Maximal Orders Having a Discrete Normalizing Classgroup.

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On a Problem of R.A. Hirschfeld.

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0. Introduction

Some concepts from algebraic geometry such as Cartier and Weil divisors of a scheme were used in a very elegant way by V.I. Danilov [2] [3] [4] in order to study the ralation between the classgroup of a normal domain R and the classgroup of R[[t]] , the ring of formal power series over R, a problem which has its roots in a conjecture of P. Samuel, cfr. [2] [5]. The strategy he uses is the following : first, one can express the class group of a normal domain R in terms of Picard groups of certain open subschemes of the affine scheme X = Spec(R). The next step is to use the good functorial properties of Picard group and the fact that there is a regular closed immersion of X into Y = Spec R [[t]], to prove the desired theorems on the open sets mentioned above and afterwards Danilov pulls the obtained information back to the class group.

In this way Danilov was able to define a natural splitting morphism for the morphism :

$$\text{Cl } R \to \text{Cl } R[[t]]$$

and to give some necessary and sufficient conditions on the normal domain R in order to ensure that this morphism is actually an isomorphism. If this is the case, we say that R has a discrete class group . If Λ is a maximal order over a normal domain domain R in some central simple K-algebra Σ, K being the field of fractions of R, one can define the normalizing class groups $\mathbb{C}l(\Lambda)$ of Λ to be the quotient group of the free Abelian group of twosided divisorial Λ-ideals modulo the subgroup of those divisorial Λ-ideals which are generated by a normalizing element (i.e. an element $x \in \Sigma$ such that $x\Lambda = \Lambda x$), cfr. e.g. [1] , [7] .

Because two sided fractional Λ-ideals extend to twosided fractional $\Lambda[[t]]$ -ideals and $\Lambda[[t]]$ is again a maximal order, cfr. [8] , one has a natural morphism :

$$\text{Cl } \Lambda \to \text{Cl } \Lambda[[t]]$$

We say that Λ has a discrete normalizing class group if this morphism is an isomorphism. The main aim of this paper is to study whether having a normalizing class group is a central property, i.e. we would like to answer the following two problems :

Problem A : if Λ is a maximal order over a normal domain R having a discrete class group, doet this imply that Λ has a discrete normalizing class group.

Problem B : If a maximal order over a normal domain R has a discrete normalizing class group, does this imply that R has a discrete class group ?

We will answer problem A affirmatively as well as Problem B is the maximal order considered is an Azumaya algebra. Even for reflexive Azumaya algebras it is not clear to the author that Problem B should be true.The main problem seems to be that torsion elements of the class groups of a normal domain not contained in Pic(R) can be killed in a reflexive Azumaya algebra. Some examples of such a situation are given.

1. The natural splitting.

Let us start by generalizing Danilov's construction of the natural splitting to maximal orders :

Theorem 1. If Λ is a maximal order over a normal domain R, then $Cl(\Lambda)$ is in a natural way a direct summand of $Cl(\Lambda[[t]])$.

Proof.

Let us first introduce some notation : $X = Spec(R)$, $Y = Spec\ R[[t]]$ and $j : X \to Y$ is the natural embedding identifying X with V(T) through the map which assigns to a prime ideal p of R the prime ideal $R[[t]]p + R[[t]]t$. It is clear that j is a regular closed immersion. Further, let $\underline{\theta}_{\Lambda[[t]]}$ denote the structure sheaf of the $R[[t]]$-algebra $\Lambda[[t]]$ over Y. If A is a divisorial ideal of $\Lambda[[t]]$, then $\underline{\theta}_A$ will be its structure sheaf as an $R[[t]]$-module. Clearly, $\underline{\theta}_A$ is a divisorial $\underline{\theta}_{\Lambda[[t]]}$-Ideal. Because maximal orders over discrete valuation rings ~~are left~~ and right principal ideal rings, we have that $(\underline{\theta}_A)_P = (\underline{\theta}_{\Lambda[[t]]})_P \cdot \alpha$ for some $\alpha_P \in \Lambda[[t]]_P$ for every height one prime ideal P of $R[[t]]$. Clearly this equality extends to some open neighbourhood of P in Y.

Now, let $P = p\,R[[t]] + t\,R[[t]]$ for some height one prime ideal p of R. We claim that $(\underline{\theta}_A)_P$ is an invertible $(\underline{\theta}_{\Lambda[[t]]})_P$-ideal. For, $\Lambda_p[[t]]$ which is a maximal order over the regular local domain $R_p[[t]]$ (because j is a regular embedding !) and $\Lambda_p[[t]]$ is of global dimansion two by [10, Prop. 5.6.]. Finally, $\Lambda_p[[t]]$ is quasi-local and therefore every left divisorial $\Lambda_p[[t]]$ - ideal is principal by the proof of [10, Th. 5.4.], So, $\Lambda_p[[t]]\,(\underline{\theta}_A)_{\hat{P}}$ is principal and by faithfully flat descent (as in the commutative case) so is $(\underline{\theta}_A)_P$.

Concluding, $\underline{\theta}_A$ is an invertible $\underline{\theta}_{\Lambda[[t]]}$ - ideal on some open set of Y which includes all height one prime ideals of $R[[t]]$ as well as all primes of the form $p\,R[[t]] + t\,R[[t]]$ for p a height one prime of R. Let $U = j^{-1}(V)$, then U is an open set of X containing all height one primes. Suppose $j_u : U \to Y$ is the composite of the embedding of U in X and j, then $j_u^{\star}(\underline{\theta}_A)$ is an invertible $(\underline{\theta}_\Lambda|U)$-ideal, where $\underline{\theta}_\Lambda$ is the structure sheaf of the R-algebra Λ on X. But then clearly $\Gamma(U, j_u^{\star}(\underline{\theta}_A))$ is a twosided divisorial Λ-ideal because $X^{(1)}(R) \subset U$. This defines the mapping

$$j_\Lambda^{\star} : Cl(\Lambda[[t]]) \to Cl(\Lambda)$$

It is also easy to see that $j^{\star}$ is a groupmorphism and that $j^{\star}$ yields a splitting for the natural map

$$i_\Lambda^{\star} : Cl(\Lambda) \to Cl(\Lambda[[t]]) \qquad \square$$

In particular, this theorem implies that $i_{\Lambda}^{\star}$ is a group-monomorphism.

2. Affirmative answer to problem A.

Proposition 1. If Λ is a maximal order over a normal domain with discrete classgroups, the Λ has a discrete normalizing class group.

Proof. Recall from [6] that the central class group, $CCl(\Lambda)$, of Λ is the quotient group of the group of two sided divisorial Λ-ideals modulo the subgroup of those which are generated by a central element. From [6] we retain that the natural sequence :

$$1 \longrightarrow Cl(R) \xrightarrow{\ \mu\ } CCl(\Lambda) \longrightarrow \bigoplus_{i=1}^{m} \mathbb{Z}/e_i\,\mathbb{Z} \longrightarrow 1$$

is exact and that $\mathrm{Coker}(\mu)$ is a finite group determined by the ramified height one primes of Λ. Our first claim is that the following commutative diagram is exact :

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Cl(R) & \xrightarrow{\ \mu\ } & CCl(\Lambda) & \longrightarrow & \bigoplus_{i=1}^{m} \mathbb{Z}/e_i\,\mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \\
1 & \longrightarrow & Cl(R[[t]]) & \xrightarrow{\ \mu'\ } & CCl(\Lambda[[t]]) & \longrightarrow & \bigoplus_{j=1}^{m'} \mathbb{Z}/f_i\,\mathbb{Z} & \longrightarrow & 1
\end{array}$$

For, let $c \in R$ be an element of the Formanek-center of Λ, cfr. e.g. [9] , then Λ_c is an Azumaya akgebra over R_c. This entails that $\Lambda[[t]]_c \cong \Lambda_c \otimes_{R_c} R_c[[t]]$ is an Azumaya algebra over $R_c[[t]]$. Therefore, the only ramified one prime ideals of $\Lambda[[t]]$ are of the form $P[[t]]$ with P a ramified height one prime of Λ. This proves that $\mathrm{Coker}\,\mu \cong \mathrm{Coker}\,\mu'$ as claimed.

Applying the snake lemma to the diagram above we get $CCl(\Lambda) \cong CCl(\Lambda[[t]])$.

If we denote with Oucent(-) the quotient group Aut(-)/In(-) where In(-) is the group of inner automorphisms, we get the following exact commutative diagram :

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Outcent}(\Lambda) & \longrightarrow & \mathrm{CCl}(\Lambda) & \longrightarrow & \mathrm{Cl}(\Lambda) & \longrightarrow & 1 \\ & & & & \downarrow\cong & & \downarrow\alpha & & \\ 1 & \longrightarrow & \mathrm{Outcent}(\Lambda[[t]]) & \longrightarrow & \mathrm{CCl}(\Lambda[[t]]) & \longrightarrow & \mathrm{Cl}(\Lambda[[t]]) & \longrightarrow & 1 \end{array}$$

Injectivity of α follows from theorem 1 whereas surjectivity follows from the diagram , finishing the proof.

An immediate consequence of this result is :

Corrollary 1 : If Λ is a maximal order over a regular domain R, then Λ has a discrete normalizing class group.

3. Problem B for Azumaya algebras.

Proposition 2. If Λ is an Azumaya algebra over a normal domain R with a discrete normalizing class group, then R has a discrete class group.

Proof. For any Azumaya algebra Γ over a normal domain S one has $\mathrm{CCl}(\Gamma) \cong \mathrm{Cl}(S)$ hance we obtain the following exact diagram :

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Outcent}(\Lambda) & \longrightarrow & \mathrm{Cl}(R) & \longrightarrow & \mathrm{Cl}(\Lambda) & \longrightarrow & 1 \\ & & \downarrow\alpha & & & & \downarrow\cong & & \\ 1 & \longrightarrow & \mathrm{Outcent}(\Lambda[[t]]) & \longrightarrow & \mathrm{Cl}(R[[t]]) & \longrightarrow & \mathrm{Cl}(\Lambda[[t]]) & \longrightarrow & 1 \end{array}$$

We claim that $\mathrm{Outcent}(\Gamma)$ consists of torsion elements of the Picardgroup of S for any Azumaya algebra over tha normal domain S. For, suppose that I is a divisorial S-ideal such that $\Gamma.I = \Gamma.m$ for some normalizing element m, then $\Gamma.I^{-1} = \Gamma.m^{-1}$ and $I.I^{-1} = S$ because for any S-ideal J we have that $\Gamma.J \cap L = J$ where L is the field of fractions of S. Therefore, I is an element of Pic(S) and moreover, taking reduced norms, we get

$$I^n = S.\mathrm{nr}(m)$$

where $n = \mathrm{p.i.d.}(\Gamma)$, showing that I is a torsion element in Pic(S), finishing the proof of our claim.

Now, let us show the the morphism Outcent$(\Lambda) \to$ Outcent $\Lambda[[t]]$ is epimorphic. If $[I] \in$ Outcent$(\Lambda[[t]])$, then $[I]$ is an element of Pic $R[[t]]$ For any normal domain R, we know that the natural morphism

$$\text{Pic } R \to \text{Pic } R[[t]]$$

is an isomorphism, showing that there exists an element $[I_o] \in$ Pic (R) such that $I_o[[t]] \cong I$. Further, by the injectivity of the morphism $Cl(\Lambda) \to Cl\ \Lambda[[t]]$ it follows that $[I_o] \in$ Outcent(Λ).
Finally surjectivity of α and the snake lemma entails that Cl(R) $\to$ Cl $R[[t]]$ is epimorphic, finishing the proof.

Recall from [11] that an algebra Λ over a normal domain R is said to be a reflexive Azumaya algebra if the natural map

$$(\Lambda \otimes \Lambda^{opp})^{**} \longrightarrow \text{End}_R(\Lambda)$$

is an isomorphism of R-algebras. For reflexive Azumaya algebras one can also prove that $CCl(\Lambda) \cong Cl(R)$ but the difficulty in extending the foregoing proposition to reflexive Azumaya algebras is that $\text{Ker}(Cl(R) \to Cl(\Lambda)) \cong$ Outcent(Λ) does not necessarily consist of elements of the Picard group of R. Let us give an example of such a situation :

Example 1. Let R be a normal domain and let I be a representant of a 2-torsion element in Cl(R). Consider :

$$\Lambda = \text{End}_R(R \oplus I) \cong \begin{pmatrix} R & I \\ I^{-1} & R \end{pmatrix}$$

then Λ is a reflexive Azumaya algebra over R(Λ is Azumaya if and only if $[I] \in$ Pic(R)). This entails that there is a well defined isomorphism :

$$\psi : \mathbb{D}(R) \to \mathbb{D}(\Lambda);\ \psi(A) = (\Lambda.A)^{**}$$

showing that every divisorial Λ-ideal is of the form :

$$\begin{pmatrix} A & I \star A \\ I^{-1} \star A & A \end{pmatrix}$$

where $A \in \mathbb{D}(R)$. ψ induces a morphism

$$\Phi : Cl(R) \to Cl(\Lambda)$$

which is really epimorphic. The class of an ideal A is killed under Φ if and only if :

$$A \oplus (I \star A) \cong R \oplus I$$

the isomorphism being one of R-modules. So, in particular, if we take $A = I$ then $\Phi([I]) = 1$ since

$$\begin{pmatrix} I & I \star I \\ I^{-1} \star I & I \end{pmatrix} = \begin{pmatrix} I & R\alpha \\ R & I \end{pmatrix} = \begin{pmatrix} R & I \\ I^{-1} & R \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$$

Where the element on the right is redily checked to be a nontrivial normalizing element of Λ.

Thus, any 2-torsion element of the class group (resp. of the Picard group) of R can be killed in the normalizing class group of a reflexive Azumaya algebra (resp. Azumaya algebra) over R of p.i. degree 2.

This construction can of course be extended to higher torsion elements

Take $[I] \in Cl(R)_n$, then let

$$\Lambda = End_R(R \oplus I \oplus I^2 \oplus \dots \oplus I^{n-1}$$

and the class of a divisorial ideal A is killed in $Cl(\Lambda)$ if and only if:

$$A \oplus (A \star I) \oplus \dots \oplus (A \star I^{n-1}) \cong R \oplus I \oplus \dots \oplus I^{n-1}$$

So, in particular, taking $A = I$ and $I \star I^{n-1} = R\alpha$ we know that $(\Lambda.I)^{**}$ is generated by the normalizing element

$$\begin{pmatrix} 0 & 0 & \dots & \alpha \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{pmatrix}$$

Conversely, it is of course easy (taking reduced norms) that the kernel of the natural morphism $Cl(R) \to Cl(\Lambda)$ consists of n-torsion elements if n = p.i. degree(Λ).

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