On a Problem of R.A. Hirschfeld.

0. Introduction.

One of the classical results in functional analysis is the Gelfand-Mazur theorem which states that every complex Banach divisor algebra is isomorphic to $\mathbb{C}$. More generally, I. Kaplansky [3] proved that every complex Von Neumann regular Banach algebra is finite dimensional over $\mathbb{C}$.

In recent years, R.A. Hirschfeld [2] and A. Verschoren [6] have extended Kaplansky's theorem in another direction.

R.A. Hirschfeld characterized all complex topological Von Neumann regular algebras which are finite dimensional over $\mathbb{C}$, and, on his suggestion, A. Verschoren extended his work to complex π-regular algebras. Verschoren obtained the following result.

Theorem 1. [6]

Let A be a complex π-regular topological algebra satisfying the following conditions :

(a) A is a Fréchet algebra (i.e. complete metric)

(b) $A/J(A)$ contains no strict field extension of $\mathbb{C}$

(c) A contains no small idempotents

then $A/J(A)$ is finite dimensional over $\mathbb{C}$.

Here, $J(A)$ denotes the (ringtheoretical) Jacobson-radical of A, cfr. e.g. [5] , and A is said to contain no small idempotents if its origin has a neighbourhood containing no nontrivial idempotent element. E.g. the open unit ball of a complex Banach algebra does not contain any idempotent.

R.A. Hischfeld asked whether one could extend Theorem 1 to complex topological algebras which are polynomial-regular, i.e. for every $a \in A$ there exists a polynomial $f(X) \in \mathbb{C}[X]$ with zero constant term and an element $b \in A$ such that $f(a).b\, f(a) = f(a)$. The main purpose of this note is to answer this question affirmatively. A secondary aim is to show that the arguments given in [2] and [6] can be shortened considerably by using the classical Artin-Weddenburn structure theorem, cfr. e.g. [1].

1. Polynomial regular Fréchet algebras.

In this section we aim to prove the following result.

Theorem 2. Let A be a Fréchet algebra with identity which is polynomial regular and contains no small idempotents, then $A/J(A)$ is semisimple Artinian.

Throughout we will assume that A is a complex polynomial regular algebra and by an idempotent we will always mean a nonzero idempotent.
If $a \in A$, we will denote by P_a the set of all polynomials $f(X) \in \mathbb{C}[X]$ with zero constant term and with a minimal number of non-zero coefficients such that $f(a).b.f(a) = f(a)$ for some $b \in A$. E.g. if e is an idempotent element, then P_e consists of monomials. Further
Further, $H(A) = \{a \in A \mid \forall x \in A, \forall f \in P_{ax} : f(ax) = 0\}$.

Lemma 1 : If A is polynomial regular, then $J(A) = H(A)$.

Proof. By definition $a \in J(A)$ implies that $1-ay$ is invertible on the left for every $y \in A$. Now, take $x \in A$, $f \in P_{ax}$ and an element $b \in A$ such that $f(ax)b\, f(ax) = f(ax)$. Then $1-f(ax)b = 1 - ax\, h(ax)c$ for some polynomial $h(X) \in \mathbb{C}[X]$ and hence there exists an element $Z \in A$ such that $Z(1-f(ax)b) = 1$. Finally, $f(ax) = Z.(1-f(ax)c)\, f(ax) = 0$ and hence $J(A) \subset H(A)$. Conversely, let $a \in H(A)$ then for every $x \in A$ and every $f(X) \in P_{ax}$ we have $f(ax) = 0$. Thus, for some $r \in \mathbb{N}$ and some polynomial h we have :

$$(\star)\quad (ax)^r(1-ax\, h(ax)) = 0$$

Now, take $e = (ax)^r h(ax)^r\ h(ax)^r$ then we get using (★) and the fact that ax and h(ax) commute :

$e^2 = (ax)^{2r}\ h(ax)^{2r} = (ax)^{2r-1}\ h(ax)^{2r-1} = \dots = (ax)^r\ h(ax)^r = e$

Because $H(A)A \subset H(A)$, e is an idempotent element of H(A). Every $f \in \mathcal{P}_e$ is monic yielding that e = 0. Therefore, ax.h(ax) is nilpotent and hence (1 - ax h(ax)) is invertible, its inverse being $1 + ax\ h(ax) + (ax)^r h(ax)^r + \dots + (ax)^{r-1} h(ax)^{r-1}$. ~~Finally, using~~ (★) we obtain that ax is nilpotent, whence 1-ax is invertible, so $a \in J(A)$.

Lemma 2. Let A be a polynomial regular Fréchet algebra which contains no small idempotents, then A contains no infinite family of commuting idempotents.

Proof. Suppose there is an infinite family of commuting idempotents then by [4] we can find an infinite family of mutually orthogonal idempotents $(p_n)_{n \in \mathbb{N}}$. Using completeness of A we can find positive real numbers $\alpha_n \in (0,1)$ such that $\|\beta p_n\| < 2^{-n}$ for each $\beta \in [0, \alpha_n]$ $\|\ \|$ being the F norm . Let $\lambda_n = \alpha_n^2$ then the sequences $\Sigma\, \lambda_n\, p_n$ and $\Sigma\, \lambda_n^{1/2}\, p_n$ are absolutely convergent. Let $a = \Sigma\, \lambda_n p_n \in A$ then there exists an element $b \in A$ and a polynomial $f(X) \in \mathbb{C}[X]$ with zero constant term such that $f(a).b.f(a) = f(a)$. Now, $g(a) = \Sigma\, g(\lambda_n)p_n.b.\ \Sigma\, g(\lambda_n)p_n = \Sigma\, g(\lambda_n)p_n$. Multiplying both terms on both sides with p_r yields $g(\lambda_r)p_r\ bg(\lambda_r)p_r = g(\lambda_r)p_r$ entailing that for every $n \in \mathbb{N}$:

$$g(\lambda_n)^{1/2}\ p_n b\ g(\lambda_n)^{1/2} p_n = p_n$$

Now, $c = \Sigma\, g(\lambda_n)^{1/2}\ p_n$ exists by the choice of the $(\lambda_n)_{n \in \mathbb{N}}$. Multiplication being jointly continuous in a complete metric algebra we obtain $c.b.c. = \lim_{N \to \infty} \sum_{n=1}^{N} p_n$. But A contains no small idempotents, hence the right side does not exist.

Proof of Theorem 2. Using the proof of lemma 1, every nonzero right ideal in A/J(A) contains a nonzero idempotent. Using a result of Kaplansky's, it will therefore be sufficient to show that A/J(A) contains no infinite number of orthogonal idempotents. Assume otherwise, then a countable subset of them can be lifted to a family of orthogonal idempotents of A by [5, VIII Prop. 4.] and the fact that J(A) is a nil ideal. But this contradicts Lemma 2.

2. Finite dimensionaly of A/J(A).

We are now in a position to answer Hirschfeld's question :

Proposition 1. Let A be a complex polynomial regular algebra satisfying satisfying the following conditions

(a) A is a Fréchet algebra

(b) A/J(A) contains no strict field extension of $\mathbb{C}$

(c) A contains no small idempotents

then A/J(A) is finite dimensional over $\mathbb{C}$

Proof.

It follows from theorem 2 and the Artin-Weddenburn result that

$$A/J(A) \cong M_{k_1}(\Delta_1) \oplus \dots \oplus M_{k_n}(\Delta_n)$$

where Δ_i is a division algebra. Condition (b) implies that $\Delta_i \cong \mathbb{C}$ for each i, finishing the proof.

Of course, condition (b) is very restrictive. In fact we have the following result, which seems to have been overlooked in [6] :

Proposition 2. If A is complex polynomial regular Fréchet algebra without small idempotents, then the following statements are equivalent :

(a) : A/J(A) contains no strict field extension of $\mathbb{C}$

(b) : A is algebraic over $\mathbb{C}$

Proof.

(a) ⇒ (b) : By proposition 1 we know that :

$$A/J(A) \cong M_{k_1}(\mathbb{C}) \oplus \ldots \oplus M_{k_n}(\mathbb{C})$$

Take any element $\overline{a} = (\alpha_1,\ldots,\alpha_n) \in A/J(A)$, then $\overline{a}$ satisfies a polynomial $f(X) \in \mathbb{C}[X]$, namely :

$$f(X) = \prod_{i=1}^{n} g_i(\alpha_i)$$

where $g_i(\alpha_i)$ is the characteristic polynomial of α_i. So, for any $a \in A$ there exists a polynomial $f(X) \in \mathbb{C}[X]$ such that $f(a) \in J(A)$. It follows from the proof of lemma 1 that $J(A)$ is a nil ideal. Therefore, there exists a natural number m such that $f(a)^m = 0$. Finally, a satisfies $f(X)^m$, finishing the proof.

(b) ⇒ (a) : $A/J(A)$ is algebraic and $\mathbb{C}$ algebraically closed.

References.

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