(\Lambda) \to \mathcal{D}^g_{\rho(\Phi)}(\Lambda(\Phi))$ defined by sending $L \to L(\Phi) = Q_\sigma(\Lambda(\Phi).L)$ is an isomorphism of groups.

Proof :

First let us check that $L(\Phi)$ is an ideal. Since $\mathcal{D}_\rho(\Lambda)$ is an Abelian group, L commutes with each I_τ and therefore we only have to check that $L.X_\tau$ is in $Q_\sigma(\Lambda(\Phi).L)$. Now, $L.X_\tau = X_\tau.(X_\tau^{-1}.L.X_\tau) = X_\tau.a_\tau^{-1}.L.a_\tau$ where $a_\tau \in N_\rho(\Lambda)$ induces ϕ_τ. Since $\Lambda.a_\tau$ and $\Lambda.a_\tau^{-1}$ are in $\mathcal{D}_\rho(\Lambda)$ we have that $Q_\sigma(\Lambda.a_\tau^{-1}.L) = (\Lambda.a_\tau^{-1}) * L = L * \Lambda.a_\tau^{-1} = Q_\sigma(L.a_\tau^{-1})$. Hence, $L.X_\tau \subset Q_\sigma(\Lambda(\Phi).L)$ follows. From $L \in \mathcal{F}_\rho(\Lambda)$ it follows that there is an $I \in \mathcal{L}(\rho)$ such that $I.L$ and $L.I$ are in $\mathcal{L}(\rho)$. By lemma 3.9 and corollary 3.24 , $Q_\sigma(\Lambda(\Phi).I) \in \mathcal{L}(\rho(\Phi))$ and moreover : $Q_\sigma(\Lambda(\Phi).I).Q_\sigma(\Lambda(\Phi).L) = (\bigoplus Q_\sigma(I_\tau.I).X_\tau), (\bigoplus Q_\sigma(I_\gamma.L).X_\gamma \subset \bigoplus Q_\sigma(I_\delta.I.L).X_\delta \subset \Lambda(\Phi)$ while $\Lambda(\Phi).I.\Lambda(\Phi).L.\Lambda(\Phi) = \bigoplus_{\tau,\gamma,\delta} I_\tau.I.I_\gamma.L.I_\delta.X_{\tau+\gamma+\delta}$ is in $\mathcal{L}(\rho(\Phi))$ and it is contained in $I(\Phi).L(\Phi)$. This entails that $I(\Phi).L(\Phi) \in \mathcal{L}(\rho(\Phi))$ and similarly one shows that $L(\Phi).I(\Phi) \in \mathcal{L}(\rho(\Phi))$. There exists an ideal $J \in \mathcal{L}(\rho)$ such that $J \subset L$, hence $J(\Phi) \subset L(\Phi)$ and $J(\Phi) \in \mathcal{F}_\rho(\Lambda(\Phi))$.

If $L(\Phi).f \subset \Lambda(\Phi)$ then $Q^g(\Lambda(\Phi)).f \subset Q^g(\Lambda(\Phi))$. But $Q^g(\Lambda(\Phi))$ is a maximal order therefore $f \in Q^g(\Lambda(\Phi))$ so we may write $f = \sum f_{\tau_i}.X_{\tau_i}$ with $f_{\tau_i} \in \Sigma$ and $\tau_0 < \dots < \tau_n$. From $L(\Phi).f \subset \Lambda(\Phi)$, $L.f_{\tau_n} \subset I_{\tau_n}$ follows , i.e. $f_{\tau_n} \in L^{-1} * I_{\tau_n}$. This means that $L(\Phi).f_{\tau_n} \subset \Lambda(\Phi)$ and $f_{\tau_n} \in (L(\Phi) : \Lambda(\Phi))$ what proves that $(L(\Phi) :_r \Lambda(\Phi))$ is a graded $\Lambda(\Phi)$- submodule of $Q^g(\Lambda(\Phi))$. Furthermore :

$(L(\Phi) :_r \Lambda(\Phi))_e = \{f.X_e, f \in \Sigma : Q_\sigma(I_\tau.L).X_\tau.f.X_e \subset I_\tau.X_\tau, \forall \tau \in G\}$
$= \{f.X_e, f \in \Sigma, I_\tau.L.\phi_\tau(f).X_\tau \subset I_\tau.X_\tau, \forall \tau \in G\}$
$= \{f.X_e, f \in \Sigma, L.\phi_\tau(f) \subset \Lambda, \forall \tau \in G\}$,

in the latter set , we have $L.\phi_e(f) = L.f \subset \Lambda$, but since $(L :_r \Lambda)$ is divisorial it is also ϕ_τ- invariant (see the first paragraph in this proof), hence $\phi_\tau(f) \in (L :_r \Lambda)$

and therefore $(L(\Phi) :_r \Lambda(\Phi))_e = (L :_r \Lambda).X_e$ follows. It is now evident that $(L(\Phi))^{**}$ is a graded $\Lambda(\Phi)$-submodule of $Q^g(\Lambda(\Phi))$ which is a divisorial $\Lambda(\Phi)$-ideal with part of degree e exactly equal to $L.X_e$.

Hence, $L(\Phi) = (L(\Phi))^{**}$ and $L(\Phi) \in \mathcal{D}^g_{\rho(\Phi)}(\Lambda(\Phi))$. Similarly, one verifies that for any B in $\mathcal{D}^g_{\rho(\Phi)}(\Lambda(\Phi))$ we have $B_e = J.X_e$ for some $J \in \mathcal{D}_\rho(\Lambda)$ such that $Q_\sigma(B) = J(\Phi)$. Since χ is clearly a groupmorphism we only have to show that $Q_\sigma(B) = B$ for every $B \in \mathcal{D}^g_{\rho(\Phi)}(\Lambda(\Phi))$. Actually, it will suffice to establish that $B' = (H :_r \Lambda(\Phi))$ for any ideal H of $\Lambda(\Phi)$ is such that $Q_\sigma(B') = B'$.

We have $H_\tau.X_\tau.B'_\gamma.X_\gamma \subset I_{\tau.\gamma}.X_{\tau.\gamma}$ with the obvious notation. Since $Q_\sigma(I_{\tau.\gamma}.X_{\tau.\gamma}) = I_{\tau.\gamma}.X_{\tau.\gamma}$ and since σ is central it follows that $H_\tau.X_\tau.Q_\sigma(B'_\gamma.X_\gamma) \subset I_{\tau.\gamma}.X_{\tau.\gamma}$, hence $Q_\sigma(B'_\gamma.X_\gamma) \subset (H :_r \Lambda(\Phi))_\gamma$ or $Q_\sigma(B') = B'$.

Let $\mathcal{P}_{\rho(\Phi)}(\Lambda(\Phi))$ be the subgroup o $\mathcal{D}_{\rho(\Phi)}(\Lambda(\Phi))$ consisting of principal ideals generated by a normalizing element of $N_{\rho(\Phi)}(\Lambda(\Phi))$ and let $\mathcal{P}^g_{\rho(\Phi)}(\Lambda(\Phi))$ be the subgroup of $\mathcal{P}_{\rho(\Phi)}(\Lambda(\Phi))$ consisting of the principal ideals generated by an homogeneous normalizing element in $N_{\rho(\Phi)}(\Lambda(\Phi))$.

Proposition 3.33 : For any $L \in \mathcal{D}_\rho(\Lambda)$ the following statements are equivalent :

(1) : $L(\Phi) \in \mathcal{P}^g_{\rho(\Phi)}(\Lambda(\Phi))$.

(2) : There exists a divisorial Λ-ideal $B \in Im(\Phi)$ such that $L = B.x$ for some $x \in N_\rho(\Lambda)$.

Proof :

(1) $\Rightarrow$ (2) : Suppose $L(\Phi) = \Lambda(\Phi).x_\tau.X_\tau$ where $x_\tau.X_\tau \in N_{\rho(\Phi)}(\Lambda(\Phi))$, then $\Lambda.\ddot{X}_e.x_\tau.\ddot{X}_\tau = x_\tau.\ddot{X}_\tau.\Lambda.\ddot{X}_e$ i.e. $\Lambda.\phi_e(\dot{\Lambda}.\phi_e(x_\tau)).c_{e,\tau} = x_\tau.\dot{\Lambda}.c_{\tau,e}$. Therefore, $\dot{\Lambda}.x_\tau = x_\tau.\Lambda$ since $\Lambda(\Phi).x_\tau \in \mathcal{D}_{\rho(\Phi)}(\Lambda(\Phi))$, i.e. $x_\tau \in N_\rho(\Lambda)$. Furthermore, $L(\Phi)_e = L.X_e = I_{\tau-1}.\phi_{\tau-1}(x_\tau).c_{\tau-1,\tau}.X_e = I_{\tau-1}.x_\tau.X_e$ (actually one may take $X_e = 1$).Hence $L = I_{\tau-1}.x_\tau$.

(2) $\Rightarrow$ (1) : If $B \in Im(\phi)$ then $B = I_\tau$ for some $\tau \in G$. We claim that $B(\Phi) = \Lambda(\Phi).X_{\tau^{-1}}$. Indeed $\Lambda(\Phi).X_{\tau^{-1}} = Q_\sigma((\Lambda(\Phi).X_{\tau^{-1}})_e.\Lambda(\Phi)) = Q_\sigma(I_\tau.\Lambda(\Phi)) = B(\Phi)$. Finally, if $L = I_\tau.x$ for some $x \in N_\rho(\Lambda)$, then $L(\Phi) = Q_\sigma(\Lambda(\Phi).I_\tau.x) = Q_\sigma(\Lambda(\Phi).I_\tau).x = \Lambda(\Phi).X_{\tau^{-1}}.x$, hence $L(\Phi) \in P^g_{\rho(\Phi)}(\Lambda(\Phi))$.

Theorem 3.34 : If we define : $Cl_\rho(\Lambda) = D_\rho(\Lambda)/P(\rho)(\Lambda)$ and $Cl^g_{\rho(\Phi)}(\Lambda(\Phi)) = D^g_{\rho(\Phi)}(\Lambda(\Phi))/P^g_{\rho(\Phi)}(\Lambda(\Phi))$ then we have the following relation :
$Cl_\rho(\Lambda) = Cl^g_{\rho(\Phi)}(\Lambda(\Phi)) \oplus \chi(Im(\phi))$
where $\chi : D_\rho(\Lambda) \to Cl_\rho(\Lambda)$ is the canonical morphism.

Proof :
The morphism of proposition 3.32 yields an exact sequence :

$$1 \to \chi(Im(\phi)) \to Cl_\rho(\Lambda) \to Cl^g_{\rho(\Phi)}(\Lambda(\Phi)) \to 1$$

where the last morphism is derived from the one of proposition 3.32 by formation of classes,using proposition 3.33. Exactness of this sequence follows from the foregoing results and the morphism $Cl_\rho(\Lambda) \to Cl^g_{\rho(\Phi)}(\Lambda(\Phi))$ may be split because taking parts of degree e in $D^g_{\rho(\Phi)}(\Lambda(\Phi))$ yields a splitting morphism :

$$Cl^g_{\rho(\Phi)}(\Lambda(\Phi)) \to Cl_\rho(\Lambda)$$

Recall that we have assumed that ρ is a central perfect kernel functor. From the foregoing section we retain that $Q_\rho[X_\tau, \phi, c]$ is a relative maximal order with respect to the kernel functor $\rho^\sim$ where $\mathcal{L}(\rho^\sim) = \{I$ ideal of $Q_\rho(\Lambda)[X_\tau, \phi, c]; C_e(I) = Q_\rho(\Lambda)\}$

Proposition 3.35 : Under the same assumptions as before , the following exact sequence is split exact :

$$1 \to D^g_{\rho(\Phi)}(\Lambda(\Phi)) \to D_{\rho(\Phi)}(\Lambda(\Phi)) \to D_{\rho^\sim}(Q_\rho(\Lambda)[X_\tau, \phi, c]) \to 1$$

Proof :

Let $\sigma^{\sim}$ be the kernel functor associated to $\rho^{\sim}$ in the usual way and consider $P \in \mathcal{P}(\rho^{\sim})$ (see the first section for notations). We claim that $P \cap \Lambda(\Phi) \in \mathcal{P}(\rho(\Phi))$. That $P \cap \Lambda(\Phi) \in \mathcal{L}(\rho(\Phi))$ follows from the fact that $Q_\rho(C_e(P \cap \Lambda(\Phi))) = C_e(P) = Q_\rho(\Lambda)$ and therefore $C_e(P \cap \Lambda(\Phi)) \in \mathcal{L}(\rho)$. That $P \cap \Lambda(\Phi)$ is is a prime ideal is also obvious since it is a central ring extension. Now, first we check that a $\rho^{\sim}$-divisorial ideal of $Q_\rho(\Lambda)[X_\tau, \phi, c]$, I say , has the property that $I \cap \Lambda(\Phi)$ is a $\rho(\Phi)$-divisorial ideal of $\Lambda(\Phi)$. Since $\Lambda(\Phi)$ is a relative maximal order satisfying the ascending chain condition on divisorial ideals, the divisorial closure of any ideal is given by letting $Q_{\sigma(\Phi)}(-)$ act on it. Now, $Q_{\sigma(\Phi)}(I \cap \Lambda(\Phi)) = I \cap \Lambda(\Phi)$ because , if $J.x \subset \Lambda(\Phi) \cap I$ for some ideal $J \in \mathcal{L}(\sigma(\Phi))$ and $x \in Q_\rho(\Lambda)[X_\tau, \phi, c]$, then $\Lambda(\Phi).x = Q_{\sigma(\Phi)}(J).x \subset Q_{\sigma(\Phi)}(I \cap \Lambda(\Phi))$ yields that $x \in \Lambda(\Phi)$ and so we have to distinguish between the following two cases :

first case : $J_g \in \mathcal{L}(\rho(\Phi))$, then $Q_\rho(\Lambda)[X_\tau, \phi, c].J.x \subset I$ and from $Q_\rho(\Lambda)[X_\tau, \phi, c].J = Q_\rho(\Lambda)[X_\tau, \phi, c]$ it then follows that $x \in I \cap \Lambda(\Phi)$.

second case : $J_g \notin \mathcal{L}(\rho(\Phi))$, then $J' = Q_\rho(\Lambda)[X_\tau, \phi, c].J \in \mathcal{L}(\sigma^{\sim})$,(because firstly, $J' \in \mathcal{L}(\rho^{\sim}) \subset P$ for some $P \in \mathcal{P}(\rho^{\sim})$ then $J \subset P \cap \Lambda(\Phi) \in \mathcal{P}(\rho(\Phi))$, a contradiction) hence $J'.x \subset I$ and also $Q_{\sigma^{\sim}}(J').x \subset I$, i.e. $x \in I$ and hence $x \in \Lambda(\Phi) \cap I$. Now, knowing that $P \cap \Lambda(\Phi)$ is divisorial we may derive from this fact that $P \cap \Lambda(\Phi) \in \mathcal{P}(\rho(\Phi))$ for if $P \cap \Lambda(\Phi)$ were not a maximal divisorial ideal contained in $\Lambda(\Phi)$ then $P \cap \Lambda(\Phi) \subset P_1$ where $P_1 \in \mathcal{P}(\rho(\Phi))$. But then $P = Q_\rho(\Lambda)[X_\tau, \phi, c].(P \cap \Lambda(\Phi))$ is properly contained in the divisorial ideal $P_1.Q_\rho(\Lambda)[X_\tau, \phi, c] = P'_1$ and this will lead to $Q_\sigma(P'_1 \cap \Lambda(\Phi)) = \Lambda(\Phi)$ and thus $P_1 = \Lambda(\Phi)$, a contradiction.

Exactness of the sequence in the statement of the proposition is now easily verified. The splitting morphism :

$$\mathcal{D}_{\rho^{\sim}}(Q_\rho(\Lambda)[X_\tau, \phi, c]) \rightarrow \mathcal{D}_{\rho(\Phi)}(\Lambda(\Phi))$$

is obtained by intersecting down divisorial ideals of $Q_\rho(\Lambda)[X_\tau, \phi, c]$ to $\Lambda(\Phi)$.

Let $CCl^g_{\rho(\Phi)}(\Lambda(\Phi))$ and $CCl_{\rho(\Phi)}(\Lambda(\Phi))$ be defined by $CCl^g_{\rho(\Phi)}(\Lambda(\Phi)) = \mathcal{D}^g_{\rho(\Phi)}(\Lambda(\Phi))/C\mathcal{P}^g_{\rho(\Phi)}(\Lambda(\Phi))$ and $CCl_{\rho(\Phi)}(\Lambda(\Phi)) = \mathcal{D}_{\rho(\Phi)}(\Lambda(\Phi))/C\mathcal{P}_{\rho(\Phi)}(\Lambda(\Phi))$, where

$CP^g_{\rho(\Phi)}(\Lambda(\Phi))$ is the group of principal ideals in $\Lambda(\Phi)$ generated by one homogeneous central element , $CP_{\rho(\Phi)}(\Lambda(\Phi))$ is the group of principal ideals in $\Lambda(\Phi)$ generated by a central element.(Note that this CCl coincides with the central classgroup Cl^c mentioned in the introduction for maximal orders).

Lemma 3.36 : There is a natural inclusion :

$$CCl^g_{\rho(\Phi)}(\Lambda(\Phi)) \to CCl_{\rho(\Phi)}(\Lambda(\Phi))$$

Proof :

We only have to verify that a graded divisorial ideal generated by a central element is actually generated by an homogeneous central element. Since the group G is ordered Abelian and torsion free this can easily be done by standard methods in graded ring theory.

Theorem 3.37 : Let Λ be a ρ – maximal order with the ascending chain condition on divisorial ideals contained in Λ , then :

$$CCl_{\rho(\Phi)}(\Lambda(\Phi)) \simeq CCl^g_{\rho(\Phi)}(\Lambda(\Phi)) \oplus CCl_{\rho^\sim}(Q_\rho(\Lambda)[X_\tau, \phi, c])$$

Proof :

From the following split exact sequence :

$$1 \to D^g_{\rho(\Phi)}(\Lambda(\Phi)) \to D_{\rho(\Phi)}(\Lambda(\Phi)) \to D_{\rho^\sim}(Q_\rho(\Lambda)[X_\tau, \phi, c]) \to 1$$

and utilizing the foregoing lemma we deduce the following exact diagram :

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & GP^g_{\rho^\sim}(\Lambda(\Phi)) & \rightarrow & CP_{\rho^\sim}(\Lambda(\Phi)) & \rightarrow & CP_{\rho^\sim}(Q_\rho(\Lambda)[X_\tau,\phi,c]) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & D^g_{\rho(\Phi)}(\Lambda(\Phi)) & \rightarrow & D_{\rho(\Phi)}(\Lambda(\Phi)) & \rightarrow & D_{\rho^\sim}(Q_\rho(\Lambda)[X_\tau,\phi,c]) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & CCl^g_{\rho(\Phi)}(\Lambda(\Phi)) & \rightarrow & CCl_{\rho(\Phi)}(\Lambda(\Phi)) & \rightarrow & CCl_{\rho^\sim}(Q_\rho(\Lambda)[X_\tau,\phi,c]) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

where exactness of the first row is easily checked and then exactness of the bottom row follows.

Because $CCl^g_{\rho(\Phi)}(\Lambda(\Phi)) \rightarrow CCl_{\rho(\Phi)}(\Lambda(\Phi))$ is a monomorphism we may apply the snake lemma to the bottom two rows and derive from this that $CCl_{\rho(\Phi)}(\Lambda(\Phi)) \rightarrow CCl_{\rho^\sim}(Q_\rho(\Lambda)[X_\tau,\phi,c])$ is epimorphic and may be splitted ; thus yielding the desired result.

Remark 3.38

(1) : If $\mathcal{L}(\rho) = \mathcal{L}(\Lambda{-}0)$ then $Q_\rho(\Lambda)[X_\tau,\phi,c]$ is nothing but $\Sigma[X_\tau,\phi,c]$ and every divisorial ideal of this ring is generated by a central element because $\Sigma[X_\tau,\phi,c]$ is an Azumaya algebra over a factorial domain. Then $CCl(Q_\rho(\Lambda)[X_\tau,\phi,c]) = 1$ and therefore $CCl^g_{\rho(\Phi)}(\Lambda(\Phi)) \simeq CCl_{\rho(\Phi)}(\Lambda(\Phi))$.

It would be interesting to know under which conditions $CCl_{\rho^\sim}(Q_\rho(\Lambda)[X_\tau,\phi,c])$ vanishes.

(2) : If Λ is a ρ − maximal order such that $Cl^g_{\rho(\Phi)}(\Lambda(\Phi)) \rightarrow Cl_{\rho(\Phi)}(\Lambda(\Phi))$ is monomorphic , then a similar argumentation as before can be applied to the normalizing class groups. One then obtains :

$$Cl_{\rho(\Phi)}(\Lambda(\Phi)) \simeq Cl^g_{\rho(\Phi)}(\Lambda(\Phi)) \oplus Cl_{\rho^\sim}(Q_\rho(\Lambda)[X_\tau,\phi,c])$$

(3) : In general, we have an exact sequence :

$$1 \rightarrow Ker(\chi) \rightarrow CCl_{\rho(\Phi)}(\Lambda(\Phi)) \rightarrow Cl_{\rho(\Phi)}(\Lambda(\Phi)) \rightarrow 1$$

where $Ker(\chi)$ is a torsion group.

4 : UNIVERSAL BIAGEBRAS ASSOCIATED WITH ORDERS

0. introduction

In [76] , M.E. Sweedler associated to every algebra A over a field K a universal measuring bialgebra $M_K(A, A)$ and its cocommutative pointed subbialgebra $H_K(A, A)$. These objects may be used in several domains , e.g. to obtain a beautiful Galois theory , cfr. [77].

Over arbitrary commutative rings , these constructions cannot be generalized and one has to restrict attention to Galois objects , as introduced in [15] , in order to get a more or less satisfactory Galois theory. However, the condition of being a Galois object, puts severe restrictions on the ringextension. A lot of 'nice' extensions , e.g. the integral closure of a Dedekind domain in a finite Galois extension of its field of fractions , do not necessarely fit into this Galois-object framework , as an example due to Janusz [31] shows.

Therefore, it would be interesting to extend Sweedler's construction to a nice class of rings , e.g. Dedekind (or Krull) domains. And in this chapter we put the first steps in this direction.

In the first section we show how one can associate to any algebra A which is finitely geneated as a module over a Dedekind domain D , a universal measuring bialgebra $M_D(A, A)$. Our constructions are similar to the ones in [76] , modulo some technical difficulties , stemming from projectivity conditions. In fact, the main reason why one can generalize the field-case constructions to algebras over Dedekind domains is that any finitely generated torsion free D-module is projective.

In the second section we restrict attention to the case of orders in central simple K-algebras Σ where K is the field of fractions of D. We mainly study the relation between $M_D(\Lambda,\Lambda)$ and $M_K(\Sigma,\Sigma)$. It turns out that $H^1_D(\Lambda,\Lambda)$ (the pointed irreducible cocommutative component of the identity) is a (not necessarely finitely generated) D-order in $H^1_K(\Sigma,\Sigma)$.

Along the lines we also prove a generalization of the classical Skolem-Noether result on automorphisms and derivations of central simple algebras.

Several examples have been included to illustrate the connection between Hopf primes (i.e. prime ideals p of D such that $H_{D_p}(\Lambda_p,\Lambda_p)$ is an order in $H_K(\Sigma,\Sigma)$) and noncommutative valuation theory.

In the third section we present a method to generalize most results of the first two sections to (reflexive) orders over Krull domains. The main idea behind this generalization is that a Krull domain behaves itself like a sheaftheoretic discrete valuation ring (on $X^{(1)}(R)$ with the induced Zariski-topology and with the inverse image of the structure sheaf under the continous map $X^{(1)}(R) \to Spec(R)$ on it).

In section 4 , the obtained bialgebras are used to obtain a Galois theory for Dedekind (and Krull) domains , which is , resembling the Galois theory of the corresponding field of fractions.

a. construction of $M_D(A,B)$.

Throughout, D will be a commutative Dedekind domain. First, we will associate to a pair of finitely generated projective D-algebras A and B (note that they need not be orders) a universal measuring D-coalgebra $M_D(A,B)$. Our construction runs along the lines of M.E. Sweedler [76] modulo some technical difficulties mainly stemming from projectivity conditions.

The main problem in generalizing Sweedler's construction to the ring case is to find a suitable substitute for A^o,(cfr. definition below). However, when D is a Dedekind domain, this problem can be succesfully solved.

Definition 4.1 : Let A be any D-algebra ,

$A^o = \{g \in A^* : Ker(g)$ contains an ideal $I : A/I$ is finitely generated and torsion free $\}$.

Remark 4.2 :A^o is a D-submodule of $A^* = Hom_D(A,D)$. Clearly, A^o is closed under scalar multiplication. The sum of any two elements of A^o is again in A^o since $A/I \cap J \to A/I \oplus A/J$ is an inclusion and therefore $A/I \cap J$ is again finitely generated and torsion free.

Proposition 4.3 : Let A, B be projective D-algebras and $f \in Alg_D(A,B)$,then :

(a) : The dual map op f ; $f^* : B^* \to A^*$ sends B^o in A^o

(b) : The map $A^* \otimes_D B^* \to (A \otimes_D B)^*$ restricts to an isomorphism $A^o \otimes_D B^o \simeq (A \otimes_D B)^o$

(c) : If $M : A \otimes_D A \to A$ is the multiplication map then $M^*(A^o) \subset A^o \otimes_D A^o$

Proof :

(a) : It is easy to show that if $b^* \in B^o$ has $J \subset Ker(b^*)$, then $Ker(f^*(b^*)) \supset f^{-1}(J)$. Further, $A/f^{-1}(J) \to B/J$ is an inclusion and therefore it is finitely generated and torsion free.

(b) : Let K be any ideal in $A \otimes_D B$ such that $(A \otimes_D B)/K$ is finitely generated and torsion free. Let $I = \{a \in A : a \otimes 1 \in K\}$ then A/I is finitely generated and torsion free , because (see part a) it is the inverse image of K under the algebra map $a \to a \otimes 1$ of A to $A \otimes_D B$. Similarly , if $J = \{b \in B : 1 \otimes b \in K\}$ then B/J is finitely generated and torsion free. Note that $A \otimes_D J + I \otimes_D B \subset K$ and by [12] , A.II.59,Prop.6 , we have :

$$A \otimes_D B/(A \otimes_D J + I \otimes_D B) \simeq A/I \otimes_D B/J$$

therefore $A \otimes_D B/(I \otimes_D B + A \otimes_D J)$ is again a finitely generated and torsion free D-module. This follows from the fact that if A' and B' are finitely generated torsion free over a Dedekind domain D , then $A \simeq I_1 \oplus ... \oplus I_n$, $B' \simeq J_1 \oplus ... \oplus J_n$ with I_i and J_j fractional ideals and therefore $A' \otimes_D B' \simeq \oplus(I_i \otimes_D J_j) \simeq \oplus(I_i . J_j)$ and the latter is thus finitely generated and torsion free. Now, suppose that $c^* \in (A \otimes_D B)^o$ with $K \subset Ker(c^*)$, I, J as above. Then c^* factorizes through $A/I \otimes_D B/J$. That is, there exists a unique C^* such that the diagram below is a commutative one :

$$\begin{array}{ccc} A \otimes_D B & \rightarrow & D \\ \downarrow & \nearrow & \\ A/I \otimes_D B/J & & \end{array}$$

Thus, $C^* \in (A/I \otimes_D B/J)^* \simeq (A/I)^* \otimes_D (B/J)^*$ (D is a Dedekind domain and [12] , A.II.80,cor.1 . Via this isomorphism , write $C^* = \sum D_i^* \otimes E_i^*$ with $D_i^* \in (A/I)^*$ and $E_i^* \in (B/J)^*$. In particular , for $a \in A/I$ and $b \in B/J$ we have :

$$< C^*, a \otimes b >= \sum < D_i^*, a > . < E_i^*, b >$$

Now, if ϕ_1, ϕ_2 are the naturalk projections $A \to A/I$ and $B \to B/J$ then the commutativity of the diagram above comes down to :

(2) : $< c^*, a \otimes b >=< C^*, \phi_1(a) \otimes \phi_2(b) >= \sum < D_i^*, \phi_1(a) > . < E_i^*, \phi_2(b) >$

Now, let $d_i^* = D_i^* o \phi_1$, then $d_i^* \in A^o$ because $I = Ker(\phi_1) \subset Ker(d_i^*)$. Similarly, $e_i^* = E_i^* o \phi_2 \in B^o$. (2) then becomes $c^* = \sum d_i^* \otimes e_i^*$, thus $(A \otimes_D B)^o \subset A^o \otimes_D B^o$.

Conversely, if $d^* \in A^o$ (resp. $e^* \in B^o$) with $I \subset Ker(d^*) : A/I$ is finitely generated torsion free (resp.$J \subset Ker(e^*) : B/J$ is finitely generated torsion free) then $A \otimes_D J + I \otimes_D B \subset Ker(d^* \otimes e^*)$ and $A \otimes_D B/(A \otimes_D J + I \otimes_D B)$ is finitely generated and torsion free. Therefore, $A^o \otimes_D B^o \subset (A \otimes_D B)^o$.

(c) : For $a^* \in A^*; a, b \in A : < M^*(a^*), a \otimes b > = < a^*, a.b >$. If $I \subset Ker(a^*)$ such that A/I is finitely generated and torsion free, then $A \otimes_D I + I \otimes_D A \subset Ker(M^*(a^*))$ and $A \otimes_D A/(A \otimes_D I + I \otimes_D A)$ is also finitely generated torsion free. Thus, $M^*(A^o) \subset (A \otimes_D A)^o = A^o \otimes_D A^o$.

Now, define $\Delta = M^* \mid A^o : A^o \to A^o \otimes_D A^o$ and $\epsilon : A^o \to D$ by $\epsilon(a^*) = < a^*, 1 >$.

Proposition 4.4 : (A^o, Δ, ϵ) is a D-coalgebra.

Proof :

Similar to a result of M.E. Sweedler [76].

If A and B are projective D-algebras and $f \in Alg_D(A, B)$, then Prop.4.3.a above states that $f^* \mid B^o$ is a map from B^o to A^o. Denote $f^o = f^* \mid B^o$. A diagram chase shows that f^o is a coalgebra map. For any D-algebra A , A^* is a left A-module with scalar multiplication defined by the rule $< b \to a^*, a > = < a^*, a.b >$ for $a^* \in A^*$, $a, b \in A$. The right action is defined by $< a^* \leftarrow b, a > = < a^*, b.a >$. These definitions make A^* into a twosided A-module.

Proposition 4.5 : Let A be a projective D-algebra . For any $f \in A^*$ the following statements are equivalent :

(a) : $f \in A^o$;

(b) : $M^*(f) \in A^o \otimes_D A^o$;
(c) : $M^*(f) \in A^* \otimes_D A^*$;
(d) : $A \to f$ is finitely generated torsion free ;
(e) : $f \leftarrow A$ is finitely generated torsion free .

Proof :

(a) $\Rightarrow$ (b) : since $M^*(A^o) \subset A^o \otimes_D A^o$ (Prop.4.3.(c))

(b) $\Rightarrow$ (c) : trivially

(c) $\Rightarrow$ (d) : let $M^*(f) = \sum a_i^* \otimes b_i^*$ where $a_i^*, b_i^* \in A^*$. By the definition of M^* we have :

$$< f, a.b >= \sum < a_i^*, a > . < b_i^*, b >$$

Hence, $b \to f = \sum a_i^* . < b_i^*, b >$, thus $(A \to f) \subset D.a_1^* + ... + D.a_n^*$ and so it is finitely generated. Now suppose that $(A \to f)$ is not torsion free , i.e. for some $b \in A, d \in D : d.(b \to f) = 0$ thus for all $a \in A : d. < f, a.b >= 0$, but this implies $< f, a.b >= 0$ for all a , or , $b \to f = 0$.

(d) $\Rightarrow$ (a) : $M = (A \to f)$ is finitely generated and torsion free. Then, $I = \{a \in A : a \to M = 0\}$ is an ideal of A with A/I is finitely generated and torsion free (because I is the kerel of the map $\phi : A \to End_D(M)$ given by $\phi(a)[m] = a \to m$. Hence, $A/I \to End_D(M)$ is an inclusion and thus A/I is finitely generated torsion free). But for any $a \in I : < f, a >=< a \to f, 1 >=< 0, 1 >= 0$. So, $I \subset Ker(f)$, whence $f \in A^o$. This proves the equivalences (a) - (d) . Obviously (e) $\Rightarrow$ (a) follows by left-right symmetry from (d) $\Rightarrow$ (a).

Proposition 4.6 : If C is a D-coalgebra such that C is a projective D-module. Let C^* be the dual algebra , then the natural map $C \to C^{**}$ maps C to C^{*o}.

Proof :

Let $c^* \in C^*, c \in C$ and c' the image of c in C^{**}. The definitions of $\to$ and of the multiplication in C^* imply :

$$c^* \rightarrow c' = \sum c'_{(1)}. < c^*, c'_{(2)} >$$

. Thus, $C^* \rightarrow c'$ is finitely generated

The injection $A^o \rightarrow A^*$ induces a map $A^{**} \rightarrow A^{o*}$. Define $\phi : A \rightarrow A^{o*}$ to be the composition map $A \rightarrow A^{**} \rightarrow A^{o*}$. Note that ϕ is an algebra morphism.

Proposition 4.7 : Let A, C be projective D-modules , A a D-algebra and C a D-coalgebra , then there is a natural one-to-one correspondence between $Alg_D(A, C^*)$ and $Coalg_D(C, A^o)$.

Proof :

Given $f \in Alg_D(A, C^*)$, let $\psi(f) \in Coalg_D(C, A^o)$ be the composite map : $C \rightarrow C^{*o} \rightarrow A^o$ where the map on the right is f^o.

If $g \in Coalg_D(C, A^o)$, let $\chi(g) \in Alg_D(A, C^*)$ be the composite map : $A \rightarrow A^{o*} \rightarrow C^*$ where the morphism on the right is g^*.

It is easily verified that $\chi(\psi(f)) = f$ and $\psi(\chi(g)) = g$, since

$$\psi(f)(c) : A \rightarrow D; a \rightarrow < f(a), c >$$

$$\chi(g)(a) : C \rightarrow D; c \rightarrow < g(c), a >$$

In the above propositions we showed that $(-)^o$ has the required properties in order to complete Sweedler's construction, this time for finitely generated projective D-algebras:

Definition 4.8 : If V is a D-module , a pair (C, ϕ) where C is a D-coalgebra and $\phi : C \rightarrow V$ a D-module morphism , is called a cofree coalgebra on V if for every projective D-coalgebra C' and D-module morphism $f : C' \to V$ there is a unique coalgebra map F making the following diagram commutative :

$$\begin{array}{ccc} C' & \rightarrow & C \\ & \searrow \quad \swarrow & \\ & V & \end{array}$$

It is clear from the definition that if C exists , it is unique up to D-coalgebra isomorphism.

Theorem 4.9 : If V is a finitely generated projective D-module , then the cofree coalgebra on V exists .

Proof :

Let $T(V^*)$ be the tensor algebra on V^* , which is a projective D-module since V^* is finitely generated projective. Let $i : V^* \to T(V^*)$ be the natural injection. Let ϕ be the composite morphism :

$$\phi : T(V^*)^o \to T(V^*)^* \to V^{**}$$

where the morphism on the right is i^*. We claim that $(T(V^*)^o, \phi)$ is the cofree coalgebra on $V^{**} \simeq V$. Let C be a projective D-coalgebra and $f : C \to V^{**}$ a D-module morphism.

Denote by F the composite $V^* \to V^{***} \to C^*$ where the right morphism is f^*. Because of the universal mapping property for $T(V^*)$, there is a unique algebra map g such that the following diagram is commutative :

$$\begin{array}{ccc} T(V^*) & & \\ \uparrow & \searrow & \\ V^* & \rightarrow & C^* \end{array}$$

Dualizing this diagram , we obtain :

$$\begin{array}{ccccc} T(V^*)^o & & & & \\ \downarrow & \nwarrow & & & \\ T(V^*)^* & & C^{*o} & & \\ \downarrow & & & \nwarrow & \\ V^{**} & \leftarrow & C^{**} & \leftarrow & C \end{array}$$

The vertical composite is nothing but $\phi : T(V^*)^o \to V^{**}$, the top diagonal composite is the unique coalgebra map $F' : C \to T(V^*)^o$ corresponding to $g : T(V^*) \to C^*$ (see Prop.4.7).

The bottom horizontal composite is $f : C \to V^{**}$ since there is a one to one correspondence between $Hom_D(C, V^{**})$ and $Hom_D(V^*, C^*)$ given by :

$$\psi : Hom_D(C, V^{**}) \to Hom_D(V^*, C^*); f \to (V^* \to V^{***} \to C^*)$$

where the morphism on the right is f^* and :

$$\chi : Hom_D(V^*, C^*) \to Hom_D(C, V^{**}); g \to (C \to C^{**} \to V^{**})$$

where the morphism on the right is g^*. Finally, the horizontal composite is $\chi(\psi(f)) = f$. Thus, $(T(V^*)^o, \phi)$ is the cofree coalgebra on $V^{**} \simeq V$.

Let us recall the definition of "measuring" . Let A, B be D-algebras, M a D-coalgebra and $\psi : M \otimes_D A \to B$ a D-module morphism. M is said to measure A to B id ψ satisfies :

(1) : $\psi(m \otimes a.a') = \sum \psi(m_{(1)} \otimes a).\psi(m_{(2)} \otimes a')$;

(2) : $\psi(m \otimes 1) = \epsilon(m).1_B$.

For all $a, a' \in A$; $m \in M$.

Theorem 4.10 : Let A, B be finitely generated projective D-algebras. There is a D-coalgebra $M = M_D(A, B)$ and a D-module morphism $\theta : M \otimes_D A \to B$ measuring A to B and with the following universal property :

If C is a projective D-coalgebra and (C, ψ) measures A to B then there is a unique coalgebra map $F : C \to M$ such that the following diagram is commutative

$$\begin{array}{ccc} M_D(A, B) \otimes_D A & \to & B \\ \downarrow & \nearrow & \\ C \otimes_D A & & \end{array}$$

Proof :
As in M.E. Sweedler [76] , using the foregoing results.

Remark 4.11 : If A is a finitely generated projective D-algebra , then $M_D(A, A)$ has a unique algebra structure such that it is a bialgebra and θ : $M_D(A, A) \otimes_D A \to A$ makes A into an $M_D(A, A)$-module.

b : bialgebras associated with D-orders.

From now on we will restrict attention to D-orders, i.e. D will be a Dedekind domain with field of fractions K , Σ a central simple K-algebra and Λ will be a subring of Σ having D as its center such that $K.\Lambda = \Sigma$. Remark that Λ is a finitely generated projective D-module since it is finitely generated and torsion free.

First, we want to investigate the connection between $M_D(\Lambda, \Lambda)$ (as defined in the foregoing section) and $M_K(\Sigma, \Sigma)$ (as defined by Sweedler in [76]).

Proposition 4.12 : $M_D(\Lambda, \Lambda) \otimes_D K$ is a subbialgebra of $M_K(\Sigma, \Sigma)$.

Proof :

Let $(M_D(\Lambda, \Lambda), \Delta, \epsilon, \mu, M)$ be the D-bialgebra as constructed in the foregoing section. We will define a K-bimodule structure on $M_D(\Lambda, \Lambda) \otimes_D K$ in the following way :

$\Delta_K : M_D(\Lambda, \Lambda) \otimes_D K \to M_D(\Lambda, \Lambda) \otimes_D K \otimes_K M_D(\Lambda, \Lambda) \otimes_D K \simeq M_D(\Lambda, \Lambda) \otimes_D M_D(\Lambda, \Lambda) \otimes_D K$

$\Delta_K(m \otimes k) = \Delta(m) \otimes k$

$\epsilon_K : M_D(\Lambda, \Lambda) \otimes_D K \to K$

$\epsilon_K(m \otimes k) = \epsilon(m).k.$

and μ_K , M_K as usual. It is easily verified that these maps are well defined and that $(M_D(\Lambda, \Lambda) \otimes_D K, \Delta_K, \epsilon_K, \mu_K, M_K)$ is a K-bialgebra. Furthermore, $\psi : M_D(\Lambda, \Lambda) \otimes_D \Lambda \to \Lambda$ is a D-measuring. Now define $\psi_K : M_D(\Lambda, \Lambda) \otimes_D K \otimes_K \Sigma \to \Sigma$ by $\psi_K(m \otimes k \otimes k'.\lambda) = k.k'.(\psi(m \otimes \lambda))$. ψ_K is well defined and a K-measuring. Applying the universal mapping property of $M_K(\Sigma, \Sigma)$ yields a unique K-coalgebra map F such that the following diagram is commutative :

It is now easy to check that $F(M_D(\Lambda,\Lambda)\otimes_D K)$ is a subbialgebra of $M_K(\Sigma,\Sigma)$.

From now on, we will identify $M_D(\Lambda,\Lambda)$ with its image in $M_K(\Sigma,\Sigma)$.

Definition 4.13 : If C is a torsion free D-coalgebra , then :

C is called irreducible if any two non-zero subcoalgebras have non-zero intersection.

C is simple if it has no non-zero subcoalgebras.

C is pointed if all simple subcoalgebras of C are free D-modules of rank one.

Lemma 4.14 : If H is a torsion free D-coalgebra and $G(H)$ is the set of its group-like elements (i.e. those elements h of H such that $\Delta(h) = h\otimes h$) , then :

(1) : $D.G(H)$ is a free D-module ;

(2) : $G(H)$ corresponds bijectively to the free subcoalgebras of rank one.

Proof :

(1) : Suppose that $D.G(H)$ is not free, hence there are $g_1,...,g_n \in G(H)$ such that $D.g_1 + ... + D.g_n$ is not free. By induction on n we may suppose however that $D.g_1 + ... + D.g_{n-1}$ is free. Thus $d_n.g_n = \sum_{i=1}^{n-1} d_i.g_i$ with $d_n \neq 0$, then :

$$\Delta(d_n.g_n) = \sum_1^{n-1} d_i.\Delta(g_i) = \sum_1^{n-1} d_i.(g_i\otimes g_i)$$

and on the other hand we have :

$$\Delta(d_n.g_n) = d_n.g_n \otimes g_n$$

hence $\sum d_n.d_i.(g_i\otimes g_i) = \sum d_i.d_j.(g_i\otimes g_j)$. Therefore, $d_n.d_i = d_i^2$, hence $d_n = d_i$ or $d_i = 0$ and further, $d_i.d_j = 0$ if $i \neq j$ so there is just one $i : d_i \neq 0$. Thus, $g_n = g_i$, done.

(2) : Let H' be a free subcoalgebra of rank one , $H' = D.h$ and thus there exists an element $d \in D$ such that $\Delta(h) = d.(h \otimes h)$. Now, take $h' = d.h$, then , $\Delta(h') = h' \otimes h'$, hence $\epsilon(h') = 1$ and this implies that d is invertible in D. Finally, $D.h' = D.h = H'$.

Recall from [76] that $H_K(\Sigma, \Sigma)$ is the maximal cocommutative pointed subcoalgebra of $M_K(\Sigma, \Sigma)$.

Definition 4.15 : $H_D(\Lambda, \Lambda) = \{m \in M_D(\Lambda, \Lambda) : m \otimes 1 \in H_K(\Sigma, \Sigma)\}$.

Proposition 4.16 : If L is a cocommutative pointed D-subcoalgebra of $M_D(\Lambda, \Lambda)$, then $L \subset H_D(\Lambda, \Lambda)$.

Proof :

Let I be a simple K-subcoalgebra of $L \otimes_D K$, and let $I' = \{i \in L : i \otimes 1 \in I\}$. Then, $0 \neq I'$ and I' is a D-subcoalgebra of L , hence there is a simple D-subcoalgebra $J = D.b \subset I'$. For, consider the set of all D-coalgebras contained in I' , then for any such D-coalgebra , $-\otimes_D K = I$ because I is a simple K-coalgebra. Now, the dual algebras of these D-coalgebras are all contained in I^* which is a finite dimensional K-algebra. Because D is a Dedekind domain , there is a maximal element among the D-algebras obtained in this way , say A . $J = A^*$ yields the desired simple D-coalgebra.

Thus, $J \otimes_D K = K.b \subset I$ and since I is simple , $K.b = I$, therefore every simple subcoalgebra of $L \otimes_D K$ is one dimensional and thus $L \otimes_D K$ is a pointed cocommutative K-subcoalgebra of $M_K(\Sigma, \Sigma)$, hence $L \otimes_D K \subset H_K(\Sigma, \Sigma)$. This finally entails that $L \subset H_D(\Lambda, \Lambda)$.

Proposition 4.17 : $H_D(\Lambda, \Lambda)$ is pointed.

Proof :

Let L be a simple D-subcoalgebra of $H_D(\Lambda, \Lambda)$, then $L \otimes_D K \subset H_K(\Sigma, \Sigma)$ a K -subcoalgebra. Since $H_K(\Sigma, \Sigma)$ is pointed, there exists an element $g \in G(H_K(\Sigma, \Sigma)) : K.g \subset L \otimes_D K$.

Let $L' = \{l \in L : l \otimes 1 \in K.g\}$, then L' is a nonzero D-subcoalgebra , hence $L = L'$. If we are able to prove that $g \in L \otimes_D D_p$ for all prime ideals p of D , then $g \in \cap L \otimes_D D_p = L$ and then $D.g \subset L$ is a D- subcoalgebra yielding that $D.g = L$, done.

Now, $L \otimes_D D_p$ is a finitely generated free D_p-module with basis say $\alpha_1, ..., \alpha_n$. Now, $\alpha_i = k.g$ for some $k \in K$, thus $\Delta(\alpha_i) = k.g \otimes g = k^{-1}.\alpha_i \otimes \alpha_i$. Since $L \otimes_D D_p$ is a D_p-coalgebra , $\Delta(\alpha_i) \in L \otimes_D D_p \otimes L \otimes_D D_p$ which has as a D_p-basis $\alpha_i \otimes \alpha_j$. Thus, finally, $k^{-1} \in D_p$, so $g \in D_p.\alpha_i \subset L \otimes_D D_p$.

Remark 4.18 : For all $m \in M_K(\Sigma, \Sigma)$, then there exists an element $d \in D$ such that $d.m : \Lambda \to \Lambda$, for , $\Lambda = D.\lambda_1 + ... + D.\lambda_n$ and $m(\lambda_i) = \sum k_{i,j}.\lambda_j$ with $k_{i,j} \in K$, so for all i we can find a suitable $d_i \in D$ such that $d_i.m(\lambda_i) \in \Lambda$. Finally, put $d = \times d_i$, then $d.m : \Lambda \to \Lambda$.

Theorem 4.19 : If $m \in H_K^1(\Sigma, \Sigma)$ (i.e. the pointed irreducible component of $H_K(\Sigma, \Sigma)$ with group like element 1 , cfr. [76]) , then there exist a $d \in D$, and a D-coalgebra $C \subset H_K(\Sigma, \Sigma)$ which is a f.g. D-module with $d.m \in C$ and C measures Λ to Λ.

Proof :

First, let $m \in C_n^+(H_K^1(\Sigma, \Sigma))$ (for notations and properties on the so called 'wedge'-terms the reader is referred to [76]).

$n = 1$: Then $\Delta(m) = m \otimes 1 + 1 \otimes m$. We may find a $d \in D$ with $d.m : \Lambda \to \Lambda$. Then, take $C = D.1 + D.(d.m)$. Then C measures Λ to Λ , $d.m \in C$ and $(C, \Delta \mid C, \epsilon \mid C)$ is a finitely generated D-coalgebra.

$n \geq 1$: Then $\Delta(m) = m \otimes 1 + 1 \otimes m + \sum n_i \otimes m_i$ with $n_i, m_i \in$

$C^+_{n-1}(H^1_K(\Sigma,\Sigma))$. By the induction hypothesis, we may find $d_i, d'_i \in D$ and C_i, C'_i finitely generated D-subcoalgebras measuring Λ to Λ such that $d_i.n_i \in C_i; d'_i.m_i \in C'_i$. Now, take $C' = \sum C_i + \sum C'_i$, then C' is a finitely generated D-subcoalgebra of $H_K(\Sigma,\Sigma)$ measuring Λ to Λ.

Further, there exists an element $d' \in D$ such that $d'.m : \Lambda \to \Lambda$. Now, take $d = d'. \times d_i. \times d'_i$ and $C = C' + D.d.m$, then C satisfies the requirements of the theorem.

Let $m \in C_n(H^1_K(\Sigma,\Sigma))$, then $m - \epsilon(m).1_\Sigma \in C^+_n(H^1_K(\Sigma,\Sigma))$, so there exist an element $d \in D$ and a subcoalgebra C with $d.(m - \epsilon(m).1_\Sigma) \in C$. Let $d'.\epsilon(m) \in D$, $dd' = d.d'$, then $dd'.m \in C$. Finally, $H^1_K(\Sigma,\Sigma) = \bigcup C_n(H^1_K(\Sigma,\Sigma))$ what finishes the proof.

Theorem 4.20 : In the situation of the foregoing theorem we have : $H^1_K(\Sigma,\Sigma) \subset H_D(\Lambda,\Lambda) \otimes_D K$.

Proof :

Let $m \in H^1_K(\Sigma,\Sigma)$, then by the foregoing theorem there is an element $d \in D$ and a finitely generated D-coalgebra $C \subset H_K(\Sigma,\Sigma)$ measuring Λ to Λ and $d.m \in C$. By the universal mapping property of $M_D(\Lambda,\Lambda)$ there is a D-coalgebra map F such that the diagram below is commutative :

$$\begin{array}{ccc} M_D(\Lambda,\Lambda) \otimes_D \Lambda & \rightarrow & \Lambda \\ \downarrow & \nearrow & \\ C \otimes_D \Lambda & & \end{array}$$

Hence, we may view $d.m$ as an element of $M_D(\Lambda,\Lambda)$ and since $d.m \otimes 1 = d.m \in H_K(\Sigma,\Sigma)$ we finally get that $d.m \in H_D(\Lambda,\Lambda)$. Finally, note that $d.m \otimes (1/d) \in H_D(\Lambda,\Lambda) \otimes_D K$.

Before we proceed with the study of the interrelation between $H_K(\Sigma,\Sigma)$ and $H_D(\Lambda,\Lambda)$ we aim to prove two results on $H_K(\Sigma,\Sigma)$ which are of some independent interest. They may be viewed as generalizations of the famous Skolem-Noether theorem which states that every K-derivation and K-automorphism of a central simple K-algebra Σ is inner.

Theorem 4.21 : Let B be a K-algebra and H be a pointed irreducible K-coalgebra with unique group-like 1 such that H measures B to B . For all natural numbers n and for all $m \in C_n^+(H)$, we may find a natural number k and an injection $\psi : B \to M_n(B)$ such that for every $b \in B$ $\psi(b)$ is an upper triangular matrix with constant diagonal element b , $\psi(b)_{1,k} = m(b)$ and $\psi(b)_{i,j} = p(b)$ with $p \in C_l^+(H)$, $l \prec n$ for all $j \prec i$.

Proof :

We proceed by induction on n .

$n = 1$: Recall from Sweedler [76] that $C_1^+(H) = P(H)$, the set of primitive elements of H. $m \in P(H)$ implies that m is a derivation on B , therefore we have an algebra morphism :

$$\psi_m : B \to M_2(B)$$

which is defned by :

$$a \to \begin{pmatrix} a & m(a) \\ 0 & a \end{pmatrix}$$

satisfying the requirements of the theorem.

$n > 1$: If $m \in C_n^+(H)$, we have $\Delta(c) = c \otimes 1 + 1 \otimes c + \sum p_i \otimes q_i$; with $p_i, q_i \in C_{n-1}^+(H)$. By the induction hypothesis we may find algebra monomorphisms ψ_{p_i}, ψ_{q_j} satisfying the requirements of the theorem.

$$\psi_{p_i} : B \to M_{k_i}(B)$$

$$\psi_{q_j} : B \to M_{l_j}(B)$$

Now, construct a mapping :

$$\psi_m : B \to M_k(B)$$

with $k = \sum(k_i + l_i) - 2.h + 1$ in the following way : let $k_0 = l_0 = 0$ and define now :

$v_a = \sum_{i=0}^{a} k_i + \sum_{j=0}^{a-1} l_j - 2.a + 1$

$w_a = \sum_{i=0}^{a}(k_i + l_i) - 2.a$

Let us now define :

$\psi_m(b)_{v_a+i, v_a+j} = \psi_{q_a}(b)_{i,j}$ for $1 \leq i \leq l_a - 1, 1 \leq j \leq l_a$

$\psi_m(b)_{w_a+i, w_a+j} = \psi_{p_a+1}(b)_{i,j}$ for $1 \leq i \leq k_{a+1}, 2 \leq j k_{a+1}$

$\psi_m(b) = b$, for all i

$\psi_m(b)_{1, w_a+i} = \psi_{p_a+1}(b)_{1,i}$ for $2 \leq i \leq k_{a+1}$

$\psi_m(b)_{k, v_a+i} = \psi_{q_a}(b)_{l_a, i}$ for $1 \leq i \leq l_{a-1}$

$\psi_m(b)_{1,k} = m(b)$

and every other entry will be zero. It is left as an easy but boring exercise to the reader to check that ψ_m is again an algebra monomorphism satisfying the requirements of the theorem.

Definition 4.22 : Let Σ be a central simple K-algebra A morphism $m \in End_K(\Sigma)$ is said to be inner , if there exist elements $a_i, a'_i \in \Sigma$ such that :

$m(a) = \sum a_i.a.a'_i$ for all $a \in \Sigma$

Theorem 4.23 (extended Skolem-Noether theorem)

All $m \in H_K(\Sigma, \Sigma)$ are inner.

Proof :

By a theorem of Kostant , $H_K(\Sigma, \Sigma) \simeq K.G \# H_K^1(\Sigma, \Sigma)$ where G is the set of all group-like elements of $H_K(\Sigma, \Sigma)$ and $H_K^1(\Sigma, \Sigma)$ is the pointed irreducible component of 1. The group-like elements are precisely the K-automorphisms of Σ and they are inner by the classical Skolem-Noether theorem. Therefore, it will be sufficient to prove that every $m \in H_K^1(\Sigma, \Sigma)$ is inner. If $m \in C_n(H_K^1(\Sigma, \Sigma))$ then $m' = m - \epsilon(m).1_\Sigma \in C_n^+(H_K^1(\Sigma, \Sigma))$, thus we can find a natural number k and an algebra morphism $\psi : \Sigma \to M_k(\Sigma)$ such that :

$$\psi(a) = \begin{pmatrix} a & & n_{i,j}(a) & m'(a) \\ & \ddots & & \\ & \underline{0} & \ddots & \\ & & & a \end{pmatrix}$$

Now, ψ is an isomorphism between Σ (embedded diagonally in $M_k(\Sigma)$) and $\psi(\Sigma)$, two simple subalgebras of the central simple K-algebra $M_k(\Sigma)$.

Furthermore, since n_{ij} and m' are in $C^+(H^1_K(\Sigma,\Sigma))$, ψ leaves K elementswise fixed , so by the Skolem-Noether theorem there exists an invertible $(x_{ij}) \in M_k(\Sigma)$ such that $\psi(a).x_{ij} = x_{ij}.a$ for all $a \in \Sigma$. For all $a \in \Sigma$ we have $a.x_{ni} = x_{ni}.a$ yielding that $x_{ni} \in K$ for every i. Since x_{ij} is invertible, we can find an index j such that $x_{nj} \neq 0$. Computing on both sides the product entry $(1,j)$ gives us :

$a.x_{1j} + \sum_{i=2}^{k-1} n_{1i}(a).x_{ij} + m'(a).x_{nj} = x_{1j}.a$,or

$m'(a) = x_{nj}^{-1}(x_{1j}.a - a.x_{1j} - \sum_{i=2}^{k-1} n_{1i}(a).x_{ij})$

Now, apply induction : $C_1^+(H^1_K(\Sigma,\Sigma))$ consists of derivations , hence they are inner , so we may assume that all n_{1i} are inner and thus m' is inner too. Finally, $m = m' + \epsilon(m).1_\Sigma$ and therefore m is inner.

Remark 4.24 : Actually, we proved that for any $m \in C^+(H^1_K(\Sigma,\Sigma))$ there exist elements $x_0, ..., x_n, y_1, ..., y_n \in \Sigma$ such that :

$m(a) = x_0.a - a.x_0 + \sum_{i=1}^{n}(x_i.a - a.x_i).y_i$ for every $a \in \Sigma$.

In view of theorem 4.20 one could expect that $H_D(\Lambda,\Lambda)$ is a (not necessarely finitely generated) D-order in $H_K(\Sigma,\Sigma)$, i.e. $H_D(\Lambda,\Lambda) \otimes_D K = H_K(\Sigma,\Sigma)$. But this is definitely not the case , even for maximal orders, as we will show in the next example :

Example 4.25 : Let Δ be a skewfield finite dimensional over its center $Z(\Delta)$ and let ψ be an automorphism of Δ such that ψ^n is inner , say $\psi^n(\delta) = x^{-1}.\delta.x$ for all $\delta \in \Delta$. The ring of skew polynomials $R = \Delta[X,\psi]$ is a maximal order over its center $Z(R) = k[t]$ where k is the subfield of $Z(\Lambda)$ which is left elementswise fixed under ψ and $t = x.X^n$. Now, for $H[\Delta,\psi] = H_{Z(R)}(R,R)$ to be a $Z(R)$-order in $H(\Delta,\psi) = H_{Z(Q)}(Q,Q)$ (where $Q = Q_{cl}(R)$) it is necessary and sufficient in view of theorem 4.20 and Kostant's theorem that $G(H[\Delta,\psi]) = G(H(\Delta,\psi))$.

$G(H(\Delta,\psi))$ is the group of all inner automorphisms of the skewfield $\Delta(X,\psi)$. Equality would mean that $\Delta[X,\psi]$ (and hence Δ) is globally invariant under every inner automorphism. By the Cartan-Hua-Brauer theorem , this would imply that either $\Delta \subset k(t)$ or $\Delta = \Delta(X,\psi)$, a contradiction unless $\Delta(X,\psi) = k(t)$. The reason why things do not work is that there are elements in $\Delta(X,\psi) - \Delta$ which are integral over k . E.g. if $\Delta = \mathbb{C}$ and if ψ is the complex conjugation , then $((X-i).i.(X-i))/(X^2-1)$ is integral over $\mathbb{R}$ in $\mathbb{C}(X,\psi)$.

Remark 4.26 : $G(H[\Delta,\psi])$ may be easily computed : if $\chi \in G(H[\Delta,\psi])$ it is easy to check that $\chi_0 = \chi \mid \Delta$ is an automorphism of Δ leaving k elementswise fixed. If $\chi(X) = a_m.X^m + \dots + a_1.X + a_0$, then , $\chi(t) = \chi_0(x).\chi(X)^n = t$ yielding that $\chi(X) = a_1.X$ and a_1 satisfies the following condition :

(*) :$\chi_0(x).a_1.\psi(a_1).\psi^2(a_1).\dots.\psi^{n-1}(a_1) = x$

Furthermore, $\chi_0(\psi(\delta)).a_1.X = \chi(\psi(\delta).X) = \chi(X.\delta) = a_1.X.\chi_0(\delta) = a_1.\psi(\chi_0(\delta)).X$, whence :

(**) : $\forall\delta \in \Delta : \psi o \chi_0(\delta) = a_1^{-1}.(\chi_0 o \psi(\delta)).a_1$.

Conversely, if χ_0 is an automorphism of Δ leaving k elementswise fixed and if $a_1 \in \Delta$ satisfies (*) and (**) then : $\chi(d_p.X^p + \dots + d_0) = \chi_0(d_p).(a_1.X)^p + \dots + \chi_0(d_0)$ determines an element of $G(H[\Delta,\psi])$. Therefore :

$$G(H[\Delta,\psi]) = \{(\chi_0, a_1) : \chi_0 \in G(H_k(\Delta,\Delta)), a_1 \in \Delta \text{ satisfying (*) and (**)}\}$$

multiplication being defined by :

$$(\chi_0, a_1).(\chi'_0, a'_1) = (\chi_0.\chi'_0, \chi_0(a'_1).a_1)$$

E.g. if $\Delta = \mathbb{C}$ and if ψ is the complex conjugation , then $G(H[\mathbb{C},\psi]) = \{(\chi_0, c) : \chi_0 \in \{1,\psi\}; c \in \mathbb{C} : \mid c \mid = 1\}$.

Theorem 4.27 : Let D be a Dedekind domain , K its field of fractions , Σ a central simple K-algebra and Λ a D-order in Σ. If G' is a finitely generated subgroup of $G(H_K(\Sigma,\Sigma))$, then, for all but a finite number of prime ideals P of D we have that $K.G' \# H^1_K(\Sigma,\Sigma) \subset H_{D_P}(\Lambda_P,\Lambda_P) \otimes K$.

Proof :

Let $G' = < \psi_1, ..., \psi_n >$. Each ψ_i is of the form $\psi_i(x) = a_i^{-1}.x.a_i, \forall x \in \Sigma$. We may find elements $d_i, d'_i \in D, \lambda_i \in \Lambda$ such that $a_i = (d_i/d'_i).\lambda_i$. If $d = \Phi d_i.d'_i \neq 0$ then all but a finite number of $P \in Spec(D)$ do not contain d , whence $\forall i$: $\psi_i(\Lambda_P) \subset \Lambda_P$. Thus, $G' \subset H_{D_P}(\Lambda_P,\Lambda_P)$ and theorem 4.20 finishes the proof.

Corollary 4.28 : If H' is a finitely generated K-subalgebra of $H_K(\Sigma,\Sigma)$ then $H' \subset H_{D_P}(\Lambda_P,\Lambda_P) \otimes K$ for all but a finite number of prime ideals P of D.

A prime ideal P of D is said to be a Hopf-prime for Λ if $H_{D_P}(\Lambda_P,\Lambda_P) \otimes K = H_K(\Sigma,\Sigma)$. We will end this section by relating Hopf-primes for orders in skewfields to valuation theory.

Definition 4.29 : A subring Γ of a skewfield Δ is called a valuation ring if it is invariant under every inner automorphism of Δ and if for every $x \in \Delta$, either $x \in \Gamma$ or $x^{-1} \in \Gamma$.

If Λ is a maximal D-order in a central K-skewfield then it is easy to check that Λ_P is a valuation ring if and only if it is invariant under every inner automorphism of Δ. As a consequence of this we have that P is a Hopf-prime for Λ if and only if Λ_P is a valuation ring ; i.e. if the P-adic valuation on K extends to a valuation on Δ.

Example 4.30 : Let D be a Dedekind domain such that its field of fractions is a global field and let Λ be a D-order in some central K-skewfield Δ. If Λ' is a maximal D- order in Δ containing Λ , then $\Lambda_P = \Lambda'_P$ for all but a finite number of prime ideals P of D. Because there are only a finite number of valuations on K which extend to Δ, there are only a finite number of Hopf-primes for Λ' and Λ.

Example 4.31 : (cfr. example 4.25) The only valuation rings in $\Delta(X, \psi)$ are he following :

$\Delta[X, \psi]_{(X)}$; $\Delta[X^{-1}, \psi^{-1}]_{(X^{-1})}$; $\Delta[X, \psi]_P$ where P is a central irreducible element. Therefore the Hopf-primes for $\Delta[X, \psi]$ correspond precisely to the central irreducible elements.

E.g. $\{(X^2 + c); c > 0\}$ is the set of Hopf-primes in $\mathbb{R}[X^2]$ for $\mathbb{C}[X, \psi]$ where ψ denotes the complex conjugation.

Example 4.32 : Let $D = \cap\{k[t]_P; P$ cental irreducible $\}$, then D is a Dedekind domain which is not semilocal. If Λ is any D-order in $\Delta(X, \psi)$ then all but a finite number of prime ideals of D are Hopf-primes for Λ.

c : extension to Krull domains.

The aim of this section is , rather than redoing everything for Krull domains instead of Dedekind domains , to present a method how most results of the foregoing section can be generalized. The rather trivial (but handy) observation behind this method is that a Krull domain can be viewed as a sheaftheoretic version of a discrete valuationring. Let us make this more precise (cfr. also II.2 for sheaftheoretic notions) : with $Spec(R)$ we will denote as usual the set of all prime ideals of R equipped with the Zariski topology. On $X^{(1)}(R)$ we put the induced topology. It is fairly easy to check from the finite character property that this induced toplology is merely the cofinite topology on $X^{(1)}(R)$. On $Spec(R)$ we put the usual structure sheaf of R , cfr. e.g. [20] , $\mathcal{O}_R$. With $\mathcal{O}_R^{(1)}$ we will denote the inverse image $i^*(\mathcal{O}_R)$ where :

$$i : X^{(1)}(R) \to Spec(R)$$

is the canonical inclusion. If $X(I) \cap X^{(1)}(R)$ is a canonical open set of $X^{(1)}(R)$ where I is an ideal of R , then it is not difficult to show that :

$$\Gamma(X(I) \cap X^{(1)}(R), \mathcal{O}_R^{(1)}) = \Gamma(X(I), \mathcal{O}_R)$$

since $\mathcal{O}_R$ is a sheaf of Krull domains and therefore its sections are determined by the stalks of $\mathcal{O}_R$ in prime ideals of height one.

Similarly, we will define for every R-module (resp. R-algebra) M (resp. A) the sheaf of modules (resp. of algebras) $\mathcal{O}_M^{(1)}$ (resp. $\mathcal{O}_A^{(1)}$) over $X^{(1)}(R)$ to be $i^*(\mathcal{O}_M)$ (resp. $i^*(\mathcal{O}_A)$). Of course , the sheafs $\mathcal{O}_M^{(1)}$ and $\mathcal{O}_A^{(1)}$ do no longer determine M and A completely. E.g. if M is an R-lattice , then $\mathcal{O}_M^{(1)} \simeq \mathcal{O}_{M^{**}}^{(1)}$. Nevertheless, reflexive R-lattices and R-algebras are completely determined by their sheafs over $X^{(1)}(R)$ by taking global sections.

Another noteworthy fact is that for R-lattices M and N :

$$\Gamma(X^{(1)}(R), \mathcal{O}_{M \otimes_R N}^{(1)}) = M \otimes_R' N$$

We will now give an example how to use this dictionary , namely the construction of the reflexive R-coalgebra A^o (i.e. with the modified tensor product in the diagrams defining the comultiplication and the counit) associated with a reflexive R-algebra A.

Let $\mathcal{HOM}_{\mathcal{O}_R^{(1)}}(\mathcal{O}_A^{(1)}, \mathcal{O}_R^{(1)})$ be the sheaf of homomorphisms from $\mathcal{O}_A^{(1)}$ to $\mathcal{O}_R^{(1)}$ (cfr. e.g. [20]) and define the subsheaf $\mathcal{A}^o$ by its sections on an open set U :

$$\Gamma(U, \mathcal{A}^o) = \{g \in \Gamma(U, \mathcal{HOM}) \mid (\mathcal{O}_A^{(1)} \mid U)/(\mathcal{KER}(g) \mid U) \text{ is a vector bundle }\}$$

i.e. a locally free sheaf of modules of finite type over $\mathcal{O}_R^{(1)} \mid U$.

It is rather straightforward to verify that this defines indeed a sheaf and if we denote with A^o its global sections , then we have for every $p \in X^{(1)}(R)$:

$$(A^o)_p = (\mathcal{A}^o)_p = (A_p)^o$$

where $(A_p)^o$ is defined as in 4.1 . Therefore, A^o is a reflexive R-module.

A sheaftheoretic analogon of proposition 4.3 is now easily derived by constructing the sheafmaps and a verification that the assumptions hold in every stalk.

Again taking global sections yields that A^o is a reflexive R-coalgebra.

d : some Galois theory for Dedekind domains.

In this section we apply the foregoing in order to obtain a rather satisfactory Galois theory for Dedekind domains. Throughout we will consider the following situation. D is a Dedekind domain having K as its field of fractions, E will be another Dedekind domain with field of fractions L such that E is a finitely generated D-module (hence E is the integral closure of D in L).

If (H, Δ, ϵ) is a D-coalgebra and $\psi : H \otimes_D E \to E$ a D-measuring , then $(H \otimes_D K, \Delta_K, \epsilon_K)$ is a K-coalgebra and $\psi_K : (H \otimes_D K) \otimes_K L \to L$ a K-measuring with :

$\Delta_K : H \otimes_D K \to H \otimes_D H \otimes_D K : h \otimes k \to \Delta(h) \otimes k$

$\epsilon_K : H \otimes_D K \to K : h \otimes k \to k.\epsilon(h)$

$\psi_K : (H \otimes_D K) \otimes_K L \to L : h \otimes k \otimes k'.e \to k.k'.\psi(h \otimes e)$

It is easy to check that all these maps are well defined.

Definition 4.33 : Define the fixed elements of an algebra A under a coalgebra C which measures A to A to be the set :

$$A^C = \{a \in A \mid c(a) = \psi(c \otimes a) = \epsilon(c).a; \forall c \in C\}$$

Proposition 4.34 : In the above situation we have : E^H is the integral closure of D in $L^{H \otimes_D K}$.

Proof :

Let L' be the field of fractions of E^H. Then $L' \subset L^{H \otimes_D K}$, because if $l' = d/d' \in L'$, then : $\psi(h \otimes k) \otimes (d/d')) = k/d'.\psi(h \otimes d) = k/d'.\epsilon(h).d = \epsilon(h \otimes k).d/d'$. Now suppose that $L' \subset L^{H \otimes_D K}$ and that this inclusion is proper. Let D' be the integral closure of D in $L^{h \otimes_D k}$, we have $D \subset E$ and for every $d' \in D'$, $h \in H$:

$$\psi(h \otimes d') = \psi_K(h \otimes 1 \otimes d') = \epsilon_K(h \otimes 1).d' = \epsilon(h).d'$$

yielding that $D' \subset E^H$ but this contradicts that $L' \subset L^{H \otimes_D K}$ were proper and therefore we obtain that $L' = L^{H \otimes_D K}$. Conversely, if $x \in L^{H \otimes_D K}$ and x is integral over D then $x \in E$ and for every $h \in H$: $\psi(h \otimes x) = \epsilon(h).x$ yielding that $x \in E^H$.

Proposition 4.35 : $H_D(E, E)$ is a D-order in $H_K(L, L)$.

Proof :

In the foregoing section we have established the inclusion : $H^1_K(L,L) \subset H_D(E,E) \otimes_D K$. Further, by a theorem of Kostant we have that $H_K(L,L) = K.G \# H^1_K(L,L)$, where G is the group of group-like elements of $H_K(L,L)$ and $\#$ denotes the smashed product , cfr. [76]. So, it remains to prove that $G \subset H_D(E,E)$.

If $g \in G$, then g is a K-automorphism of L. If $e \in E$, then there exist elements $d_0, \ldots, d_n \in D$ such that : $d_n.e^n + \ldots + d_1.e + d_0 = 0$, hence , $d_n.g(e)^n + \ldots + d_1.g(e) + d_0 = 0$. Since E is the integral closure of D in L we have that $g(e) \in E$. Thus $D.G$ is a

cocommutative pointed measuring bialgebra and by the universal property of $M_D(E,E)$ we then have that $D.G \to M_D(E,E)$ is an inclusion. Finally, because $D.G$ is pointed, $D.G$ maps into $H_D(E,E)$.

Now, let $H \subset H_D(E,E)$ then by definition : $H \otimes_D K \subset H_K(L,L)$ and this entails by the theorem of Kostant that : $H \otimes_D K \simeq K.G(H \otimes_D K) \# H^1_K(H \otimes_D K)$.

Now, we are able, as in the foregoing section to prove that $G(H) = G(H \otimes_D K)$. Put, $H^1 = \{h \in H : h \otimes 1 \in H^1(H \otimes_D K)\}$. Clearly, H^1 is a D-subcoalgebra of H and $H^1 \otimes_D K = H^1(H \otimes_D K)$.

Definition 4.36 : Let E be a ringextension of D such that E is a finitely generated D-module.

E is called a Galois extension with Galois group G if there is a representation of G by D-automorphisms of E leaving D as the fixed ring.

E is called a purely inseparable extension if for every $x \in E$ there is a natural number e with $x^{p^e} \in D$, p being the characteristic of D.

Remark 4.37 : Our definition of a Galois extension is not the same as the one given in De Meyer-Ingraham [19] . E.g. the extension $\mathbb{Z}[\sqrt{2}]$ of $\mathbb{Z}$ is Galois in the sense of the foregoing definition but is not in the sense of [19].

Theorem 4.38 : (Galois theorem for Dedekind domains)

Let D be a Dedekind domain of characteristic p , H a cocommutative bialgebra measuring a Dedekind extension E of D , $H \subset H_D(E,E)$, $G = G(H)$ and H^1 as above , then :

(a) : E^{H^1} is Galois over E^H;

(b) : $E^{D.G}$ is purely inseparable over E^H;

(c) : E and $E^{H^1} \otimes_{E^H} E^{D.G}$ have the same field of fractions L.

Proof :

(a) : By Prop.4.34 ,E^{H^1} is the integral closure of D in $L^{H^1_K(H \otimes_D K)}$. Now by a result of [a1] , we know that $L^{H^1_K(H \otimes_D K)}$ is a Galois field extension over $L^{H \otimes_D K}$. Hence, there are $L^{H \otimes_D K}$-auromorphisms of $L^{H^1_K(H \otimes_D K)}$ leaving exactly $L^{H \otimes_D K}$ fixed. As in the proof of Prop.d.2 we know that all these automorphisms map E^{H^1} into itself. Thus, the elements of E^{H^1} which are fixed under all these automorphisms form precisely E^H.

(b) : $E^{D.G}$ is the integral closure of D in $L^{K.G}$. Now, again using a result of Sweedler's [a1] we know that $L^{K.G}$ is purely inseparable over $L^{H \otimes_D K}$. If $x \in E^{D.G}$ then there exist elements $d_{n-1}, \ldots, d_0 \in D$ such that : $x^n + \ldots + d_1.x + d_0 = 0$ hence there is a natural number such that $x^{p^e} \in L^{H \otimes_D K}$ and furthermore , $(x^n + \ldots + d_0)^{p^e} = 0$ yielding that x^{p^e} is in the integral closure of D in $L^{H \otimes_D K}$, i.e. in E^H.

(c) : Because $L^{H^1_K(H \otimes_D K)}$ and $L^{K.G}$ are lineary independent over $L^{H \otimes_D K}$ there is an isomorphism $E^{H^1} \otimes E^{D.G} \simeq E^{H^1}.E^{D.G}$. Finally, the field of fractions of $E^{H^1}.E^{D.G}$ equals $L^{H^1_K(H \otimes_D K)}.L^{K.G} = L$.

PART III : THE NORMALIZING CLASSGROUP OF A MAXIMAL ORDER :

1. INTRODUCTION :

In (commutative) algebraic geometry , divisors are used to study the intrinsic geometrical properties of schemes , cfr. e.g. [28].

These geometrical concepts were used in a very elegant way by V.I. Danilov , cfr. [17] and [18] , in order to study the relation between the classgroup of a normal domain R and the classgroup of $R[[t]]$, the ring of formal power series over it , a problem which has its roots in a conjecture of P. Samuel , cfr. e.g. [22].

The strategy he uses is the following : first, one may express the classgroup of a normal domain in terms of the Picard groups of certain open subvarieties of $Spec(R)$. The next step is then to use the good functorial and cohomological properties of these Picard groups in order to prove the desired theorems on the open sets and afterwards Danilov pulls the obtained information back to the classgroup. In this way Danilo was able to define a natural splitting morphism for the inclusion :

$$Cl(R) \to Cl(R[[t]])$$

and to give some necessary and sufficient conditions on the normal domain R in order to ensure that this morphism is an isomorphism.

In this chapter we try to generalize some of these results to the normalizing classgroup of maximal orders over Krull domains.

In section two we recall the constructions of the affine schemes $O_\Lambda^{(nc)}$ and $O_\Lambda^{(bi)}$ associated to an order Λ , due to F. Van Oystaeyen and A. Verschoren , cfr. [83] and [90].

Usually these schemes are quite different causing some problems with respect to their functorial behaviour. Using a result of Chamarie's [13] , we show that both schemes coincide if Λ is a maximal order over a Krull domain R. Moreover, in this case there is a good connection between these rather obscure noncommutative schemes and the usual central scheme of the R- algebra Λ , cfr. e.g. [20].

In the third section we generalize the notions of Weil and Cartier divisors to these noncommutative schemes and study their interrelation. This appraoch enables us to generalize Danilov's main tool , i.e. expressing the classgroup in terms of Picard groups of open subvarieties , to the normalizing classgroup $Cl(\Lambda)$ of a maximal order Λ.

Furthermore, this yields a cohomological interpretation of this classgroup and conversely we present ringtheoretical interpretations (such as the type number and the genera of a maximal order) to the cohomology pointed sets occuring in this description.

In the last section we will apply this machinery to a chartacterization of those locally factorial Krull domains R with field of fractions K for which all maximal R-orders in $M_n(K)$ are conjugated , as well as to study the relation between $Cl(\Lambda)$ and $Cl(\Lambda[[t]])$.

2. : THE AFFINE SCHEME OF A MAXIMAL ORDER

This section is devoted to the construction of structure sheaves associated to orders. The first attempt to define a structure sheaf of a left Noetherian ring is due to Murdoch and Van Oystaeyen [56],cfr. also [83].The main advantedge of their approach (contrary to the Golan-Raynaud-Van Oystaeyen sheaf of [27]) is that one recovers the ring by taking global sections. This fact enables us to study the ring in a local-global manner,cfr. e.g. [82]. This sheaftheoretic machinery developes into 'non-commutative algebraic geometry,an introduction',[90], of F. Van Oystaeyen and A. Verschoren. In this work (mostly concerned with the p.i. case) two types of sheaves seem to be of interest , namely the module- and the bimoduletype corresponding to whether the localization used is localization in Λ-mod or the relative localization in bi(Λ) , the category of Artin bimodules , i.e. twosided Λ-modules M such that $M = \Lambda.Z_\Lambda(M)$ where $Z_\Lambda(M) = \{m \in M : \forall\lambda \in \Lambda : \lambda.m = m.\lambda\}$ cfr e.g. [62]. The bimodule sheaf is most likely to behave in a nice functorial way with respect to ring extensions in the sense of C. Procesi [62], i.e. ringextensions $\Lambda \subset \Gamma$ such that $\Gamma = \Lambda.Z_\Lambda(\Gamma)$, whereas the module sheaf contains more information than the former.

We will briefly scetch the construction for an R-order Λ over a Krull domain R in some central simple K-algebra Σ , for more details the reader is referred to [90]. As a topological space we take $Y = Spec(\Lambda)$, the set of all twosided prime ideals of Λ equipped with the usual Zariski topology. That is, a typical open set is of the form $Y(I) = \{P \in Y : I \not\subseteq P\}$ for some (twosided) ideal $I$of Λ . On Y we define a presheaf of rings $\mathcal{O}_\Lambda^{(nc)}$ in the following way

$$\Gamma(Y(I), \mathcal{O}_\Lambda^{(nc)}) = \{q \in \Sigma : \exists L \in \mathcal{L}(I) : L.q \subseteq \Lambda\}$$

where $\mathcal{L}(I) = \{L \leq_l \Lambda : \exists J \leq \Lambda, I \subset radJandJ \subset L\}$. If this filter defines an idempotent kernel functor , then $\Gamma(Y(I), \mathcal{O}_\Lambda^{(nc)}) = Q_I(\Lambda)$. Of course, restriction morphisms are inclusions.

The following facts are now easily verified :

(1) :$\mathcal{O}_\Lambda^{(nc)}$ is actually a sheaf of maximal orders ;

(2) : the stalk of $\mathcal{O}_\Lambda^{(nc)}$ at a point P equals $Q_{\Lambda-P}(\Lambda)$, the Murdoch-Van Oystaeyen localization at a prime ideal P , i.e. $Q_{\Lambda-P}(\Lambda) = \{q \in \Sigma : \exists I \leq_2 \Lambda, I \not\subset P, I.q \subset \Lambda\}$;

(3) : the ring Λ may be recovered as the ring of global sections , i.e. $\Gamma(X, \mathcal{O}_\Lambda^{(nc)}) = \Lambda$.

Lemma 2.1 : If Λ is a maximal R-order in Σ and if U is an open set of Y such that $X^{(1)}(\Lambda) \subset U$, then $\Gamma(U, \mathcal{O}_\Lambda^{(nc)}) = \Lambda$.

Proof For any $P \in X^{(1)}(\Lambda), \Lambda \subset \Gamma(U, \mathcal{O}_\Lambda^{(nc)}) \subset Q_{\Lambda-P}(\Lambda)$, whence $\Lambda \subset \Gamma(U, \mathcal{O}_\Lambda^{(nc)}) \subset \cap\{Q_{\Lambda-P}(\Lambda); P \in X^{(1)}(\Lambda)\}$, so we are left to prove that this intersection equals Λ .

So let $q \in \cap\{Q_{\Lambda-P}(\Lambda); P \in X^{(1)}(\Lambda)\}$, then for any $P \in X^{(1)}(\Lambda)$ there exists an ideal I_P such that $I_P.q \subset \Lambda$, whence $\sum I_P.q \subset \Lambda$. Finally , $\sum I_P$ is an ideal which is not contained in any minimal prime ideal and therefore $\Lambda.q = (\sum I_P)^d.q \subset \Lambda$ whence $q \in \Lambda$. Note that $I^d = (I : \Lambda) : \Lambda$.

Important notational remark 2.2 :

Some caution is in order if Λ is not Noetherian , for then the filters $\mathcal{L}(I)$ need not be a priori idempotent. We will recall here Chamarie's approach to bypass this problem , see [13] Chap.IV and Chap.V for more details :

If $\mathcal{L}^2(\sigma)$ is a multiplicatively closed filter of non-zero twosided ideals of Λ , then one can define :

$$\Lambda_{(\sigma)} = \{x \in \Sigma \mid \exists I \in \mathcal{L}^2(\sigma) : I.x \subset \Lambda\}$$

If we define a filter of right ideals $\mathcal{L}(\sigma^r)$, to be the set of all essential right ideals of Λ such that : $\forall x \in \Lambda \mid \overline{(I : x)}$ contains an ideal of Λ , where $\overline{(.)} = ((.)^{-1})^{-1}$, then by [13] lemme 4.2.2 this filter is idempotent and $\Lambda_{(\sigma)} = Q_{\sigma^r}(\Lambda)$.

In a similar way we may define an idempotent filter of left ideals $\mathcal{L}(\sigma^l)$ such that $\Lambda_{(\sigma)} = Q_{\sigma^l}(\Lambda)$.

By abuse of notation we will then denote :

$$Q_\sigma(\Lambda) = \Lambda_{(\sigma)}$$

It is clear that whenever $\mathcal{L}^2(\sigma) \subset \mathcal{L}^2(\tau)$, then $\mathcal{L}(\sigma^r) \subset \mathcal{L}(\tau^r)$ and $\mathcal{L}(\sigma^l) \subset \mathcal{L}(\tau^l)$ which is necessary in order to have a well defined restriction morphism.

As mentioned before, the assignment $\Lambda \vdash (Y, \mathcal{O}_\Lambda^{(nc)})$ does not have nice functorial properties in general. Even if $f : \Lambda \to \Gamma$ is an extension of rings in the sense of C. Procesi , then this extension does not induce in general a morphism between the ringed spaces $(Y_\Lambda, \mathcal{O}_\Lambda^{(nc)})$ and $(Y_\Gamma, \mathcal{O}_\Gamma^{(nc)})$. This flow may be avoided by introducing so called bimodule structure sheaves $\mathcal{O}_\Lambda^{bi}$. These sheaves are constructed by means of bimodule localizations as in [62]. Nevertheless one can define $\mathcal{O}_\Lambda^{bi}$ roughly to be the sheafification of the presheaf with sections $\Lambda . Z_\Lambda(\Gamma(U, \mathcal{O}_\Lambda^{(nc)}))$ on the open set U and these bimodule sheaves have a functorial behaviour with respect to extensions. The next lemma (which is essentially due to M.Chamarie) is therefore of crucial importance for the functorial properties of Cartier and Weil divisors defined later on.

Proposition 2.3 : If Λ is a maximal R-order, then $\mathcal{O}_\Lambda^{(nc)} \simeq \mathcal{O}_\Lambda^{bi}$.

Proof : Clearly , $\mathcal{O}_\Lambda^{bi}$is a subsheaf of $\mathcal{O}_\Lambda^{(nc)}$, therefore it is sufficient to check that their stalks are isomorphic at every point $P \in Spec(\Lambda)$. Now, $Q_{\Lambda - P}(\Lambda) = \Lambda \otimes R_p$where $p = P \cap R$, R the center of Λ , yielding that $Q_{\Lambda-P}(\Lambda) = Q^{bi}_{\Lambda - P}$ finishing the proof.

Remark 2.4 :

If Λ is a maximal R-order , then one can define the structure sheaf of a divisorial Λ-ideal A , $\mathcal{O}_A^{(nc)}$ in a similar manner. For any open set $Y(I)$ we have a multiplicatively closed filter of twosided ideals $\mathcal{L}(I)$ and associated to it the idempotent filters of one-sided ideals $\mathcal{L}(I^r)$ and $\mathcal{L}(I^l)$ defined above. By [13] , proof of lemme 4.2.2 , we know that $Q_{I^r}(A) = Q_{I^l}(A)$ is a divisorial ideal of $Q_I(\Lambda)$. Again, by abuse of notation we will write :

$$Q_I(A) = Q_{I^r}(A) = Q_{I^r}(A)$$

and take for the sections of $\mathcal{O}_A^{(nc)}$ on the open set $Y(I)$:

$$\Gamma(Y(I), \mathcal{O}_A^{(nc)}) = Q_I(A)$$

Again, restriction morphisms will be inclusions. $\mathcal{O}_A^{(nc)}$ is readily checked to be a sheaf of divisorial ideals with stalks $Q_{\Lambda-P}(A)$.

Although Weil and Cartier divisors associated to Λ must be defined on these noncommutative structure sheafs, we will usually benefit from the relation between $\mathcal{O}_\Lambda^{(nc)}$ and the central scheme of Λ , $\mathcal{O}_\Lambda$, given below which simplifies matters a lot.

With $\mathcal{O}_R$ we will denote the usual structure sheaf of R over $X = Spec(R)$ and $\mathcal{O}_\Lambda$ denotes the structure sheaf of Λ over $X = Spec(R)$ as defined e.g. in [20].Before studying upon the relation between $\mathcal{O}_\Lambda^{(nc)}$ and $\mathcal{O}_\Lambda$, let us recollect the definition of the direct image and the inverse image of a sheaf , cfr. e.g. [26].

Let X and Y be two topological spaces and let $f : X \to Y$ be a continous morphism. Let $\mathcal{A}$ and $\mathcal{B}$ be two sheaves (of groups, of rings,...) over X and Y resp. We call a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ compatible with f every continous morphism g (of groups,of rings,...) from the étaled spaces $\mathcal{A}$ to $\mathcal{B}$ such that the diagram below is commutative :

$$\begin{array}{ccc} \mathcal{A} & \rightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ X & \rightarrow & Y \end{array}$$

We say that g is an f-homomorphism from $\mathcal{A}$ to $\mathcal{B}$. In [26] it is proved that for given X,Y,f and $\mathcal{B}$, there exists a sheaf (of groups, of rings,...) over X , $f^*(\mathcal{B})$, and an f-homomorphism $f^*(\mathcal{B}) \to \mathcal{B}$ such that every f-homomorphism $\mathcal{A} \to \mathcal{B}$ factorizes through a homomorphism of sheaves (of groupes, of rings, ...) $\mathcal{A} \to f^*(\mathcal{B})$. $f^*(\mathcal{B})$ is said to be the inverse image of $\mathcal{B}$.

Let $f : X \to Y$ be a continous morphism between topological spaces and let $\mathcal{A}$ be a sheaf (of groups, of rings,...) over X. We can define a sheaf (of groups,of rings,...) , $f_*(\mathcal{A})$ over Y by taking for its sections :

$$\Gamma(V, f_*(\mathcal{A})) = \Gamma(f^{-1}(V), \mathcal{A})$$

for every open set V of Y. It is straightforward to check that $f_*(\mathcal{A})$ is actually a sheaf. $f_*(\mathcal{A})$ is said to be the direct image of $\mathcal{A}$ under the continuous morphism f. The assignment : $\mathcal{A} \vdash f_*(\mathcal{A})$ is a covariant functor from the category of sheaves (of groups, of rings,...) over X to the category of sheaves (of groups, of rings, ...) over Y.

Now, the ringextension $R \subset \Lambda$ induces a continuous morphism $i : Y \to X$ defined by $i(P) = P \cap R$.

Proposition 2.5 : If Λ is a maximal R-order in a central simple K-algebra Σ ,then with notations as before :

(1) : $i_*(\mathcal{O}_\Lambda) \simeq \mathcal{O}_\Lambda^{(nc)}$;

(2) : $i^*(\mathcal{O}_\Lambda^{(nc)}) \simeq \mathcal{O}_\Lambda$.

Proof :

It follows from the result of M. Chamarie cited above that :

$$Q_{\Lambda-P}(\Lambda) = Q^{bi}_{\Lambda-P}(\Lambda) = \Lambda.R_p$$

where $p = P \cap R$. Therefore, $i_*(\mathcal{O}_\Lambda)$ is a subsheaf of $\mathcal{O}_\Lambda^{(nc)}$ and all their stalks are isomorphic , whence $i_*(\mathcal{O}_\Lambda) \simeq \mathcal{O}_\Lambda^{(nc)}$. Similarly, $\mathcal{O}_\Lambda$ is a subsheaf of $i^*(\mathcal{O}_\Lambda^{(nc)})$ and all their stalks are isomorphic, finishing the proof.

3 : CARTIER AND WEIL DIVISORS OF MAXIMAL ORDERS

In this section we will introduce Cartier and Weil divisors associated to a maximal R-order Λ and study their relation. In the third subsection we will present a ringtheoretical interpretation of some cohomology pointed sets which appear in this study.

(1) : CARTIER DIVISORS :

a. : Cartier divisors on noncommutative schemes

Throughout this section, Λ will be a maximal R-order in a central simple K-algebra Σ. We will consider the following two sheaves of not necessarely Abelian groups.

A : The sheaf of units , $\mathcal{U}_\Lambda^{(nc)}$, which is defined in the obvious way , i.e. $\Gamma(V, \mathcal{U}_\Lambda^{(nc)}) = U(\Gamma(V, \mathcal{U}_\Lambda^{(nc)}))$ for every open set V of Y and restriction morphisms are inclusions. It is straightforward to check that $\mathcal{U}_\Lambda^{(nc)}$ is indeed a sheaf.

B : The sheaf of normalizing elements , $\mathcal{N}_\Lambda^{(nc)}$, which is defined by : $\Gamma(V, \mathcal{N}_\Lambda^{(nc)}) = N(\Gamma(V, \mathcal{O}_\Lambda^{(nc)})) = \{q \in \Sigma : \Gamma(V, \mathcal{O}_\Lambda^{(nc)}).q = q.\Gamma(V, \mathcal{O}_\Lambda^{(nc)})\}$ and restriction morphisms are inclusions.

In order to check that $\mathcal{N}_{\Lambda}^{(nc)}$ is a sheaf, we need the following technical result

Lemma 3.1 : Let Λ be a maximal R-order and let $\mathcal{L}(\sigma) \subset \mathcal{L}(\tau)$ be two multiplicatively closed filters of ideals of Λ and $q \in N(Q_\sigma(\Lambda))$ then $q \in N(Q_\tau(\Lambda))$.

Proof :

Passing from σ (resp. τ) to σ^l (resp. τ^l) as defined in the remarks 2.2 and 2.4 we may suppose that the filters of left ideals $\mathcal{L}(\sigma^l) \subset \mathcal{L}(\tau^l)$ are both idempotent.

By a result of Chamarie's (proof of lemme 4.2.2) we know that the localization maps $Q_{\sigma^l}(.)$ and $Q_{\tau^l}(.)$ induce groupepimorphisms :

$$Q_{\sigma^l}(.) : \mathcal{D}(\Lambda) \to \mathcal{D}(Q_\sigma(\Lambda))$$

$$Q_{\tau^l}(.) : \mathcal{D}(\Lambda) \to \mathcal{D}(Q_\tau(\Lambda))$$

Throughout we will use the notation described in 2.4 .

Therefore, if $q \in N(Q_\sigma(\Lambda))$ then there exists a divisorial Λ-ideal M such that $Q_\sigma(M) = Q_\sigma(\Lambda).q,$, so it will be sufficient to prove that $Q_\tau(M) = Q_\tau(\Lambda).q$. Let $x \in Q_\tau(M)$ then there exists an ideal $I \in \mathcal{L}^2(\tau)$ such that I.x $\subset$ M $\subset Q_\sigma(M) = Q_\sigma(\Lambda).q$. Clearly, q being a normalizing element of the Goldie ring $Q_\sigma(\Lambda)$, q is invertible in Σ whence : $I.x.q^{-1} \subset Q_\Sigma(\Lambda) \subset Q_\tau(\Lambda), i.e. x.q^{-1} \in Q_\tau(\Lambda)$ and thus $x \in Q_\tau(\Lambda).q$.

Conversely, if $x \in Q_\tau(\Lambda)$ then $I.x \subset A$ for some ideal $I \in \mathcal{L}^2(\tau)$ whence $I.x.q \subset A.q \subset Q_\sigma(M) \subset Q_\tau(M)$ finishing the proof.

Proposition 3.2 : $\mathcal{N}_{\Lambda}^{(nc)}$ is a sheaf of groups and the stalk of $\mathcal{N}_{\Lambda}^{(nc)}$ at a point P equals $N(Q_{\Lambda-P}(\Lambda))$.

Proof : In view of the foregoing lemma , $\mathcal{N}_{\Lambda}^{(nc)}$ is a presheaf which is clearly separated (since all restriction morphisms are inclusions). Therefore we are left to prove the gluing property. So, let $\{U_i; i \in I\}$ be an open covering of U and let $q \in \Gamma(U_i, \mathcal{N}_\Lambda^{(nc)})$ for every $i \in I$. Then, $q.\Gamma(V, \mathcal{O}_\Lambda) = q.(\cap\Gamma(U_i, \mathcal{O}_\Lambda)) = \cap(q.\Gamma(U_i, \mathcal{O}_\Lambda)) = \cap(\Gamma(U_i, \mathcal{O}_\Lambda).q) = \Gamma(V, \mathcal{O}_\Lambda).q$ whence $q \in \Gamma(V, \mathcal{O}_\Lambda)$.

Finally, let us calculate the stalk of $\mathcal{N}_\Lambda^{(nc)}$ at a point P. Clearly, $(\mathcal{N}_\Lambda^{(nc)})_P \subset N(Q_{\Lambda-P}(\Lambda))$ by the foregoing lemma. Conversely, let $q \in N(Q_{\Lambda-P}(\Lambda))$, then there exists a divisorial Λ-ideal M such that $Q_{\Lambda-P}(M) = Q_{\Lambda-P}(\Lambda).q$. Therefore, $(\mathcal{O}_M)_P = Q_{\Lambda-P}(\Lambda).q$ and likewise $(\mathcal{O}_{M^{-1}})_P = Q_{\Lambda-P}(\Lambda).q^{-1}$. Choose a neighborhood V of P such that $q \in \Gamma(V, \mathcal{O}_M)$ and $q^{-1} \in \Gamma(V, \mathcal{O}_{M^{-1}})$. Then, $q^{-1}.\Gamma(V, \mathcal{O}_\Lambda).q \subset q^{-1}.\Gamma(V, \mathcal{O}_M) \subset \Gamma(V, \mathcal{O}_\Lambda)$ whence $\Gamma(V, \mathcal{O}_\Lambda).q \subset q.\Gamma(V, \mathcal{O}_\Lambda)$ and likewise $q.\Gamma(V, \mathcal{O}_\Lambda) \subset \Gamma(V, \mathcal{O}_\Lambda).q$yielding that $q \in \Gamma(V, \mathcal{N}_\Lambda^{(nc)})$, finishing the proof.

If R is a commutative Krull domain, $\mathcal{N}_R$ is of course the constant sheaf with sections K^* , the nonzero elements of the field of fractions K of R. If A is not commutative , $\mathcal{N}_\Lambda^{(nc)}$ is not necessarely constant as the following example shows :

Example 3.3 : Let $A = \mathbb{C}[X, -]$ where - denotes the complex conjugation . Then A is a p.i. Dedekind ring with center $\mathbb{R}[X^2]$. In [79] it is proved that $\{x^2 + c; c > 0\}$ is precisely the set of those prime ideals of $\mathbb{R}[X^2]$ such that the valuation extends to a valuation in $\mathbb{C}(X, -)$. This implies that for the corresponding prime ideals of A , $(\mathcal{N}_\Lambda^{(nc)})_P = \mathbb{C}(X, -)$. Now, suppose $\mathcal{N}_\Lambda^{(nc)}$ were constant , then $N(\Lambda) = \mathbb{C}(X, -)$. Combining results of [38] and [39] this would entail that every localization at a prime ideal is a valuationring, a contradiction .

Later on we will show that $\mathcal{N}_\Lambda^{(nc)}$ is not a constant sheaf unless all maximal R-orders in Σ are conjugated.

Clearly, $\mathcal{U}_\Lambda^{(nc)}$ is a normal subsheaf of $\mathcal{N}_\Lambda^{(nc)}$, so we can form its quotient sheaf $\mathcal{C}_\Lambda^{(nc)} = \mathcal{N}_\Lambda^{(nc)}/\mathcal{U}_\Lambda^{(nc)}$ which is a sheaf of abelian (!) groups because for any maximal R-order Λ , $\mathcal{D}(\Lambda)$ is abelian so $[N(\Lambda), N(\Lambda)] \subset U(\Lambda)$ and this entails that $[\mathcal{N}_\Lambda^{(nc)}, \mathcal{N}_\Lambda^{(nc)}] \subset \mathcal{U}_\Lambda^{(nc)}$.

In analogy with the commutative case, we define :

Definition 3.4 : A *Cartier divisor* on Y is a global section of the sheaf $\mathcal{C}_\Lambda^{(nc)}$. Thinking of the properties of quotient sheaves, one sees that a Cartier divisor on Y can be described by giving an open cover $\{U_i; i \in I\}$ of Y and for every $i \in I$ an element $n_i \in \Gamma(U_i, \mathcal{N}_\Lambda^{(nc)})$ such that for all i,j in I : $n_i.n_j^{-1} \in \Gamma(U_i \cap U_j, \mathcal{U}_\Lambda^{(nc)})$.

A Cartier divisor is said to be *principal* if it is in the image of the natural map $\Gamma(X, \mathcal{N}_\Lambda^{(nc)}) \to \Gamma(X, \mathcal{C}_\Lambda^{(nc)})$.

Two Cartier divisors are *linearly equivalent* if their quotient (which is defined locally) is principal. The abelian (!) group of Cartier divisor classes on Y will be denoted by CaCl(Y), the Cartier classgroup of Y.

Thus CaCl(Y) is determined by the exact sequence :

$$\Gamma(X, \mathcal{N}_\Lambda^{(nc)}) \to \Gamma(X, \mathcal{C}_\Lambda^{(nc)}) \to CaCl(Y) \to 1$$

In a similar manner one can define the Cartier classgroup of an open subvariety U of Y , CaCl(U) , by the following sequence :

$$\Gamma(V, \mathcal{N}_\Lambda^{(nc)} \mid U) \to \Gamma(V, \mathcal{C}_\Lambda^{(nc)} \mid U) \to CaCl(U) \to 1$$

In case R is commutative , CaCl(U) is nothing but the Picardgroup of the open subvariety U , cfr. e.g. [28] .

We will now briefly discuss the cohomological and functorial properties of these CaCl(U). Before relating the Cartier classgroup to cohomology let us extend some wellknown definitions to sheaves of not necessarely Abelian groups.

For proofs and more details on non-Abelian cohomology we refer the reader to [25] and [52].

Let $\mathcal{G}$ be a sheaf of groups on a topological space Y and let $\mathcal{U} = \{U_i; i \in I\}$ be an open covering of Y. A 1-cocycle for $\mathcal{U}$ with values in $\mathcal{G}$ is a family $\{g_{ij}; i, j \in I\}$, $g_{ij} \in \Gamma(U_i \cap U_j, \mathcal{G})$ such that for every triple $i, j, k \in I$:

$$(g_{ij} \mid U_i \cap U_j \cap U_k).(g_{jk} \mid U_i \cap U_j \cap U_k) = (g_{ik} \mid U_i \cap U_j \cap U_k).$$

Two cocycles g and g' are cohomologous if there is a family $\{h_i, i \in I\}$ with $h_i \in \Gamma(U_i, \mathcal{G})$ such that :

$$g'_{ij} = (h_i \mid U_i \cap U_j).g_{ij}.(h_j \mid U_i \cap U_j)^{-1}.$$

This is easily checked to be an equivalence relation and the set of cohomology classes is written $H^1(\mathcal{U}_X, \mathcal{G})$. It is a pointed set with a distinguished element $\{g_{ij}; i, j \in I$ where $g_{ij} = 1$ for all i and j }. The pointed set $H^1(X, \mathcal{G})$ is then defined to be $\lim H^1(\mathcal{U}_X, \mathcal{G})$ where the direct limit is taken over all open coverings of Y . The main result in this setting is :

Proposition 3.5 : [25,52] : To any exact sequence of sheaves of groups $1 \to \mathcal{G}\prime \to \mathcal{G} \to \mathcal{G}\prime\prime \to 1$ there is associated an exact sequence of pointed sets

$$1 \to \Gamma(X, \mathcal{G}\prime) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{G}\prime\prime) \to$$

$$\to H^1(X, \mathcal{G}\prime) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{G}\prime\prime)$$

Applying this result to the exact sequence of sheaves of groupes: $1 \to \mathcal{U}_\Lambda^{(nc)} \to \mathcal{N}_\Lambda^{(nc)} \to \mathcal{C}_\Lambda^{(nc)} \to 1$, it is straightforward to deduce the following

Proposition 3.6 : If A is a maximal R-order , then :
(a) : $1 \to CaCl(Y) \to H^1(X, \mathcal{U}_\Lambda^{(nc)}) \to H^1(X, \mathcal{N}_\Lambda^{(nc)})$;
(b) : If U is an open set of Y , then :
$1 \to CaCl(U) \to H^1(V, \mathcal{U}_\Lambda^{(nc)} \mid U) \to H^1(V, \mathcal{N}_\Lambda^{(nc)} \mid U)$.

If R is commutative , CaCl(Y) = Pic(R) . Now, if Λ is a maximal R-order we can define Pic(Λ) to be the quotient group of the group of invertible Λ-ideals by the subgroup op those invertible Λ-ideals which are generated by a normalizing element. Pic(Λ) is ,of course, an Abelian group. In order to relate Pic(Λ) to CaCl(Y) we have to impose a technical condition on Λ :

Definition 3.7 : A maximal R-order Λ is said to be locally weak-factorial iff $\mathrm{Pic}(Q_{\Lambda-P}(\Lambda)) = 1$ for every $P \in \mathrm{Spec}(\Lambda)$.

Proposition 3.8 : If Λ is a locally weak-factorial maximal R-order , then $\mathrm{Pic}(\Lambda) = \mathrm{CaCl}(Y)$.

Proof : Let M be an invertible Λ-ideal , then $Q_{\Lambda-P}(M)$ is an invertible $Q_{\Lambda-P}(\Lambda)$-ideal for every $P \in Y$. Λ being locally weak-factorial this entails that $Q_{\Lambda-P}(M) = Q_{\Lambda-P}(\Lambda).n_P$ for some $n_P \in N(Q_{\Lambda-P}(\Lambda))$. As in the proof of Prop. 3.2 one can lift these equalities to a neighborhood V_P of P and it is clear that $\{(V_P, n_P)\}$ determines a Cartier divisor on Y. Thus, we have a well defined map from $I(\Lambda)$, the group of invertible Λ-ideals , to Cart(Y) , the group of all Cartier divisors on Y , which is readily checked to be a groupmorphism. Furthermore , the induced morphism $\mathrm{Pic}(\Lambda) \to \mathrm{CaCl}(Y)$ is monomorphic. Let us check that it is also epimorphic.

If $\{(V_i, n_i)\}$ determines a Cartier divisor on Y , then we can find a finite number among the V_i , say $\{V_1, ..., V_n\}$ such that $Y = \cup_{i=1}^n V_i$ (because Y is quasi-compact). Now, if $V_i = X(I_i)$, then $Q_{I_i}(P_{i_1}^{n_{i_1}} * ... * P_{i_{k_i}}^{n_{i_{k_i}}}) = Q_{I_i}(\Lambda).n_i$ for every i . Thus we can find a divisorial Λ-ideal M such that for every i = 1 , ... , n : $Q_{I_i}(M) = Q_{I_i}(\Lambda).n_i$ and furthermore : $\Gamma(V_i, O_M) = \Gamma(V_i, O_\Lambda).n_i$ for every $i \in \{1, ..., n\}$.

Finally, we have to check that M is invertible. Consider $\tilde{O}_{M^{-1}}$, then in every stalk one obtains : $Q_{\Lambda-P}(M^{-1}).Q_{\Lambda-P}(M) = Q_{\Lambda-P}(\Lambda)$ whence $M^{-1}.M = \cap Q_{\Lambda-P}(M.M^{-1}) = \cap Q_{\Lambda-P}(M).Q_{\Lambda-P}(M^{-1}) = \cap Q_{\Lambda-P}(\Lambda) = \Lambda$ finishing the proof.

This is probably the proper place to state some questions about the splitting of prime ideals in maximal order rings. If $Kdim(\Lambda) = 1$ (i.e. if Λ is a Dedekind prime p.i. ring), it is well known that there are no prime ideals of the center which split up in the ring, i.e. Λ is so called Zariski-central cfr. [85]. For arbitrary maximal order, it is known [13] that height one prime ideals satisfy the unique-lying-over property with respect to the center (Λ may therefore be called 'divisorial Zariski central') but in general Λ is not Zariski central. The first example of such a situation was constructed by M. Ramras [64]. He gives a maximal order over a regular local ring of global dimension 2 such that there are exactly two maximal ideals lying over the central radical.

First, we will present another method for constructing split- examples. The remarkable (?) thing about this class of examples is that the problem reduces entirely to commutative field-theory. Let Λ be any maximal order and suppose that $P \in Spec(\Lambda)$ lies uniquely over its center C (it follows from [13] that this property is equivalent with : $C(P)$ satisfies the left and right Ore- conditions). It is rather easy to verify that the fiber of the extension $\Lambda \to \Lambda[t]$ in P equals $Spec(Q(\Lambda/P)[t]$ whereas the central fiber in $p = P \cap C$ equals $Spec(Q(C/p)[t]$. Therefore, the fiber in P does not split up over its center if and only if $Z(Q(\Lambda/P))$ is a purely inseparable field extension of $Q(C/p)$. Split-examples are now easily constructed :

Take $A = \mathbb{C}[X, -]$ and $P = (X)$, then $A/P = \mathbb{C}$ and $C/p = \mathbb{R}$. Let $f(t)$ be an irreducible polynomial over $\mathbb{R}$ which splits up over $\mathbb{C}$, e.g. $t^2 + 1 = (t+i).(t-i)$, then $(X, t+i)$ and $(X, t-i)$ are two prime ideals of $A[t]$ which lie over the central prime $(X^2, t^2 + 1)$.

QUESTION A : If Λ is a maximal order with center C and let $p \in Spec(C)$. Are there only a finite number of prime ideals of Λ lying over p ?

QUESTION B : If Λ is a maximal order, $P_1, P_2 \in Spec(\Lambda)$ such that $P_1 \cap C = P_2 \cap C$. Under which conditions on Λ does this imply that $pid(\Lambda/P_1) = pid(\Lambda/P_2)$

It is quite easy to construct counterexamples to question B :

Let A be a divisorial R-ideal which is not invertible for some commutative Krull domain R and consider the maximal R-order :

$$\Lambda \simeq End_R(R \oplus R \oplus A) \simeq \begin{pmatrix} R & R & A \\ R & R & A \\ A^{-1} & A^{-1} & R \end{pmatrix}$$

and take a prime ideal P of R such that $A.A^{-1} \subset P$, then it is clear that :

$$P_1 = \begin{pmatrix} P & P & A \\ P & P & A \\ A^{-1} & A^{-1} & R \end{pmatrix}$$

$$P_2 = \begin{pmatrix} R & R & A \\ R & R & A \\ A^{-1} & A^{-1} & P \end{pmatrix}$$

are prime ideals of Λ lying over P and furthermore :

$$p.i.d.(\Lambda/P_1) = 2; p.i.d.(\Lambda/P_2) = 1$$

Let us now turn to the functorial properties. Let Λ be a maximal R-order in a central simple K-algebra Σ and let Γ be a maximal S-order in a central simple L-algebra Θ. Now, suppose that :

$$\phi : \Lambda \to \Gamma$$

is a monomorphic ringextension in the sense of C. Procesi , then it follows that $\phi(R) \subset S$ and that ϕ can be extended to a ringmorphism :

$$\Phi : \Sigma \to \Theta$$

From [62] we retain that :

$$i_\phi : Spec(\Gamma) \to Spec(\Lambda); i_\phi(P) = \phi^{-1}(P)$$

is a continuous morphism. Furthermore, it follows from [90] that there is a morphism of ringed spaces :

$$(i_\phi, \Delta) : (Spec(\Gamma), \mathcal{O}_\Gamma^{bi}) \to (Spec(\Lambda), \mathcal{O}_\Lambda^{bi})$$

This means that $\Delta : \mathcal{O}_\Lambda^{bi} \to (i_\phi)_*(\mathcal{O}_\Gamma^{bi})$ is a morphism of sheaves of rings over $Spec(\Lambda)$. So, in particular we have for any open set U of $Spec(\Lambda)$ the following commutative diagram :

$$\begin{array}{ccc} \Lambda & \to & \Gamma(U, \mathcal{O}_\Lambda^{bi}) \\ \downarrow & & \downarrow \\ \Gamma & \to & \Gamma(i_\phi^{-1}(U), \mathcal{O}_\Gamma^{bi}) \end{array}$$

where the horizontal morphisms are the restrictionmorphisms which are central extensions .

These facts entail that $\Delta(U)$ is a ringextension in the sense of Procesi and of course we have that $\Delta(U) = \Phi \mid \Gamma(U, \mathcal{O}_\Lambda^{bi})$. Furthermore, Δ induces a morphism between the sheaves of groups :

$$\Delta : \mathcal{U}_\Lambda^{(nc)} \to (i_\phi)_*(\mathcal{U}_\Gamma^{(nc)}$$

and Φ induces a morphism between the sheaves of groups :

$$\Phi : \mathcal{N}_\Lambda^{(nc)} \to (i_\phi)_*(\mathcal{N}_\Gamma^{(nc)})$$

Combining all these facts it is fairly easy to verify that for every open set U of $Spec(\Lambda)$, there is a canonical groupmorphism :

$$CaCl(U) \to CaCl(i_\phi^{-1}(U))$$

Therefore, in particular we obtain a groupmorphism :

$$\phi : CaCl(\Lambda) \to CaCl(\Gamma)$$

b. : Cartier divisors on central schemes

Let $\mathcal{O}_\Lambda$ denote the usual structure sheaf of the maximal R-order Λ on $X = Spec(R)$. Then, clearly one can define :

A : The sheaf of units ; $\mathcal{O}_\Lambda^*$ in the obvious way , i.e.

$$\Gamma(V, \mathcal{O}_\Lambda^*) = U(\Gamma(V, \mathcal{O}_\Lambda))$$

for every open subvariety V of $X = Spec(R)$ and restriction morphisms are inclusions.

B : The sheaf of normalizing elements , $\mathcal{N}_\Lambda$, which is defined by :

$$\Gamma(V, \mathcal{N}_\Lambda) = N(\Gamma(V, \mathcal{O}_\Lambda))$$

and restriction morphisms are inclusions. As above one can prove that $\mathcal{N}_\Lambda$ is actually a sheaf.

Furthermore, in view of Prop. 2.5 it is now quite easy to verify :

Proposition 3.9 : If Λ is a maximal R-order in a central simple K-algebra Σ , then with notations as before :
(1) : $i_*(\mathcal{O}_\Lambda^*) = \mathcal{U}_\Lambda^{(nc)}$; $i^*(\mathcal{U}_\Lambda^{(nc)}) = \mathcal{O}_\Lambda^*$
(2) : $i_*(\mathcal{N}_\Lambda) = \mathcal{N}_\Lambda^{(nc)}$; $i^*(\mathcal{N}_\Lambda^{(nc)}) = \mathcal{N}_\Lambda$

Similarly, one can define the sheaf of Cartier divisors of Λ on $X = Spec(R)$ to be the quotient sheaf :

$$\mathcal{C}_\Lambda = \mathcal{N}_\Lambda / \mathcal{O}_R^*$$

and the Cartier classgroup of an open subvariety V of $X = Spec(R)$ $CaCl_c(V)$ will be defined by the exact sequence below :

$$\Gamma(V, \mathcal{N}_\Lambda \mid V) \to \Gamma(V, \mathcal{C}_\Lambda \mid V) \to CaCl_c(V) \to 1$$

and again it is rather trivial to verify :

Proposition 3.10 : If Λ is a maximal R-order in a central simple K-algebra Σ , then with notations as before we have :
(1) : $i_*(\mathcal{C}_\Lambda) = \mathcal{C}_\Lambda^{(nc)}$; $i^*(\mathcal{C}_\Lambda^{(nc)}) = \mathcal{C}_\Lambda$
(2) : $CaCl_c(V) = CaCl(i^{-1}(V))$ for every open subvariety V of $Spec(R)$.

These facts will enable us to restrict attention to the central schemes which are usually easier to handle.

(2) : WEIL DIVISORS :

Having defined what Cartier divisors are , let us now look at Weil divisors. Again we will introduce them first on the noncommutative schemes and afterwards we will show that it is enough to study them on the central schemes.

a : Weil divisors on noncommutative schemes

If U is an open set of $Y = Spec(\Lambda)$, then we will denote by $X^{(1)}(U)$ the set $X^{(1)}(\Lambda) \cap U$, i.e. the minimal nonzero prime ideals of Λ lying inside U and with $Div(U)$ we denote the free Abelian group generated by the set $X^{(1)}(U)$. E.g. in the case that $U = Y$, then $Div(U)$ is nothing but $\mathcal{D}(\Lambda)$, the group of twosided divisorial Λ- ideals.

The assignment $U \to Div(U)$ defines a flabby sheaf (i.e. a sheaf such that all the restrictionmorphisms are epimorphic) on Y which we will denote in the sequel by $\mathcal{D}_{\Lambda}^{(nc)}$. There is a canonical sheafmorphism :

$$\phi : \mathcal{N}_{\Lambda}^{(nc)} \to \mathcal{D}_{\Lambda}^{(nc)}$$

defined in the following way : if $n \in \Gamma(X(I), \mathcal{N}_{\Lambda}^{(nc)})$, then $Q_I(\Lambda).n$ is a divisorial $Q_I(\Lambda)$-ideal. Using the notation of 2.4 and lemme 4.2.2 of [13] we know that :

$$Q_I(-) : \mathcal{D}(\Lambda) \to \mathcal{D}(Q_I(\Lambda))$$

is epimorphic with kernel the group generated by $X^{(1)}(\Lambda) \cap \mathcal{L}(\sigma_I)$ it follows that $Q_I(\Lambda).n = Q_I(P_1)^{k_1} * \dots * Q_I(P_l)^{k_l}$ where $P_i \in X^{(1)}(\Lambda) \cap X(I)$. Clearly , the maps :

$$\phi_I : \Gamma(X(I), \mathcal{N}_{\Lambda}^{(nc)}) \to \Gamma(X(I), \mathcal{D}_{\Lambda}^{(nc)})$$

such that $\phi_I(n) = \sum k_j.P_j$ are well defined groupmorphisms compatible with the restriction morphisms , i.e. if $X(J) \subset X(I)$, then the diagram below is a commutative one :

$$\begin{array}{ccc} \Gamma(Y(I), \mathcal{N}_\Lambda^{(nc)}) & \rightarrow & \Gamma(Y(I), \mathcal{D}_\Lambda^{(nc)}) \\ \downarrow & & \downarrow \\ \Gamma(Y(J), \mathcal{N}_\Lambda^{(nc)}) & \rightarrow & \Gamma(Y(J), \mathcal{D}_\Lambda^{(nc)}) \end{array}$$

where the vertical morphism are the restriction morphisms.

Lemma 3.11 : If Λ is a maximal R-order in Σ , then the following sequence of sheaves on $Y = Spec(\Lambda)$ equipped with the Zariski topology is exact :

$$1 \rightarrow \mathcal{U}_\Lambda^{(nc)} \rightarrow \mathcal{N}_\Lambda^{(nc)} \rightarrow \mathcal{D}_\Lambda^{(nc)}$$

Proof :

For every prime ideal P of Λ we have the natural map described above :

$$\phi_P : N(Q_{\Lambda-P}(\Lambda)) \rightarrow \mathcal{D}(Q_{\Lambda-P}(\Lambda))$$

and it is straightforward to check that its kernel equals $U(Q_{\Lambda-P}(\Lambda))$, finishing the proof.

Definition 3.12 : A Weil divisor on an open subscheme U of $Y = Spec(\Lambda)$ is an element of $Div(U)$ and $Cl(U)$, the classgroup of the open subscheme U , is defined by the sequence :

$$1 \rightarrow \Gamma(U, \mathcal{U}_\Lambda^{(nc)} \mid U) \rightarrow \Gamma(U, \mathcal{N}_\Lambda^{(nc)} \mid U) \rightarrow Div(U) \rightarrow Cl(U) \rightarrow 1$$

Of course, if $U = Y$, then $Cl(U) = Cl(\Lambda)$, the Chamarie- or normalizing-classgroup of the maximal R-order Λ , i.e. $Cl(\Lambda)$ is the quotient of $\mathcal{D}(\Lambda)$ by $\mathcal{P}(\Lambda)$

, the subgroup of those divisorial Λ-ideals which are generated by one element , which is then of course a normalizing element.

Having related the classgroup to Weil divisors and the Picardgroup to Cartier divisors (at least whenever the Picardgroup coincides with the cohomological Picardgroup) our next aim will be to find a relation between $Cl(U)$ and $CaCl(U)$ for certain open sets U of $Y = Spec(\Lambda)$. The diagram of sheaves of groups below is exact and commutative :

$$\begin{array}{ccccccccc} 1 & \to & \mathcal{U}_\Lambda^{(nc)} & \to & \mathcal{N}_\Lambda^{(nc)} & \to & \mathcal{C}_\Lambda^{(nc)} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathcal{U}_\Lambda^{(nc)} & \to & \mathcal{N}_\Lambda^{(nc)} & \to & \mathcal{D}_\Lambda^{(nc)} & & \end{array}$$

where $\psi : \mathcal{C}_\Lambda^{(nc)} \to \mathcal{D}_\Lambda^{(nc)}$ is the induced morphism. By taking sections on an open subscheme U we obtain that the diagram below is a commutative and exact one of groups :

$$\begin{array}{ccccccccccc} 1 & \to & \Gamma(U, \mathcal{U}_\Lambda^{(nc)}) & \to & \Gamma(U, \mathcal{N}_\Lambda^{(nc)}) & \to & Cart(U) & \to & CaCl(U) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \Gamma(U, \mathcal{U}_\Lambda^{(nc)}) & \to & \Gamma(U, \mathcal{N}_\Lambda^{(nc)}) & \to & Div(U) & \to & Cl(U) & \to & 1 \end{array}$$

where $\theta(U) : CaCl(U) \to Cl(U)$ is the induced morphism. This morphism which arises here from sheafmanipulation seems to have been overlooked even in the commutative case for non-Noetherian Krull domains. In every proof known to the author one uses the fact that $Y = Spec(\Lambda)$ is a (locally) Noetherian space in order to determine a Cartier divisor on an open subvariety U of Y by a finite covering which is necessary to determine $\psi(U)$. It follows from the above argument that this presents no real problem. But let us give another, more constructive and less sheaftheoretic proof of this :

Proposition 3.13 : If Λ is a maximal R-order in Σ and if U is an open set of $Y = Spec(\Lambda)$, then there is a natural monomorphism :

$$\psi(U) : Cart(U) \to Div(U)$$

which induces a monomorphism :

$$\theta(U) : CaCl(U) \to Cl(U)$$

Proof :

Let a Cartier divisor on U be defined by the set $\{(V_i, n_i)\}$ where V_i is an open subset of U and $n_i \in \Gamma(V_i, \mathcal{N}_{\Lambda}^{(nc)})$. Note that for any ideal I of Λ , $X^{(1)}(\Lambda) - (X^{(1)}(\Lambda) \cap X(I))$ is a finite set since I contains a normalizing (even a central) element , say n and of course we have $X(\Lambda.n) \subset X(I)$ and furthermore, $X^{(1)}(\Lambda) - (X^{(1)}(\Lambda) \cap X(\Lambda.n))$ is clearly a finite set. Therefore we can choose a finite number among the V_i's , say $V_1, ..., V_k$ such that :

$$X^{(1)}(\Lambda) \cap U = (X^{(1)}(\Lambda) \cap V_1) \cup .... \cup (X^{(1)}(\Lambda) \cap V_k)$$

Since a Cartier divisor depends only on $X^{(1)}(\Lambda) \cap U$ it is therefore determined by $\{(V_i, n_i); 1 \leq i \leq k\}$. The proof proceeds now as in the Noetherian case , cfr. e.g. Hartshorne [28].

The next proposition investigates when the monomorphism :

$$\theta(U) : CaCl(U) \to Cl(U)$$

is actually an isomorphism. We say that Λ is locally factorial in an open subscheme U of $Y = Spec(\Lambda)$ if and only if $Cl(Q_{\Lambda - P}(\Lambda)) = 1$ for every $P \in U$.

Proposition 3.14 : If Λ is a maximal R-order in Σ and if U is an open subscheme of $Y = Spec(\Lambda)$, then :

$$\theta(U) : CaCl(U) \to Cl(U)$$

is an isomorphism if and only if Λ is locally factorial in U.

Proof :

Suppose that Λ is locally factorial in U and let $\sum n_i.P_i \in Div(U)$. If $A = P_1^{n_1}*...*P_k^{n_k}$, then $Q_{\Lambda-P}(A) = Q_{\Lambda-P}(\Lambda).n_p$ for some $n_p \in N(Q_{\Lambda-P}(\Lambda))$ for every $P \in U$. Similarly, $Q_{\Lambda-P}(A^{-1}) = Q_{\Lambda-P}(\Lambda).n_p^{-1}$. Now,take a sufficiently small neighborhood V_P of P such that $n_p \in \Gamma(V_P, \mathcal{O}_A^{(nc)})$ and $n_p^{-1} \in \Gamma(V_P, \mathcal{O}_{A^{-1}}^{(nc)})$, then it is fairly easy to verify that $n_p \in \Gamma(V_P, \mathcal{N}_\Lambda^{(nc)})$ and therefore $\{(V_P, n_p)\}$ defines a Cartier divisor on U such that its image under $\psi(U)$ equals $\sum n_i.P_i$.

Conversely, suppose that $\theta(U) : CaCl(U) \to l(U)$ is epimorphic , or equivalently that $\psi(U) : Cart(U) \to Div(U)$ is epimorphic. Let $P \in U$ and take $A \in \mathcal{D}(Q_{\Lambda-P}(\Lambda))$, then $A = Q_{\Lambda-P}(P_1)^{n_1}*...*Q_{\Lambda-P}(P_k)^{n_k}$ for some $P_i \in X^{(1)}(\Lambda) \cap U$. Let $\{(V_i, n_i)\}$ determine a Cartier divisor on U such that its image under $\phi(U)$ equals $\sum n_i.P_i$. If $P \in V_i$, then this implies that $A = Q_{\Lambda-P}(\Lambda).n_i$ showing that $Cl(Q_{\Lambda-P}(\Lambda)) = 1$, finishing the proof.

Therefore, in particular , if Λ is locally factorial in $Y = Spec(\Lambda)$, then $Cl(\Lambda) = CaCl(Y)$ and we obtain a cohomological interpretation of $Cl(\Lambda)$. Unfortunately being locally factorial is a rather restrictive condition on Λ. Therefore, one would like to extend the foregoing proposition to a larger class of rings, e.g. those for which $Pic(Q_{\Lambda-P}(\Lambda)) = 1$ for every $P \in Y$. The next result presents a noncommutative generalization of Danilov's main tool :

Theorem 3.15 : If Λ is a maximal R-order in Σ and if $Pic(Q_{\Lambda-P}(\Lambda)) = 1$ for every $P \in Spec(\Lambda)$, then there exists a filtered family of open subsets $\{U_i\}$ of $Y = Spec(\Lambda)$ such that :

$$Cl(\Lambda) = \varinjlim CaCl(U_i)$$

Proof :

For any divisorial Λ-ideal A_i we will define the set :

$$U_i = \{P \in Spec(\Lambda) : Q_{\Lambda-P}(A_i) \text{ is an invertible } Q_{\Lambda-P}(\Lambda) \text{ -ideal } \}$$

First, we claim that that U_i is an open set in the Zariski- topology. For, if $Q_{\Lambda-P}(A_i).Q_{\Lambda-P}(A_i^{-1}) = Q_{\Lambda-P}(\Lambda)$, then there exist elements $f_j \in Q_{\Lambda-P}(A_i)$ and $g_j \in Q_{\Lambda-P}(A_i^{-1})$ such that $\sum f_j.g_j = 1$.

All these elements live on a sufficiently small neighborhood V_P of P. Now, let $P_1 \in V_P$, then $Q_{\Lambda-P_1}(A_i).Q_{\Lambda-P_1}(A_i^{-1})$ is an ideal of $Q_{\Lambda-P_1}(\Lambda)$ which contains 1 , therefore $P_1 \in U_i$, finishing the proof of our claim.

Furthermore, for every divisorial Λ-ideal A_i , $X^{(1)}(\Lambda) \subset U_i$, yielding that $Div(U_i) = \mathcal{D}(\Lambda)$. Also, it follows from lemma 2.1 that $\Gamma(U_i, \mathcal{U}_\Lambda^{(nc)}) = U(\Lambda)$ and $\Gamma(U_i, \mathcal{N}_\Lambda^{(nc)}) = N(\Lambda)$. Therefore, $Cl(U_i) = Cl(\Lambda)$ for every U_i , so there is a canonical monomorphism by Prop.3.14 , $CaCl(U_i) \to Cl(\Lambda)$ yielding a monomorphism :

$$\varinjlim \; CaCl(U_i) \to Cl(\Lambda)$$

We are left to check that this map is epimorphic and in order to do this , it is clearly sufficient to check that the morphism :

$$\varinjlim \; Cart(U_i) \to \mathcal{D}(\Lambda)$$

is surjective. Therefore, let $A_i \in \mathcal{D}(\Lambda)$, then by the definition $Q_{\Lambda-P}(A_i)$ is invertible for every $P_i \in U_i$. Since we have assumed that $Pic(Q_{\Lambda-P}(\Lambda)) = 1$, this means that $Q_{\Lambda-P}(A_i) = Q_{\Lambda-P}(\Lambda).n_p$ for some normalizing element $n_P \in N(Q_{\Lambda-P}(\Lambda))$. Again, for every P one can extend this equality to some open neighborhood V_P of P and $\{(V_P, n_P)\}$ describes an element of $Cart(U_i)$, finishing the proof.

In the next section we will show how one may weaken the condition :

$$Pic(Q_{\Lambda-P}(\Lambda)) = 1$$

for all $P \in Spec(\Lambda)$. Instead of taking the direct limit over the special set of open sets U_i , it is clear that :

$$Cl(\Lambda) = \varinjlim \; CaCl(U)$$

where the direct limit is taken over all open subschemes U of $Y = Spec(\Lambda)$ such that $X^{(1)}(\Lambda) \subset U$.

These results provide us with a cohomological interpretation of the classgroup. We will now investigate the functorial properties of Weil divisors.

Let Λ be a maximal R-order in a central simple K-algebra Σ and let Γ be a maximal S-order in a central simple L-algebra Θ. Now, suppose that :

$$\phi : \Lambda \rightarrow \Gamma$$

is a ringextension in the sense of C. Procesi , then it follows that $\phi(R) \subset S$ and that ϕ can be extended to a ringmorphism :

$$\Phi : \Sigma \rightarrow \Theta$$

Therefore,

$$i_\phi : Spec(\Gamma) \rightarrow Spec(\Lambda)$$

is a continuous morphism. Furthermore, we impose that $\phi : \Lambda \rightarrow \Gamma$ satisfies pas d'éclatement , i.e.

$$ht(\phi^{-1}(P)) \leq 1$$

if $ht(P) = 1$. This entails that there is a morphism of sheaves of groups :

$$\Delta_D : \mathcal{D}_\Lambda^{(nc)} \rightarrow (i_\phi)_*(\mathcal{D}_\Gamma^{(nc)})$$

which is defined in the following way. Let $Y_\Gamma(J)$ be a typical open set of $Spec(\Gamma)$ and let $i_\phi^{-1}(Y_\Gamma(J)) = Y_\Lambda(I)$, then there is a ringmorphism :

$$\Gamma(Y_\Lambda(I), \mathcal{O}_\Lambda^{(nc)}) = Q_I(\Lambda) \rightarrow \Gamma(Y_\Gamma(J), \mathcal{O}_\Gamma^{(nc)}) = Q_J(\Gamma)$$

(using the notation of remark 2.2) which satisfies pas d'éclatement. Therefore , this morphism induces a natural morphism on the divisors (cfr. part I) and we obtain :

$$\begin{array}{ccc} \mathcal{D}(Q_I(\Lambda)) & \rightarrow & \mathcal{D}(Q_J(\Gamma)) \\ \uparrow & & \downarrow \\ \Gamma(Y_\Gamma(I), \mathcal{D}_\Lambda^{(nc)}) & \rightarrow & \Gamma(Y_\Gamma(J), \mathcal{D}_\Gamma^{(nc)}) \end{array}$$

and the composite bottom morphism yields the desired morphism of sheaves of groups over $Spec(\Lambda)$.

Moreover, this morphism makes the following exact diagram of sheaves of groups over $Spec(\Lambda)$ into a commutative one :

$$\begin{array}{ccccccc} 1 & \rightarrow & \mathcal{U}_\Lambda^{(nc)} & \rightarrow & \mathcal{N}_\Lambda^{(nc)} & \rightarrow & \mathcal{D}_\Lambda^{(nc)} \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \rightarrow & (i_\phi)_*(\mathcal{U}_\Gamma^{(nc)}) & \rightarrow & (i_\phi)_*(\mathcal{N}_\Gamma^{(nc)}) & \rightarrow & (i_\phi)_*(\mathcal{D}_\Gamma^{(nc)}) \end{array}$$

and consequently there is for every open set U of $Spec(\Lambda)$ a canonical groupmorphism :

$$Cl(U) \rightarrow Cl(i_\phi^{-1}(U))$$

Therefore, in particular we obtain a groupmorphism :

$$Cl(\Lambda) \rightarrow Cl(\Gamma)$$

if we take $U = Spec(\Lambda)$.

(b) : Weil divisors on the central schemes

Again, let $\mathcal{O}_\Lambda$ denote the usual structure sheaf of the maximal R-order Λ on $X = Spec(R)$. Then, one may define a sheaf $\mathcal{D}_\Lambda$ of groups on $Spec(R)$ by taking for its sections on an open subvariety U of $X = Spec(R)$ the free Abelian group generated by the prime ideals of $X^{(1)}(\Lambda)$ lying over the prime ideals in $X^{(1)}(R) \cap U$.

The classgroup of Λ on the open subvariety U of $X = Spec(R)$ will then be defined by the exact sequence :

$$1 \to \Gamma(U, \mathcal{O}_\Lambda^* \mid U) \to \Gamma(U, \mathcal{N}_\Lambda \mid U) \to \Gamma(U, \mathcal{D}_\Lambda \mid U) \to Cl_c(U) \to 1$$

In combination with the results on Cartier divisors on the central schemes it is now easy to prove :

Proposition 3.16 : If Λ is a maximal R-order in a central simple K-algebra Σ , then with notations as before we have :

(1) : $i_*(\mathcal{D}_\Lambda) = \mathcal{D}_\Lambda^{(nc)}$; $i^*(\mathcal{D}_\Lambda^{(nc)}) = \mathcal{D}_\Lambda$

(2) : $Cl_c(U) = Cl(i^{-1}(U))$ for every open subvariety U of $X = Spec(R)$.

(3) : RINGTHEORETICAL INTERPRETATION :

The cohological interpretation of the normalizing classgroup as presented in the foregoing section has the advantadge that one can now apply cohomology-theory (i.e. mainly a lot of exact sequences) to the study of the behaviour of this classgroup under ringextensions.

On the other hand, this approach has two major disadvantadges :

(a) : Translating our questions in terms of cohomology pointed sets, one seems to lose grip on what actually goes on.

(b) : We have not been able to prove for a sufficienctly large class of maximal orders (e.g. the Noetherian ones) that they are locally weak-factorial.

In this section we avoid both problems for maximal orders by :

(a) : presenting a ringtheoretical interpretation of the cohomology pointed sets lim $H^1(U, \mathcal{O}^*_\Lambda)$ and lim $H^1(U, \mathcal{N}_\Lambda)$;

(b) : extending Theorem 3.15 .

The main result of this section is :

Theorem 3.17 : If Λ is a maximal order over a Krull domain R , then the following sequence is exact :

$$1 \to Cl(\Lambda) \to \underrightarrow{lim}(H^1(U, \mathcal{O}^*_\Lambda)) \to \underrightarrow{lim}(H^1(U, \mathcal{N}_\Lambda)) \to 1$$

where the direct limit is taken over all open sets U of Spec(R) such that $X^{(1)}(R) \subset U$.

The proof follows immediatly from the three propositions below. Let us first extend some classical definitions to orders over Krull domains.

One of the main obstacles in a noncommutative normalization- theory is presented by the fact that maximal orders are (usually) not unique. Therefore one could ask whether it is possible to construct all maximal orders from a given one.

Clearly, conjugation defines an equivalence relation on the set of maximal orders and the set of orders conjugated to Λ is of course described by the set $\Sigma^*/N(\Lambda)$. By $t_R(\Sigma)$ we denote the set of equivalence classes of maximal orders over R in Σ. $(t_R(\Sigma))$ is said to be the type-number of Σ over R.

Now, we will extend some classical results of Roggenkamp [71] on genera of lattices to the case of orders over Krull domains.

So, Λ will be an R-order in some central simple K- algebra Σ. If M and N are two torsion free left Λ- modules which are divisorial R-lattices. Then M and N are said to lie in the same genus , notation $M \vee N$, if and only if $M_p \simeq N_p$ as left Λ_p-modules for every $p \in X^{(1)}(R)$. We will write :

$$\mathcal{G}(M) = \{N \in \Lambda - \text{mod a divisorial } R\text{- lattice} : N \vee M\}$$

and with $g(M)$ we will denote the pointed set of left Λ- module isomorphism classes in $\mathcal{G}(M)$ (the distinguished element of $g(M)$ will of course be the class of M).

We will now relate $\mathcal{G}(M)$ to idèles in Σ. Recall that an idèle in Σ is a family $\{x_p; p \in X^{(1)}(R)\}$ where the $x_p \in \Sigma$ and for all but a finite number of p we have $x_p = 1$.

Lemma 3.18 : There is a one-to-one correspondence between the elements in $\mathcal{G}(M)$ and the idèles in Σ.

Proof :

Let $M \vee \Lambda$, then we may assume (up to isomorphism) that $K.M = \Sigma$. Because both M and Λ are R-lattices this implies that $M_p = \Lambda_p$ for allmost all $p \in X^{(1)}(R)$. For the finitely many exceptions we have $M_{p'} \simeq \Lambda_{p'}$ i.e. $M_{p'} = \Lambda_{p'}.a_{p'}$ for some element $a_{p'} \in \Sigma$. Taking $a_p = 1$ if $M_p = \Lambda_p$ we can define the map :

$$\psi : \mathcal{G}(M) \to \text{idèles}_R(\Sigma)$$

by : $\psi(M) = \{a_p; p \in X^{(1)}(R)\}$. First , we claim that this map is injective , for , if $\psi(M) = \psi(M')$, then $M = \cap M_p = \cap M'_p = M'$. Furthermore, ψ is also surjective , for , given an idèle in Σ , $\{a_p; p \in X^{(1)}(R)\}$, we will define $M_p = \Lambda_p.a_p$. Let $\Lambda' = \cap\{\Lambda_p; a_p = 1\}$, then $N = \Lambda' \cap M_{p_1} \cap \dots \cap M_{p_k}$ where $\{p_1, ..., p_k\}$ are the finitely many height one primes for which the corresponding $a_p \neq 1$,is a left Λ-module which is a divisorial R-lattice such that $N_p = M_p$ for all $p \in X^{(1)}(R)$ since R_p is a flat R-module and hence tensoring with R_p commutes with finite intersections.

Definition 3.19 : Two idèles in Σ , $\{a_p; p \in X^{(1)}(R)\}$ and $\{b_p; p \in X^{(1)}(R)\}$ are said to be equivalent if and only if :

$$\psi^{-1}(\{a_p; p \in X^{(1)}(R)\}) \simeq \psi^{-1}(\{b_p; p \in X^{(1)}(R)\})$$

Hence there is a one-to-one correspondence between elements in $h(\Lambda)$ and equivalence classes of idèles in Σ.

Proposition 3.20 : If Λ is a maximal order over a Krull domain R , then the following sequence of pointed sets is exact :

$$1 \to Cl(\Lambda) \to h(\Lambda) \to t_R(\Sigma) \to 1$$

Proof :

The map of pointed sets :

$$\phi : Cl(\Lambda) \to h(\Lambda)$$

is of course given by sending the class $[I]$ of a divisorial Λ-ideal I to the isomorphism $< I >$ of I in $h(\Lambda)$. This map is a monomorphism of pointed sets , for , if $< I > = < \Lambda >$ then one can extend the left Λ-module isomorphism $I \to \Lambda$ to a Σ-linear isomorphism $\Sigma \to \Sigma$ showing that $I = \Lambda.n$. Because Λ is a maximal order and I is a twosided divisorial Λ-ideal this entails that n is a normalizing element , so $[I] = 1$ in $Cl(\Lambda)$. Further, the map of pointed sets :

$$\psi : h(\Lambda) \to t_R(\Sigma)$$

is given by sending an isomorphism class $< A >$ of a left divisorial Λ-ideal A to the conjugacy-class of :

$$O_r(A) = \{x \in \Sigma : A.x \subset A\}$$

in $t_R(\Sigma)$. Let us first check that this map is well-defined. If $< A > = < B >$ then by an argument as above , $A = B.x$ for some $x \in \Sigma^*$.This entails that $x^{-1}.O_r(B).x \subset O_r(A)$. Finally, because $O_r(A)$ and $O_r(B)$ are both maximal orders, this inclusion is an equality and therefore they are conjugated.

The sequence is exact in $h(\Lambda)$. For, if $O_r(L) = x^{-1}.\Lambda.x$ for some $x \in \Sigma^*$ then , because $\Lambda = x.O_r(L).x^{-1} \subset O_r(L.x^{-1})$ and Λ is a maximal order , $O_r(L.x^{-1}) = \Lambda$ showing that $L.x^{-1}$ is a twosided divisorial Λ-ideal. Therefore, $< L > = < L.x^{-1} >$ and $Ker(\psi) \subset Im(\phi)$ and the inverse inclusion is of course trivial since ψ does not depend upon the choice of the representative.

Finally, we have to check that ψ is epimorphic. So, let Γ be a representative of a class in $t_R(\Sigma)$. Then it is fairly easy to check (cfr. e.g. Fossum [21]) that $(\Lambda :_r \Gamma)$ is a divisorial R-lattice which is a left Λ-ideal and a right Γ-ideal , entailing that $O_r((\Lambda :_r \Gamma)) = \Gamma$ because Γ is a maximal order and clearly $\Gamma \subset O_r((\Lambda :_r \Gamma))$, finishing the proof of the proposition.

Note that the sequence above is merely an exact sequence of pointed sets. So, in general one cannot conclude from this sequence that when all the sets occuring are finite that the number of isomorphism classes of left Λ-ideals is the product of the number of elements of $Cl(\Lambda)$ with the number of conjugacy-classes of maximal orders. The main problem is that $Cl(\Lambda)$ does depend upon the choice of Λ. In the table below we will present some examples of such situations. We take Σ to be a quaternion-algebra over $\mathbb{Q}$ and Λ a maximal $\mathbb{Z}$-order in Σ with prime discriminant

p. h will denote the class- number (i.e. the number of elements of $h(\Lambda)$) , t will be the type-number (i.e. the number of elements of $T_{\mathbb{Z}}(\Sigma)$) and h_1 (resp. h_2) will denote the number of conjugacy classes of maximal orders having $Cl(\Gamma) \simeq 1$ (resp. $Cl(\Gamma) \simeq \mathbb{Z}/2.\mathbb{Z}$) :

$p =$	$h =$	$t =$	$h_2 =$	$h_1 =$
2	1	1	0	1
3	1	1	0	1
5	1	1	0	1
7	1	1	0	1
11	2	2	0	2
13	1	1	0	1
17	2	2	0	2
19	2	2	0	2
23	3	3	0	3
29	3	3	0	3

$p =$	$h =$	$t =$	$h_2 =$	$h_1 =$
31	3	3	0	3
37	3	2	1	1
41	4	4	0	4
43	4	3	1	2
47	5	5	0	5
53	5	4	1	3
59	6	6	0	6
61	5	4	1	3
67	6	4	2	2
71	7	7	0	7
73	6	4	2	2
79	7	6	1	5
83	8	7	1	6
89	8	7	1	6
97	8	5	3	2

We will now give a cohomological interpretation of the pointed sets $h(\Lambda)$ and $t_R(\Sigma)$

Proposition 3.21 : If Λ is a maximal order over a Krull domain R then :

$$h(\Lambda) \simeq \varinjlim H^1_{Zar}(U, \mathcal{O}^*_\Lambda \mid U)$$

where the direct limit is taken over all open subvarieties U of $Spec(R)$ such that $X^{(1)}(R) \subset U$.

Proof :

Let L be a left Λ-ideal which is a divisorial R-lattice , then $L^{-1} = (L :_r \Lambda)$ is a right Λ-ideal which is a divisorial R-lattice. By $\mathcal{O}_L$ (resp. $\mathcal{O}_{L^{-1}}$) we will denote the structure sheaf of L (resp. of L^{-1}) over $Spec(R)$. For any $p \in X^{(1)}(R)$, it is clear that :

$$(\mathcal{O}_L)_p = L_p = \Lambda_p.a_p$$

because Λ_p is both a left- and right-principal ideal ring. Similarly, one obtains :

$$(\mathcal{O}_{L^{-1}})_p = (L^{-1})_p = a_p^{-1}.\Lambda_p$$

Now, take a neighborhood V_p of p such that $a_p \in \Gamma(V_p, \mathcal{O}_L)$ and $a_p^{-1} \in \Gamma(V_p, \mathcal{O}_{L^{-1}})$ then it is fairly easy to check that :

$$(\mathcal{O}_L) \mid V_p \simeq (\mathcal{O}_\Lambda \mid V_p).a_p$$

If we define $U = \cup\{V_p; p \in X^{(1)}(R)\}$, then $\{(V_p, a_p)\}$ defines an element of $\Gamma(U, \Sigma^*/\mathcal{O}^*_\Lambda)$ where Σ^* denotes the constant sheaf over $Spec(R)$ with sections Σ^*. Writing out the long exact (Zariski)-cohomology sequence associated to the exact sequence :

$$1 \to \mathcal{O}^*_\Lambda \to \Sigma \to \Sigma/\mathcal{O}^*_\Lambda \to 1$$

one finds :

$$\Gamma(U, \Sigma^*) \to \Gamma(U, \Sigma^*/\mathcal{O}^*_\Lambda) \to H^1_{Zar}(U, \mathcal{O}^*_\Lambda) \to 1$$

In this way,we associate to every left divisorial Λ- ideal L an element of lim $H^1_{Zar}(U, \mathcal{O}^*_\Lambda)$. It follows from the exact sequence above that the elements associated with L and L' coincide if and only if $L = L'.x$ for some $x \in \Gamma(U, \Sigma^*) = \Sigma^*$.

Conversely, with every element of lim $H^1_{Zar}(U, \mathcal{O}^*_\Lambda)$ we may associate in a natural way an isomorphism class of left divisorial Λ-ideals by choosing an element in $\Gamma(U, \Sigma^*/\mathcal{O}^*_\Lambda)$ which generates it , say $\{(V_o, a_p)\}$ and then defining the left $\mathcal{O}_\Lambda \mid U$-Ideal $\mathcal{O}_L \mid U$ locally by :

$$\mathcal{O}_L \mid V_p = (\mathcal{O}_\Lambda \mid V_p).a_p$$

this yields a well defined sheaf and then taking its sections $\Gamma(U, \mathcal{O}_L)$ we obtain a left divisorial Λ- ideal because $X^{(1)}(R) \subset U$, finishing the proof.

Proposition 3.22 : If Λ is a maximal order over a Krull domain R , then

$$t_R(\Sigma) \simeq \varinjlim H^1_{Zar}(U, \mathcal{N}_\Lambda \mid U)$$

where the direct limit is taken over all open subvarieties U of $Spec(R)$ such that $X^{(1)}(R) \subset U$.

Proof :

Let Γ be any maximal R-order in Σ. With $\mathcal{O}_{(\Gamma:_r\Lambda)}$ (the conductor) we denote the presheaf which assigns to an open set U of $Spec(R)$ the sections :

$$\Gamma(U, \mathcal{O}_{(\Gamma:_r\Lambda)}) = \{x \in \Sigma : \Gamma(U, \mathcal{O}_\Gamma.x \subset \Gamma(U, \mathcal{O}_\Lambda)\}$$

An easy computation shows that $\mathcal{O}_{(\Gamma:_r\Lambda)}$ is actually a sheaf of left $\mathcal{O}_\Gamma$-ideals and right $\mathcal{O}_\Lambda$-ideals. Furthermore $\mathcal{O}^{-1}_{(\Gamma:_r\Lambda)}$ which is defined by its sections $\Gamma(U, \mathcal{O}^{-1}_{(\Gamma:_r\Lambda)}) = \Gamma(U, \mathcal{O}_{(\Gamma:_r\Lambda)})^{-1}$ is also a sheaf and a left $\mathcal{O}_\Lambda$-ideal and a right $\mathcal{O}_\Gamma$-ideal.

Now, let p be any height one prime ideal of R. Since both Λ_p and Γ_p are maximal orders over the discrete valuation ring R_p , they are conjugated , i.e.

$s_p^{-1}.\Gamma_p.s_p = \Lambda_p$ for some element $s_p \in \Sigma^*$. We claim that there exists a neighborhood V_p of p such that :

$$s_p^{-1}.(\mathcal{O}_\Gamma \mid V_p).s_p = \mathcal{O}_\Lambda \mid V_p$$

Both $\mathcal{O}_{(\Gamma:_r\Lambda)}$ and $\mathcal{O}^{-1}_{(\Gamma:_r\Lambda)}$ are sheaves , so s_p and s_p^{-1} live on a sufficiently small neighborhood V_p of p. Therefore, $s_p.\Gamma(V_p, \mathcal{O}_\Lambda) \subset \Gamma(V_p, (\mathcal{O}^{-1}_{(\Gamma:_r\Lambda)})$ hence $\Gamma(V_p, \mathcal{O}_\Lambda).s_p^{-1} \subset \Gamma(V_p, \mathcal{O}^{-1}_{(\Gamma:_r\Lambda)}) = \Gamma(V_p, \mathcal{O}_{(\Gamma:_r\Lambda)})^{-1} \subset (s_p.\Gamma(V_p, \mathcal{O}_\Lambda))^{-1} = \Gamma(V_p, \mathcal{O}_\Lambda).s_p^{-1}$ and similarly one obtains : $\Gamma(V_p, \mathcal{O}_{(\Gamma:_r\Lambda)}) = s_p.\Gamma(V_p, \mathcal{O}_\Lambda)$. Therefore, our claim is proved.

Now, let $U = \cup\{V_p; p \in X^{(1)}(R)\}$ is an open set containing $X^{(1)}(R)$ and $\{(V_p, s_p)\}$ describes a section in $\Gamma(U, \Sigma^*/\mathcal{N}_\Lambda)$. Consider the exact sequence of sheaves of pointed sets with respect to the Zariski topology :

$$1 \to \mathcal{N}_\Lambda \to \Sigma^* \to \Sigma^*/\mathcal{N}_\Lambda \to 1$$

Taking sections over U yields the following exact cohomology sequence :

$$1 \to N(\Lambda) \to \Sigma^* \to \Gamma(U, \Sigma^*/\mathcal{N}_\Lambda) \to H^1_{Zar}(U, \mathcal{N}_\Lambda) \to 1$$

Therefore, the section $\{(V_p, s_p)\}$ determines an element in $H^1_{Zar}(U, \mathcal{N}_\Lambda)$ (and so in $\lim H^1_{Zar}(U, \mathcal{N}_\Lambda)$) which is different from the distinguished element in $H^1_{Zar}(U, \mathcal{N}_\Lambda)$ if and only if Γ is not conjugated to Λ.

Conversely , let $s \in \lim H^1_{Zar}(U, \mathcal{N}_\Lambda)$ and choose an open set U of $Spec(R)$ containing $X^{(1)}(R)$ and an element $s(U) \in H^1_{Zar}(U, \mathcal{N}_\Lambda)$ which represents s. Using the above exact sequence , $s(U)$ is determined by some section in $\Gamma(U, \Sigma^*/\mathcal{N}_\Lambda)$. Such a section is given by a set of couples $\{(U_i, s_i)\}$ where U_i is an open cover of U , $s_i \in \Gamma(U_i, \Sigma^*)$ for every i and $s_i^{-1}.s_j \in \Gamma(U_i \cap U_j, \mathcal{N}_\Lambda)$ for all i and j. On U we will now define the twisted sheaf of maximal orders $\mathcal{O}_\Gamma \mid U$ by putting :

$$\mathcal{O}_\Gamma \mid U_i = s_i.(\mathcal{O}_\Lambda \mid U_i).s_i^{-1}$$

. It is now quite easy to show that $\Gamma = \Gamma(U, \mathcal{O}_\Gamma)$ is a maximal R-order in Σ and this finishes the proof.

4 : SOME APPLICATIONS

In this section we will give some applications of the theory developed above. Using the cohomological interpretation of the type number we study the conjugation of maximal orders in matrixrings over locally factorial Krull domains. In a second application we study maximal orders having a discrete normalizing classgroup.

(1) : CONJUGATION OF MAXIMAL ORDERS :

Maury and Raynaud [50] asked the following question :

(Problème 12) : Let Λ be a maximal R-order in Σ. Is every other maximal R-order in Σ isomorphic to Λ (by an inner automorphism) ? If not, what can one say about the isomorphism classes ?

In fact , in the original statement of the problem , Σ is a skewfield. First, we will present some counterexamples to the original question. The rest of this section is devoted to the more general question when Σ is an arbitrary central simple algebra. Special attention is given at the case when $\Sigma = M_n(K)$. It will turn out that in this case , the isomorphism classes of maximal orders (resp. of Azumaya-algebras) are closely related to module theoretic questions in R-mod.

1. some counterexamples :

polynomial rings over skewfields :

If Δ be a skewfield of finite dimension over its center K then $\Delta[t]$ is a maximal order over $K[t]$. Moreover, it is well known that every twosided ideal of $\Delta[t]$ is generated by a central (hence normalizing) element , entailing that :

$$Cl^c(\Delta[t]) = Cl(\Delta[t]) = 1$$

Furthermore, E. Jespers and P. Wauters have shown in [33] that for any maximal R-order Λ , $Cl^c(\Lambda) \simeq Cl^c(\Lambda[t])$, entailing that $Cl(\Delta[t,s]) = 1$. Therefore, it follows from the exact sequence of pointed sets :

$$1 \to Cl(\Delta[t,s]) \to h(\Delta[t,s]) \to T_{K[t,s]}(\Delta[t,s]) \to 1$$

that $h(\Delta[t,s]) = 1$ if all maximal $K[t,s]$-orders in $\Delta(t,s)$ were conjugated to $\Delta[t,s]$. So, in particular this would entail that every projective left ideal of $\Delta[t,s]$ would be free.

As is well known , Ojanguren and Sridharan have proved in [59] that for evcery skewfield Δ there exists a projective non-free left $\Delta[t,s]$-ideal , L . Clearly, $O_r(L)$ is then a maximal $K[t,s]$-order in $\Delta(t,s)$ which is not conjugated to $\Delta[t,s]$.

maximal orders in quaternionalgebras :

As is clear from the table given in the previous section, maximal orders over $\mathbb{Z}$ in quaternionalgebras over $\mathbb{Q}$ (with prime discriminant) ,yield another large class of counterexamples to the original question of Maury and Raynaud.

In the rest of this chapter we will restrict attention to conjugation-problems of maximal orders in matrixrings.

2. conjugation in matrixrings :

Recall that a Krull domain R is said to be locally factorial if R_p is a unique factorization domain for every $p \in Spec(R)$. In this part we aim to characterize those locally factorial Krull domains R with field of fractions K such that all maximal R- orders in $M_n(K)$ are conjugated and in general we will show how to calculate the number of isomorphism classes.

By PGL_n we denote $Aut(P^n_R)$, the automorphism scheme of the n-dimensional projective space over R , i.e, PGL_n is the sheafification of the presheaf which assigns $PGL_n(\Gamma(U, \mathcal{O}_R))$ to an open set U of $Spec(R)$. For more details, the reader is referred to Milne [52].

Proposition 4.1 : If R is a locally factorial Krull domain and if $\Lambda = M_n(R)$ then $H^1_{Zar}(U, \mathcal{N}_\Lambda) = H^1_{Zar}(U, PGL_n)$ for every open set U of $Spec(R)$.

Proof :

If we assign to an open set U of $Spec(R)$ the group $GL_n(\Gamma(U, \mathcal{O}_R)).K^* \subset GL_n(K)$, then this defines a presheaf of groups on $Spec(R)$. We will denote its sheafification by $GL_n.K^*$. Clearly, this sheaf is a subsheaf of $\mathcal{N}_\Lambda$. First, we will prove that all their stalks are isomorphic.

If $p \in Spec(R)$ and if $x \in N(M_n(R_p))$, then $M_n(R_p).x = M_n(A)$ for some divisorial R_p-ideal A. Because R_p is a unique factorization domain , $A = R_p.k$ for some $k \in K^*$, yielding that $x \in GL_n(R_p).k$. Therefore, $GL_n.K^* = \mathcal{N}_\Lambda$.

The following sequence of sheaves of groups is exact :

$$1 \to K^* \to GL_n.K^* \to GL_n/GL_n \cap K^* \cong PGL_n \to$$

where K^* is the constant sheaf with sections K^*.

Taking sections over U yields the following long exact cohomology sequence :

$$1 \to \Gamma(U, K^*) \to \Gamma(U, \mathcal{N}_\Lambda) \to \Gamma(U, PGL_n) \to 1$$

$$1 \to H^1_{Zar}(U, \mathcal{N}_\Lambda) \to H^1_{Zar}(U, PGL_n) \to 1$$

finishing the proof.

Corollary 4.2 : If R is a locally factorial Krull domain with field of fractions K , then :

$$t_R(M_n(K)) \simeq \varinjlim H^1_{Zar}(U, PGL_n)$$

where the direct limit is taken over all open sets U containing $X^{(1)}(R)$.

Proof :
Follows immediatly from proposition 3.22 and the foregoing proposition.

Let us now apply this result to some special classes of locally factorial Krull domains.

(a) : Dedekind domains.

Proposition 4.3 : If R is a Dedekind domain, then all maximal R-orders in $M_n(K)$ are conjugated if and only if the morphism :

$$(-)^n : Cl(R) \to Cl(R)$$

sending $[A]$ to $[A]^n$ is an epimorphism. More generally, the elements of $Coker((-)^n)$ are in one-to-one correspondence with the conjugacy-classes of maximal R-orders in $M_n(K)$.

Proof :

By a sheaf version of the Skolem-Noether theorem, cfr e.g. Milne [52], the following sequence of sheaves of groups is exact :

$$1 \to \mathcal{O}_R^* \to GL_n \to PGL_n \to 1$$

So, writing out the long exact cohomology sequence entails :

$$H^1_{Zar}(X, \mathcal{O}_R^*) \to H^1_{Zar}(X, GL_n) \to H^1_{Zar}(X, PGL_n) \to H^2_{Zar}(X, \mathcal{O}_R^*)$$

Now, because R is a Dedekind domain , it has Krull dimension one and hence $H^2_{Zar}(U, \mathcal{O}_R^*)$ vanishes. Furthermore, by [52] we know that $H^1_{Zar}(X, GL_n)$ is the set of isomorphism classes of projective rank n R-modules , a set which we will denote by $Proj_n(R)$. By Steinitz' result , cfr. e.g. [53] , any projective rank n R-module is isomorphic to $J_1 \oplus \ldots \oplus J_n$ for some fractional R-ideals J_i . Further, $H^1_{Zar}(X, \mathcal{O}_R^*) = Pic(R) = Cl(R)$ and $\delta : H^1_{Zar}(X, \mathcal{O}_R^*) \to H^1_{Zar}(X, GL_n)$ is defined by sending $[I]$ to $[I \oplus \ldots \oplus I]$ because δ derives from the sheafmorphism $\mathcal{O}_R^* \to GL_n$ assigning locally the diagonal matrix $diag(\alpha)$ to a unit α. By Corollary 4.2 all maximal orders in $M_n(K)$ are conjugated if and only if δ is epimorphic. That is, for any fractional R-ideals $J_1, \ldots, J_n$, there exists a fractional R-ideal I such that $J_1 \oplus \ldots \oplus J_n \simeq I \oplus \ldots \oplus I$. Applying again Steinitz' theorem we find : $J_1 \ldots J_n \simeq I^n$ finishing the first part of the proof because every fractional R-ideal J must have an n^{th}-root.

As for the second part; the set of conjugacy classes of maximal orders in $M_n(K)$ is in one-to-one correspondence with $Coker(\delta)$. By Steinitz' result two projective rank n R-modules $I_1 \oplus \ldots \oplus I_n$ and $J_1 \oplus \ldots \oplus J_n$ are isomorphic if and only if $I_1 \ldots I_n \simeq J_1 \ldots J_n$ entailing that $Coker(\delta) \simeq Coker((-)^n)$, finishing the proof.

Remark 4.4 : F. Van Oystaeyen suggested the following independent ringtheoretical proof of this result. Because $M_n(R)$ is an Azumaya algebra , all maximal R-orders in $M_n(K)$ are Azumaya algebras , cfr. e.g. Reiner [67] or

Roggenkamp [70]. Furthermore, the natural morphism :

$$[-\otimes_R K] : Br(R) \to Br(K)$$

is monomorphic whence any maximal R-order is of the form $End_R(P)$ where $P \in Proj_n(R)$. Applying again the theorem of Steinitz to the condition $End_R(P) \simeq M_n(R)$ yields the same comdition on $Cl(R)$.

Another interpretation of Proposition 4.3 is :

Corollary 4.5 : If R is a Dedekind domain with field of fractions K, then the following sequence of Abelian groups is exact :

$$1 \to Cl(R)_n \to Cl(R) \to Cl(R) \to t_R(M_n(K)) \to 1$$

where the morphism $Cl(R) \to Cl(R)$ is given by sending a class $[I]$ to $[I^n]$.

It is now quite easy to construct counterexamples to the question of Maury and Raynaud in the case of maximal orders in matrixrings :

Example 4.6 : Let $R = \mathbb{Z}[\sqrt{-5}]$, then $Cl(R) = \mathbb{Z}/2.\mathbb{Z}$ and as a generator we may take the nonprincipal ideal $I = (2, 1+\sqrt{-5})$. Now, let $\Lambda = End_R(R \oplus I)$, then :

$$\Lambda \simeq \begin{pmatrix} \mathbb{Z}[\sqrt{-5}] & (2,1+\sqrt{-5}) \\ (2,1+\sqrt{-5})^{-1} & \mathbb{Z}[\sqrt{-5}] \end{pmatrix}$$

and it is easy to verify that Λ cannot be conjugated to $M_2(\mathbb{Z}[\sqrt{-5}]$.

This example presents also a counterexample to the following question : in the commutative case we know that the fixed ring of a Krull domain under a finite group of automorphisms is again a Krull domain , does a similar result holds also for maximal orders over Krull domains ? Let ϕ be the natural non-trivial automorphism on $\mathbb{Q}(\sqrt{-5})$ and by Φ we denote its extension to $M_2(\mathbb{Q}(\sqrt{-5})$. It is easy to verify that $\phi(I) = I$ yielding that Φ is also an automorphism of Λ. Calculating the fixed ring yields :

$$\Lambda^{\Phi} \simeq \begin{pmatrix} \mathbb{Z} & 2.\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

and this ring is definitely not a maximal order. It would be interesting to know whether the fixed ring of a maximal order over a Krull domain is always a tame order.

More generally, for any ring of integers R in an algebraic number field K all maximal R-orders in $M_n(K)$ are conjugated if and only if $Cl(R)$ contains no n-torsion. This follows immediatly from the fact that $Cl(R)$ is finite , so injectivity of $(-)^n$ implies surjectivity and conversely.

Example 4.7 : The exact sequence of corollary 4.5 makes it possible to compute the type number in many cases. In the examples below we calculate

$$T_{\mathbb{Z}[\sqrt{-m}]}(M_n(\mathbb{Q}(\sqrt{-m})))$$

in function of n and m :

$m =$	$Cl(\mathbb{Z}[\sqrt{-m}] =$	$n = 2$	$n = 3$
5	2	2	1
23	3	1	3
29	6	2	3

The asymptotical classgroup of a maximal order Λ , $Cl_{\infty}(\Lambda)$, which was introduced by the author in [41] is defined in the following way :

Let $\{M_n(\Lambda); n \in \mathbb{N}\}$ be filtered by :

$$M_n(\Lambda) \leq M_m(\Lambda) \text{ iff } n.a = m$$

and the corresponding natural morphisms :

$$M_n(\Lambda) \to M_a(M_n(\Lambda))$$

Clearly, all these ringextensions satisfy pas d'éclatement and hence there is a filtered family :

$$\{Cl(M_n(\Lambda)); n \in \mathbb{N}\}$$

The asymptotical classgroup is now defined to be :

$$Cl_\infty(\Lambda) \simeq \varinjlim Cl(M_n(\Lambda))$$

For a more K-theoretic approach to this asymptotical classgroup the reader is referred to [41].

Proposition 4.8 : If R is a commutative Dedekind domain , then :

$$Cl_\infty(R) \simeq Cl(R) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Proof :

We have seen earlier that :

$$Ker(Cl(R) \to Cl(M_n(R)))$$

consists of n-torsion elements.

Conversely, if $[I]$ is an n-torsion element in $Cl(R)$ then :

$$I \oplus \dots \oplus I \simeq I^n \oplus \dots \oplus R \simeq R \oplus \dots \oplus R$$

yielding that $M_n(I)$ is generated by a normalizing element. Therefore, the kernel of the epimorphic morphism :

$$Cl(R) \to Cl_\infty(R)$$

consists precisely of the torsion elements, finishing the proof.

(b) : locally factorial Krull domains.

We will need the following two lemmas :

Lemma 4.9 : If R is a locally factorial Krull domain , then $H^2_{Zar}(U, \mathcal{O}^*_R) = 1$ for every open set U in the Zariski- topology.

Proof :

Because R is locally factorial , Weil and Cartier divisors coincide, entailing exactness of the following sequence :

$$1 \to \mathcal{O}^*_R \to K^* \to \mathcal{D}_R \to 1$$

where $\mathcal{D}_R$ is the sheaf of Weil divisors on $Spec(R)$ and exactness in the Zariski topology. Because K^* is a constant sheaf and $\mathcal{D}_R$ is a flabby sheaf their higher cohomology- groups vanish. Writing out the long exact cohomology sequence then entails that $H^2_{Zar}(U, \mathcal{O}^*_R) = 1$.

Lemma 4.10 : If R is any Krull domain, then $\lim H^1_{Zar}(U, GL_n)$, where the direct limit is taken over all open sets U containing $X^{(1)}(R)$, is the set of isomorphism classes of reflexive R-modules which are free of rank n in any height one prime ideal of R.We will denote this set of isomorphism classes by $Ref_n(R)$.

The proof of this lemma is classical ,cfr. e.g. [52] p.134 .

Theorem 4.11 : If R is a locally factorial Krull domain then all maximal orders in $M_n(K)$ are conjugated if and only if the map from $Cl(R)$ to $Ref_n(R)$ sending $[I]$ to $[I \oplus \ldots \oplus I]$ is surjective.

Proof :

Consider the following exact sequence :

$$\varinjlim H^1(U,\mathcal{O}_R^*) \to \varinjlim H^1(U,GL_n) \to \varinjlim H^1(U,PGL_n) \to \varinjlim H^2(U,\mathcal{O}_R^*)$$

where the direct limits are taken over all open sets U in the Zariski topology containing xer. By lemma 4.9 we know that $\lim H^2(U,\mathcal{O}_R^*) = 1$. Further, b Danilov's theorem cfr. e.g. [17] (or lemma d.2) , $\varinjlim H^1(U,\mathcal{O}_R^*) = Cl(R)$ and $\lim H^1(U,GL_n) = Ref_n(R)$.

These facts imply that $\lim H^1(U,PGL_n) = 1$ if and only if the map from $Cl(R)$ to $Ref_n(R)$ sending $[I]$ to $[I \oplus ... \oplus I]$ is epimorphic .

Remark 4.12 : Of course, the condition for all maximal orders to be conjugated is a very stringent one. E.g. if R is a unique factorization domain, then all maximal orders in $M_n(K)$ are conjugated if and only if every reflexive R-module (e.g. every projective R-module) is free. It is already clear from this remark that there will be a strong connection between conjugateness of maximal orders in matrixrings and questions like the Bass- Quillen conjecture.

As a corollary of the foregoing result , we aim to recover the following classical result which was first proved by M. Ramras [64] :

Proposition 4.13 : If R is a regular local ring of $gldim(R) \leq 2$, then all maximal R-orders in $M_n(K)$ are conjugated.

Proof :

We have to check that $H^1_{Zar}(U,PGL_n) = 1$ where $U = X(m)$, m being the unique maximal ideal of R. Now, as a special case of lemma d.2 above we find that $H^1_{Zar}(U,GL_n)$ is $Ref_n(R)$. Because $gldim(R) \leq 2$, reflexive R-modules are projective whence $Ref_n(R) = Proj_n(R)$ and $Ref_1(R) = Pic(R)$. Finally, R being local implies that $Pic(R) = Proj_n(R) = 1$ and theorem 4.11 finishes the proof.

Corollary 4.14 : If R is a regular local ring of $gldim(R) \leq 2$, then all maximal R-orders in $M_n(K)$ are Azumaya algebras.

Before ending this section let us mention that M. Van den Bergh has been able to generalize Proposition 4.8 to the higher dimensional case.

Theorem 4.15 : If R is a commutative Krull domain of finite Krull dimension , then :

$$Cl_\infty(R) \simeq Cl(R)/Tors(Pic(R))$$

(2) : MAXIMAL ORDERS WITH A DISCRETE NORMALIZING CLASSGROUP

As was already mentioned in the introduction , most of the technical machinery described above was developed in order to study the relation between the normalizing classgroup $Cl(\Lambda)$ of a maximal order Λ over a normal domain R and that of the ring of formal power series $\Lambda[[t]]$ over it which is a maximal order over $R[[t]]$ by a result of H. Marubayashi [47]. Because $\Lambda \to \Lambda[[t]]$ is clearly an extension satisfying pas d'éclatement there is a natural morphism :

$$Cl(\Lambda) \to Cl(\Lambda[[t]])$$

by virtue of the results on p.143 .

Definition 4.16 :

A maximal order Λ over a normal domain R is said to have a discrete normalizing classgroup if $Cl(\Lambda) \simeq Cl(\Lambda[[t]])$.

In this section we aim to study whether having a discrete normalizing classgroup is a central property , i.e. we would like to answer the next two problems

Problem 4.17 : If Λ is a maximal order over a normal domain R having a discrete classgroup , does this imply that Λ has also a discrete normalizing classgroup ?

Problem 4.18 : If a maximal order over a normal domain R has a discrete normalizing classgroup , does this imply that R has a discrete classgroup ?

We will answer Problem 4.17 affirmatively as well as Problem 4.18 if the maximal order is an Azumaya algebra. Even if Λ is a reflexive Azumaya algebra it is not at all clear that Problem 4.18 should be true. The main problem seems to be that torsion elements of the classgroup of a normal domain can be killed in a reflexive Azumaya algebra. Some examples of such a situation are given.

Let us start by generalizing Danilov's construction of the natural splitting morphism to maximal orders :

(a) : the natural splitting

Theorem 4.19 :

If Λ is a maximal order over a normal domain R , then $Cl(\Lambda)$ is a direct summand of $Cl(\Lambda[[t]])$.

Proof :

We will merely scetch the proof , details are left to the reader.

Let X and Y denote respectively $Spec(R)$ and $Spec(R[[t]])$, then :

$$j : f(t) \to f(0)$$

induces a closed regular immersion $X \to Y$ which identifies X with $V(T) = \{P \in Y \mid (T) \subset P\}$. It is clear that j induces in a natural way morphisms of sheaves of groups :

$$j_1^* : \mathcal{O}^*_{\Lambda[[t]]} \to \mathcal{O}^*_{\Lambda}$$

$$j_2^* : \mathcal{N}_{\Lambda[[t]]} \to \mathcal{N}_{\Lambda}$$

and these morphisms cause the following exact commutative diagram :

$$\begin{array}{ccccccccc} 1 & \to & Cl(\Lambda) & \to & \lim H^1(U, \mathcal{O}^*_\Lambda) & \to & \lim H^1(U, \mathcal{N}_\Lambda) & \to & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \to & Cl(\Lambda[[t]]) & \to & \lim H^1(V, \mathcal{O}^*_{\Lambda[[t]]}) & \to & \lim H^1(V, \mathcal{N}_{\Lambda[[t]]}) & \to & 1 \end{array}$$

Furthermore, the inclusion $R \to R[[t]]$ induce a natural morphism $i : Y \to X$ which induces morphisms of sheaves of groups :

$$i_1^* : \mathcal{O}_\Lambda^* \to \mathcal{O}_{\Lambda[[t]]}^*$$

$$i_2^* : \mathcal{N}_\Lambda \to \mathcal{N}_{\Lambda[[t]]}$$

leading to the following exact commutative diagram :

$$\begin{array}{ccccccccc} 1 & \to & Cl(\Lambda) & \to & \lim H^1(U, \mathcal{O}_\Lambda^*) & \to & \lim H^1(U, \mathcal{N}_\Lambda) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & Cl(\Lambda[[t]]) & \to & \lim H^1(V, \mathcal{O}_{\Lambda[[t]]}^*) & \to & \lim H^1(V, \mathcal{N}_{\Lambda[[t]]}) & \to & 1 \end{array}$$

and a carefull investigation of these two diagrams learns that the induced morphism $Cl(\Lambda[[t]]) \to Cl(\Lambda)$ yields a natural splitting for the map $Cl(\Lambda) \to Cl(\Lambda[[t]])$.

In particular this theorem implies that $Cl(\Lambda) \to Cl(\Lambda[[t]])$ is a monomorphism.

(b) : solution to problem 4.17

Theorem 4.20 :
If Λ is a maximal order over a Krull domain R with a discrete classgroup , then Λ itself has a discrete normalizing classgroup.

Proof :
First, we will show that Λ has a discrete central classgroup. As was remarked earlier , the cokernel of the injective natural morphism from the classgroup of a Krull domain to the central classgroup of a maximal order consists of a finite group which determines the ramification of height one prime ideals.

Therefore, we obtain the following exact commutative diagram :

$$\begin{array}{ccccccccc} & & 1 & & & & & & \\ & & \downarrow & & & & & & \\ 1 & \rightarrow & Cl(R) & \rightarrow & Cl^c(\Lambda) & \rightarrow & \bigoplus \mathbb{Z}/e_p.\mathbb{Z} & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & Cl(R[[t]]) & \rightarrow & Cl^c(\Lambda[[t]]) & \rightarrow & \bigoplus \mathbb{Z}/e_Q.\mathbb{Z} & \rightarrow & 1 \\ & & \downarrow & & & & & & \\ & & 1 & & & & & & \end{array}$$

so it will suffice to prove that the ramification-groups are isomorphic. Let Q be any height one prime ideal of $\Lambda[[t]]$, then either $Q \cap \Lambda = 0$ or $Q \cap \Lambda$ is an height one prime ideal of Λ since $R \rightarrow R[[t]]$ is an extension satisfying pas d'éclatement. Suppose first that $Q \cap \Lambda = 0$, then the localization of $\Lambda[[t]]$ at Q is a localization of $\Sigma[[t]]$ and is therefore an Azumaya algebra showing that Q is not ramified with respect to the center.

Therefore, the only ramified height one prime ideals are of the form $P[[t]]$ where $P \in X^{(1)}(\Lambda)$. It is easy to verify that $P[[t]]$ is ramified if and only if P is ramified and their ramification- indices are equal.

Therefore, $\bigoplus \mathbb{Z}/e_p.\mathbb{Z} \simeq \bigoplus \mathbb{Z}/e_Q.\mathbb{Z}$ and consequently $Cl^c(\Lambda) \simeq Cl^c(\Lambda[[t]])$.

Finally, we have the following exact commutative diagram :

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ 1 & \rightarrow & Outcent(\Lambda) & \rightarrow & Cl^c(\Lambda) & \rightarrow & Cl(\Lambda) & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & Outcent(\Lambda[[t]]) & \rightarrow & Cl^c(\Lambda[[t]]) & \rightarrow & Cl(\Lambda[[t]]) & \rightarrow & 1 \\ & & & & \downarrow & & & & \\ & & & & 1 & & & & \end{array}$$

Surjectivity of the natural morphism $Cl(\Lambda) \rightarrow Cl(\Lambda[[t]])$ follows from the above diagram , whereas injectivity follows from Th.1 , finishing the proof of the theorem.

A immediate consequence of this result is :

Corollary 4.21 :

If Λ is a maximal order over a regular domain R , then Λ has a discrete normalizing classgroup.

(c) : problem 4.18 for azumaya algebras

Proposition 4.22 :

If Λ is an Azumaya algebra over the Krull domain R , then R has a discrete classgroup if and only if Λ has a discrete classgroup.

Proof :

One implication is clear from the foregoing theorem. Now, suppose that Λ has a discrete normalizing classgroup, then we have the exact commutative diagram :

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \to & Outcent(\Lambda) & \to & Cl(R) & \to & Cl(\Lambda) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Outcent(\Lambda[[t]]) & \to & Cl(R[[t]]) & \to & Cl(\Lambda[[t]]) & \to & 1 \\
 & & & & & & \downarrow & & \\
 & & & & & & 1 & &
\end{array}$$

First, we claim that $Outcent(-)$ consists of torsion elements of the Picardgroup. For, let Γ be an Azumaya algebra over the Krull domain S and suppose that I is a divisorial S-ideal such that $\Gamma.I = \Gamma.n$ for some normalizing element n , then $\tilde\Gamma.\tilde I^{-1} = \tilde\Gamma.n^{-1}$ and $\tilde I.\tilde I^{-1} = \tilde S$ since for any $\tilde S$-ideal $\tilde J$ we have $\tilde\Gamma.\tilde J \cap \tilde L = \tilde J$, where L is the field of fractions of S. Therefore, I is an element of the Picardgroup of S and moreover taking reduced norms we obtain : $I^m = S.nr(n)$ where $m = p.i.d.(\Gamma)$ showing that I is a torsion element , finishing the proof of our claim.

Now, let us show that $Outcent(\Lambda) \to Outcent(\Lambda[[t]])$ is an epimorphic map. If $[I] \in Outcent(\Lambda[[t]])$, then $[I]$ is an element of $Pic(R[[t]])$. Now, for any Krull domain R we know that the natural morphism :

$$Pic(R) \to Pic(R[[t]])$$

is an isomorphism, showing that there exists an element $[I_0] \in Pic(R)$ such that $I_0[[t]] \simeq I$. Further, by the injectivity of the morphism $Cl(\Lambda) \to Cl(\Lambda[[t]])$ it follows that $[I_0] \in Outcent(\Lambda)$, finishing the proof.

The difficulty in extending this result to reflexive Azumaya algebras is that $Ker(Cl(R) \to Cl(\Lambda))$ consists not necessarely of elements of the Picard-group of R.Let us give an example of such a situation :

Example 4.22 :

Let R be a Krull domain and let I be a representant of a 2-torsion element in $Cl(R)$. Now, consider :

$$\Lambda = End_R(R \oplus I) \simeq \begin{pmatrix} R & I \\ I^{-1} & R \end{pmatrix}$$

then Λ is a reflexive Azumaya algebra over R (Λ is Azumaya if and only if $[I] \in Pic(R)$). This entails that there is a well defined isomorphism :

$$\psi : D(R) \to D(\Lambda); \psi(A) = (\Lambda.A)^{**}$$

showing that every divisorial Λ-ideal is of the form :

$$\begin{pmatrix} A & I * A \\ I^{-1} * A & A \end{pmatrix}$$

where $A \in D(R)$. ψ induces a morphism :

$$\Phi : Cl(R) \to Cl(\Lambda)$$

which is clearly epimorphic. The class of an ideal A is killed under Φ if and only if :

$$A \oplus (I * A) \simeq R \oplus I$$

the isomorphism being one of R-modules. So, in particular , if we take $A = I$ then $\Phi([I]) = 1$ because :

$$\begin{pmatrix} I & I*I \\ I^{-1}*I & I \end{pmatrix} = \begin{pmatrix} I & R.\alpha \\ R & I \end{pmatrix} = \begin{pmatrix} R & I \\ I^{-1} & R \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$$

where the element on the right is readily checked to be a nontrivial normalizing element of Λ.

Thus, any 2-torsion element of the classgroup (resp. of the Picard- group) of R can be killed off in the normalizing classgroup of a reflexive Azumaya algebra (resp. Azumaya algebra) over R of $p.i.d.(\Lambda) = 2$.

This construction can of course be extended to higher torsion elements , for , if $[I] \in Cl(R)_n$, then take :

$$\Lambda = End_R(R \oplus I \oplus I^2 \oplus \ldots \oplus I^{n-1})$$

and the class of a divisorial ideal A is killed in $Cl(\Lambda)$ if and only if :

$$A \oplus (A * I) \oplus \ldots \oplus (A * I^{n-1}) \simeq R \oplus I \oplus \ldots \oplus I^{n-1}$$

So, in particular , taking $A = I$ and $I * I^{n-1} = R.\alpha$ we know that $(\Lambda.I)^{**}$ is generated by the normalizing element :

$$\begin{pmatrix} 0 & 0 & \cdots & \alpha \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Conversely, it is of course easy (taking reduced norms) that the kernel of the natural map :

$$Cl(R) \to Cl(\Lambda)$$

consists of n-torsion elements if $p.i.d.(\Lambda) = n$.

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