

Smooth Maximal Orders  
in Quaternion Algebras, I.

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## 0. Introduction.

In their fundamental paper on maximal orders [4], M. Auslander and O. Goldman studied the structure theory of maximal orders in two specific situations : over Dedekind domains in arbitrary central simple algebras and over regular domains in full matrix rings.

Whereas L. Silver [25] and r. Fossum [7] generalized some results to maximal orders over normal (resp. Krull) domains, M. Ramras [21,22,23] continued on the path taken by Auslander and Goldman, i.e. the study of maximal orders (preferably with finite global dimension) over regular local domains in arbitrary central simple algebras.

Renewed interest in this rather restricted but important class of maximal orders came with the publication of two recent papers by M. Artin [2], [3]. The first deals with the Zariski local structure of maximal orders over a smooth surface (i.e. the study of the number of conjugacy classes) whereas the second describes the étale local structure of the Brauer-Severi scheme associated to a maximal order over a Dedekind domain. Smooth (or strongly regular) maximal orders were introduced and studied jointly with M. Van den Bergh [15] in an attempt to grasp the contents of [2]. Later, it turned out that for this class of maximal orders one can easily describe the étale local structure of the (weak) Brauer-Severi scheme.

In this paper I have chosen to restrict attention to smooth maximal orders in quaternion algebras in order to make matters as concrete as possible. Further, it sometimes simplifies notation considerably while preserving the heart of the more general arguments which will appear elsewhere [14] [15].

Much of what follows is joint work with M. Van den Bergh.

# 1. Smooth maximal orders in quaternion algebras

In this section we will introduce smooth maximal orders and study their Zariski local structure in quaternion algebras. Since the definition differs slightly from that of strongly regular orders in [15] we will briefly recall both definitions.

Let  $\Lambda$  be a tame order over a normal domain  $R$ , i.e. a reflexive  $R$ -order such that  $\Lambda_p$  is hereditary for every height one prime  $p$  of  $R$ , in some central simple  $K$ -algebra  $\Sigma$ ,  $K$  being the field of fractions.

Let  $D = \{D_1, \dots, D_n\}$  be a set of Weil divisors of  $\Lambda$ , i.e. a set of divisorial  $\Lambda$ -ideals, then we define the Rees ring of  $\Lambda$  associated to the set of Weil divisors  $D$ ,  $\Lambda[D]$  to be the  $\mathbb{Z}^{(n)}$ -graded subring of  $\Sigma[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ , where  $\deg(X_i) = (0, \dots, 1, \dots, 0)$ , whose part of degree  $(m_1, \dots, m_n)$  is given by

$$\Lambda[D](m_1, \dots, m_n) = (D_1^{m_1} \star \dots \star D_n^{m_n}) X_1^{m_1} \dots X_n^{m_n}$$

These Rees rings were first introduced by F. Van Oystaeyen in the  $\mathbb{Z}$ -graded commutative case [27] and were subsequently generalized in e.g. [11, 12, 16].

Definition 1. [15] with notation as above.  $\Lambda$  is said to be a strongly regular order iff there exists a set of Weil divisors  $D$  such that  $\Lambda[D]$  is an Azumaya algebra over a regular center and every  $D_i \in D$  is an invertible  $\Lambda$ -ideal.

For a more intrinsic characterization of strongly regular orders and the relation to the other regularity conditions, the reader is referred to [15]. Let us now turn to the archetype-example : let  $P_c = \{p_1, \dots, p_n\}$  be the finite set of height one prime ideals of  $R$  which ramify in the maximal (!) order  $\Lambda$  with ramification indices say  $\{e_1, \dots, e_n\}$  and let

$P = \{P_1, \dots, P_n\}$  be the uniquely determined height one prime ideals of  $\Lambda$  which lie over the ramified central primes. It is now fairly easy to compute the center of  $\Lambda[P]$ . It turns out to be the  $\mathbb{Z}^{(n)}$ -graded subring  $R[P]$  of  $K[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$  whose part of degree  $(m_1, \dots, m_n)$  is given by

$$R[P](m_1, \dots, m_n) = (p_1^{\lfloor \frac{m_1}{e_1} \rfloor} \star \dots \star p_n^{\lfloor \frac{m_n}{e_n} \rfloor}) X_1^{m_1} \dots X_n^{m_n}$$

where  $\lfloor \frac{q}{b} \rfloor$  denotes the least integer  $\geq \frac{q}{b}$ .

Throughout we will assume that the maximal order  $\Lambda$  satisfies :

(Et<sub>1</sub>) : For every height one prime ideal  $p$  of  $R$  there exists an étale extensions  $R_p \rightarrow S(p)$  which splits  $\Sigma$ .

This condition is always satisfied in case the residue fields  $\mathbb{K}(p)$  are perfect, cfr. [24].

The main application of the graded construction given above to the theory of maximal orders satisfying (Et<sub>1</sub>) is that it basically reduces questions about these rather arbitrary maximal orders to graded questions about graded reflexive Azumaya algebras [19] i.e. graded algebras  $A$  over a graded normal domain  $S$  such that

$$(A \otimes_S A^{\text{opp}})^{\star\star} \cong \text{END}_S(A)$$

where the tensorproduct and  $\text{END}_S(A)$  is graded in the obvious way (cfr. [18]) and the isomorphism is gradation preserving. This fact follows from :

Proposition 1 : [12], [13]

With notations and assumptions as above, we have :

(1) :  $\Lambda[P]$  is a graded reflexive Azumaya algebra with center  $R[P]$  which is a normal domain.

(2) : there is an equivalence of categories between left reflexive  $\Lambda$ -modules and graded reflexive  $\Lambda[P]$ -modules.

(3) : if every  $P_i \in P$  is an invertible  $\Lambda$ -ideal, then there is an equivalence of categories between  $\Lambda$ -mod and  $\Lambda[P]$ -gr, the category of all  $\mathbb{Z}^{(n)}$ -graded left  $\Lambda[P]$ -modules.

Definition 2 : A maximal order  $\Lambda$  over a regular domain  $R$  is said to be smooth iff  $\Lambda$  is strongly regular with respect to the set of Weil divisors  $P$ .

Using Prop. 1.3. it is clear that both smooth and strongly regular orders have finite global dimension. If  $\Lambda$  is a maximal order over a regular local domain of  $\text{gldim}(R) \leq 2$  every  $P_i \in P$  is invertible because  $P_i$  is a reflexive  $R$ -module, hence free whence projective as a  $\Lambda$ -module by [21, Prop.3.5]. Let us give an example due to M. Van den Bergh showing that a strongly regular maximal order need not be smooth.

Example 1 : let  $R$  be the affine cone  $\mathbb{C}[X,Y,Z]/(XY-Z^2)$ , then the classgroup of  $R$  is  $\mathbb{Z}/2\mathbb{Z}$  and is generated by the ruling  $p = (Y,Z)$ . Let  $\Lambda$  be the reflexive Azumaya algebra over  $R$

$$\Lambda = \text{End}_R(R \oplus p) \cong \begin{pmatrix} R & p \\ p^{-1} & R \end{pmatrix}$$
 and let  $D = \left\{ \begin{pmatrix} p & p \star p \\ p^{-1} \star p & p \end{pmatrix} \right\} = \Lambda \cdot \begin{pmatrix} 0 & y \\ 1 & 0 \end{pmatrix}$  then  $\Lambda[D]$  is the  $\mathbb{Z}$  graded ring :

$$\Lambda[D] = \Lambda \left[ \begin{pmatrix} 0 & Y \\ 1 & 0 \end{pmatrix} X_1, \begin{pmatrix} 0 & 1 \\ Y^{-1} & 0 \end{pmatrix} X_1^{-1} \right]$$

which is readily checked to be an Azumaya algebra with a regular center

$$R[D] : \dots \oplus (Y^{-1}) X_1^{-2} \oplus p^{-1} X_1^{-1} \oplus R \oplus p X_1 \oplus (Y) X_1^2 \oplus p(Y) X_1^3 \oplus \dots$$

Therefore,  $\Lambda$  is strongly regular but not smooth since  $R$  is singular.

From now on, we will restrict attention to smooth maximal orders over a regular domain  $R$  in a quaternion algebra  $\Sigma$ . First, we aim to study the Zariski local structure. For simplicity's sake we will assume that

height one prime ideals of  $\Lambda$  lying over ramified prime ideals are generated by a normalizing element. However, we do not know whether this condition is always satisfied.

Lemma 1 : If  $\Lambda$  is a smooth maximal order over a regular local domain  $R$  in  $\Sigma$ , then  $\# P \leq 2$ .

Proof.

Let  $n = \# P$ , then, because  $\Lambda[P]$  is a  $\mathbb{Z}^{(n)}$ -graded Azumaya algebra over  $R[P]$ , which is a graded local domain with unique graded maximal ideal  $m[P] = \sum_{\sigma \in H} m \cdot R[P]_{\sigma} + \sum_{\tau \in G \setminus H} R[P]_{\tau}$  where  $G = \mathbb{Z}^{(n)}$  and  $H = 2\mathbb{Z} \oplus \dots \oplus 2\mathbb{Z}$ , we must have that  $\Lambda[P]/\Lambda[P] \cdot m[P]$  is a  $\mathbb{Z}^{(n)}$ -graded central simple algebra of dimension 4 over the  $\mathbb{Z}^{(n)}$ -graded field

$$R[P]/m[P] = R/m[X_1^2, X_1^{-2}, \dots, X_n^2, X_n^{-2}]$$

Because every prime ideal  $P_i$ ,  $1 \leq i \leq n$ , is supposed to be generated by a normalizing element, an easy calculation shows that

$$\Lambda[P]/\Lambda[P] \cdot m[P] \cong \bigoplus_{0 \leq i_j < 2} \Lambda/(\Lambda m + P_1 + \dots + P_n) X_1^{i_1} \dots X_n^{i_n}$$

the isomorphism being one of graded  $R/m[X_1^2, X_1^{-2}, \dots, X_n^2, X_n^{-2}]$ -modules.

Calculating dimensions on both sides yields :

$$(F_1) \quad 4 = 2^n \cdot \dim_{R/m} (\Lambda/(\Lambda m + P_1 + \dots + P_n))$$

This equality immediately implies that  $n \leq 2$ ;

This lemma also holds for smooth maximal orders in division algebras of dimension  $p^2$ ,  $p$  a prime number. Therefore, it seems to me that for high ( $\geq 3$ ) dimensions, smooth maximal orders are only a first approximation for "nice" regular maximal orders.

If  $\dim(\Sigma) \neq p^2$  one can have a worse ramification divisor. E.g. it is perfectly possible to have smooth maximal orders over regular local

domains of dimension 4 in central simple algebras of dimension 16 with 4 central ramified height one primes, each having ramification index 2. Let us recall the definition of a set of regular divisors with normal crossings [10]. We say that a set of Weil divisors  $D = \{D_i; i \in I\}$  has strictly normal crossings if for every prime ideal  $q \in R$  lying in  $U \text{ supp}(D_i)$  we have : if  $I_q = \{i; q \in \text{supp}(D_i)\}$ , then for  $i \in I_q$  we have that  $D_i = \sum_{\lambda} \text{div}(x_{i,\lambda})$  with  $x_{i,\lambda} \in R_q$  and  $\{(x_{i,\lambda})_{i,\lambda}\}$  form part of a regular system of parameters in  $R_q$ . We say that the set  $D = \{D_i; i \in I\}$  has normal crossings if for every  $q \in U \text{ supp}(D_i)$  there exists an étale neighbourhood  $\text{Spec}(S)$  of  $q$  in  $\text{Spec}(R)$  such that the family of inverse images of the  $\{D_i; i \in I\}$  on  $S$  have strictly normal crossings. Finally, a divisor  $D$  of  $R$  is called regular at  $q \in \text{supp}(D)$  if the subscheme  $D$  of  $\text{Spec}(R)$  is regular at  $q$ . The divisor  $D$  is called regular if it is regular everywhere.

Lemma 2 : If  $\Lambda$  is a smooth maximal order over a regular domain  $R$ , then  $P_c$  is a set of regular divisors with normal crossings.

Proof.

Let  $P_c = \{p_1, \dots, p_n\}$ , then by lemma 1 we know that for every prime ideal  $m \in \text{Spec}(R)$ ,  $N_m = \{i \leq n \mid m \in \text{supp}(p_i)\} \leq 2$ .

Let us first consider the case :  $N_m = 1$ . Then  $R[P]_m$  must be a regular and graded domain. After a possible renumeration we have :

$$R[P]_m \cong R[\{p_1\}][X_2, X_2^{-1}, \dots, X_n, X_n^{-1}]_m$$

so we may assume that :

$$\hat{R} : (p^{-1})X^{-2} \oplus R_m X^{-1} \oplus R_m \oplus (p)X \oplus (p)X^2 \oplus X^3 \oplus \dots$$

is a regular graded domain which is graded local with unique maximal graded ideal

$$\hat{m} : (p^{-1}m)_m X^{-2} \oplus R_m X^{-1} \oplus m_m \oplus (p)_m X \oplus (pm)_m X^2 \oplus \dots$$

By a result of [15], this is equivalent with

$$\text{gr.dim}_{\hat{R}/\hat{m}}(\hat{m}/\hat{m}^2) = \text{gldim}(R_m)$$

Calculating  $(\hat{m})^2$  gives us :

$$(\hat{m})^2: (m^2+p)^{-1} p_m X^{-2} \oplus m_m X^{-1} \oplus (m^2+p)_m \oplus (mp)_m X \oplus (m^2+p)_m X^2 \oplus \dots$$

and therefore

$$\hat{m}/(\hat{m})^2 \cong m/m^2+p [X^2, X^{-2}] \oplus R/m [X^2, X^{-2}] X^{-1}$$

as graded  $\hat{R}/\hat{m} = R/m [X^2, X^{-2}]$ -modules. Hence, we must have :

$$\dim_{R/m}(m/m^2+p) = \text{gldim}(R_m)-1$$

and this is equivalent to saying that  $(p) \nsubseteq m^2$ , i.e.  $m$  is a regular point on the subscheme determined by  $(p)$ .

Let us now turn to the case :  $N_m = 2$ . Again, after a possible renumeration, we have :

$$R[P]_m \cong R[\{p,q\}][X_3, X_3^{-1}, \dots, X_n, X_n^{-1}]$$

so we may assume that  $\hat{R} = R[\{p,q\}]_m$  is a regular domain which is graded local with unique maximal ideal  $\hat{m}$  :

$$\begin{array}{cccccc} mp^{-1}q & \oplus & q & \oplus & mq & \oplus & pq & \oplus & mpq \\ & \oplus & & \oplus & & \oplus & & \oplus & \\ p^{-1}q & \oplus & q & \oplus & q & \oplus & pq & \oplus & pq \\ & \oplus & & \oplus & & \oplus & & \oplus & \\ mp^{-1} & \oplus & R & \oplus & m & \oplus & p & \oplus & mp \\ & \oplus & & \oplus & & \oplus & & \oplus & \\ p^{-1} & \oplus & R & \oplus & R & \oplus & p & \oplus & p \\ & \oplus & & \oplus & & \oplus & & \oplus & \\ mp^{-1}q^{-1} & \oplus & q^{-1} & \oplus & mq^{-1} & \oplus & pq^{-1} & \oplus & mpq^{-1} \end{array}$$



Again, calculating  $(\hat{m})^2$  gives us :

$$\begin{array}{cccccc}
 A_p^{-1} & \oplus & m_q & \oplus & A_q & \oplus & m_{pq} & \oplus & A_{pq} \\
 \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 m_p^{-1} q & \oplus & q & \oplus & m_q & \oplus & p_q & \oplus & m_{pq} \\
 \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 A_p^{-1} & \oplus & m & \oplus & A & \oplus & m_p & \oplus & A_p \\
 \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 m_p^{-1} & \oplus & R & \oplus & m & \oplus & p & \oplus & m_p \\
 \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 A_p^{-1} q^{-1} & \oplus & m_q^{-1} & \oplus & A_q^{-1} & \oplus & m_{pq}^{-1} & \oplus & A_{pq}^{-1}
 \end{array}$$

where  $A = m^{2+p+q}$ ,  $R = R_m$  and we have omitted all powers of  $X_1$  and  $X_2$ .

From this description it follows that :

$$\begin{aligned}
 \hat{m}/\hat{m}^2 &\cong m/m^{2+p+q} [X_1^2, X_1^{-2}, X_2^2, X_2^{-2}] \\
 &\oplus R/m [X_1^2, X_1^{-2}, X_2^2, X_2^{-2}] X_1 \\
 &\oplus R/m [X_1^2, X_1^{-2}, X_2^2, X_2^{-2}] X_2
 \end{aligned}$$

the isomorphism being one as graded  $\hat{R}/\hat{m} = R/m [X_1^2, X_1^{-2}, X_2^2, X_2^{-2}]$ -modules.

hence, we must have :

$$\dim_{R/m} (m/m^{2+p+q}) = \text{gldim} (R/R_m) - 2$$

or equivalently that  $\{p, q\}$  is a part of a regular  $m$ -sequence in  $R_m$ . In particular it follows that  $m$  is a regular point on the subscheme determined by  $\{p\}$  (resp. by  $\{q\}$ ). Finally, [10, lemma 1.8.4.] finishes the proof.

**Definition 3** : If  $R$  is a regular domain with field of fractions  $K$ , if  $\Sigma$  is a quaternion algebra over  $K$  and if  $P_c$  is the set of ramified height one primes of  $R$  in  $\Sigma$ , we say that  $R$  has a regular ramification divisor with normal crossings in  $\Sigma$  iff  $P_c$  is a set of regular divisors with normal crossings s.t. for every  $m \in \text{Spec}(R)$ ,  $N_m$  (as defined in the proof of lemma 2)  $\leq 2$ .

Using the proofs of lemma 1 and 2, one can prove :

Proposition 1 : If  $\Lambda$  is a maximal order over a regular domain  $R$  in a quaternion algebra  $\Sigma$ , then  $\Lambda$  is smooth iff :

(a).  $R$  has a regular ramification divisor with normal crossings in  $\Sigma$ .

(b). For every  $m \in \text{Spec}(R)$  one of the following cases occur :

CASE 0 :  $N_m = 0$ , i.e.  $\Lambda_m$  is Azumaya over  $R_m$

CASE 1 :  $N_m = 1$  and  $\dim_{R/m} (\Lambda/\Lambda_m + P) = 2$

CASE 2 :  $N_m = 2$  and  $\dim_{R/m} (\Lambda/\Lambda_m + P + Q) = 1$

where  $P$  (resp.  $Q$ ) is the height one prime of  $\Lambda$  lying over the ramified central prime  $p$  (resp.  $q$ ).

Let us give a geometric interpretation of condition (a) for a maximal order on a smooth surface. Let  $\{p_1, \dots, p_n\}$  be the ramified height one primes, then each  $p_i$  can be viewed as a curve on the surface.  $N_m \leq 2$  then says that there are no three such curves which intersect at one point. Furthermore, each curve must be nonsingular and in an intersection point of two curves the tangent lines may not coincide.

As we will see later, CASE 0 (resp. 1,2) corresponds to CASE 1.1.(i) (resp. (ii), (iii)) of [2], whereas CASE 1.1 (iv) cannot occur as a smooth maximal order.

Let us give some examples of smooth maximal orders.

Example 2: Let  $\Lambda = \mathbb{C}[X, -]$  be the skew polynomial ring over  $\mathbb{C}$  where  $-$  denotes conjugation. It is clear that  $\Lambda$  is a maximal order with center  $R = \mathbb{R}[t]$  where  $X^2 = t$ . It follows that  $P = \{(X)\}$ , so  $\Lambda[P]$  is the  $\mathbb{Z}$ -graded ring.

$$(X^{-2})X_1^{-2} \oplus (X^{-1})X_1^{-1} \oplus \Lambda \oplus (X)X_1 \oplus (X^2)X_1^2 \oplus \dots$$

and  $R[P]$  is the  $\mathbb{Z}$ -graded ring :

$$(t^{-1})X_1^{-2} \oplus R X_1^{-1} \oplus R \oplus (t) X_1 \oplus (t) X_1^2 \oplus \dots$$

For every prime ideal  $q \neq (X) \in \text{Spec } \mathbb{R}[t]$ ,  $\Lambda_q$  is Azumaya over  $R_q$  whence  $\Lambda[P]_q$  is Azumaya over  $R_q$ . In the ramified prime we have :

$$\dim_{\mathbb{R}[t]/(t)} (\mathbb{C}[(X), -]/(X)) = \dim_{\mathbb{R}} (\mathbb{C}) = 2$$

i.e. case 1. So,  $\Lambda$  is smooth over  $R$ .

Further,  $R[P]_{(t)}/t[P]_{(t)} \cong \mathbb{R}[t_1, t_1^{-1}]$  where  $t_1 = t X_1^2$  and

$\Lambda[P]_{(t)}/\Lambda[P]t[P]_{(t)}$  is the  $\mathbb{Z}$ -graded central simple algebra

$\mathbb{C}[Y_1, Y_1^{-1}, -]$  with  $Y_1 = XX_1$  over  $\mathbb{R}[t_1, t_1^{-1}]$ .

Example 3 : Let  $R$  be a regular local domain of dimension 2 and suppose

$x$  and  $y$  generate the maximal ideal  $m$ . Let  $\Sigma$  be the quaternion-algebra

$(x, y)_K$  and let  $\Lambda = R[1, i, j, ij]$ , i.e.  $\Lambda$  is  $R$ -free with generators

$1, i, j, ij$  and with relations :

$$i^2 = x, j^2 = y \text{ and } ij = -ji$$

In [20] it was checked that  $\Lambda$  is a maximal  $R$ -order. Clearly  $P_c = \{(x), (y)\}$

which is a set of regular ramification indices with normal crossings.

Further,  $\Lambda[P]$  is a  $\mathbb{Z} \oplus \mathbb{Z}$ -graded ring which can be visualized (omitting

powers of  $X_1$  and  $X_2$ ) as :

$$\begin{array}{ccccccccc} (i^{-2}j^2) & \oplus & (i^{-1}j^2) & \oplus & (j^2) & \oplus & (ij^2) & \oplus & (i^2j^2) \\ \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\ (i^{-2}j) & \oplus & (i^{-1}j) & \oplus & (j) & \oplus & (ij) & \oplus & (i^2j) \\ \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\ (i^{-2}) & \oplus & (i^{-1}) & \oplus & \Lambda & \oplus & (i) & \oplus & (i^2) \\ \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\ (i^{-2}j^{-1}) & \oplus & (i^{-1}j^{-1}) & \oplus & (j^{-1}) & \oplus & (ij^{-1}) & \oplus & (i^2j^{-1}) \\ \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\ (i^{-2}j^{-2}) & \oplus & (i^{-1}j^{-2}) & \oplus & (j^{-2}) & \oplus & (ij^{-2}) & \oplus & (i^2j^{-2}) \end{array}$$

and its center  $R[P]$  is the  $\mathbb{Z} \oplus \mathbb{Z}$ -graded ring which looks like :

$$\begin{array}{cccccc}
 (x^{-1}y) & \oplus & (y) & \oplus & (y) & \oplus & (xy) & \oplus & (xy) \\
 \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 (x^{-1}y) & \oplus & (y) & \oplus & (y) & \oplus & (xy) & \oplus & (xy) \\
 \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 (x^{-1}) & \oplus & R & \oplus & R & \oplus & (x) & \oplus & (x) \\
 \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 (x^{-1}) & \oplus & R & \oplus & R & \oplus & (x) & \oplus & (x) \\
 \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 (x^{-1}y^{-1}) & \oplus & (y^{-1}) & \oplus & (y^{-1}) & \oplus & (xy^{-1}) & \oplus & (xy^{-1})
 \end{array}$$

and  $\dim_{R/m} (\Lambda/\Lambda m + (i) + (j)) = \dim_{R/m} (R/m) = 1$ , so  $\Lambda$  is a smooth maximal  $R$ -order of case 2.

Further,  $R[P]/m[P] = R/m[Y_1^2, Y_1^{-2}, Y_2^2, Y_2^{-2}]$  where  $Y_1 = iX_1$  and  $Y_2 = jX_2$ . Whereas  $\Lambda[P]/\Lambda[P]m[P]$  is the  $\mathbb{Z} \oplus \mathbb{Z}$ -graded central simple algebra:  $R/m[Y_1, Y_1^{-1}, Y_2, Y_2^{-1}]$  with  $Y_1 Y_2 = -Y_2 Y_1$ . Its part of degree  $(0,0)$  equals  $R/m$  corresponding to the fact that  $\Lambda$  is quasi-local with maximal ideal  $M = (i, j)$ .

Example 4 : Let  $F$  be any field with characteristic unequal to 2. Let

$R = F[X, Y]_{(X, Y)}$  where  $X$  and  $Y$  are indeterminates over  $F$ . Then  $R$  is regular local of dimensions two and has field of fractions  $K = F(X, Y)$ . Let  $\Sigma$  be the quaternion algebra  $(X, 1+Y)_K$  and let  $\Lambda$  be the  $R$ -free order  $R[1, i, j, ij]$  then  $\Lambda$  is a maximal order [20, p. 471]. Then  $P = \{(i)\}$ ,  $P_c = \{(x)\}$  and  $(x) \nsubseteq m^2$ . Because  $\dim_{R/m} (\Lambda/\Lambda m + (i)) = \dim_F (F \oplus F) = 2$ ,  $\Lambda$  is smooth. Further  $R[P]/m[P] \cong F[Y_1^2, Y_1^{-2}]$  where  $Y_1 = iX_1$  and  $\Lambda[P]/\Lambda[P]m[P]$  is the  $\mathbb{Z}$ -graded algebra

$$(F \oplus F\epsilon)[Y_1, Y_1^{-1}, \varphi] \xrightarrow{\alpha} M_2(F[Y_1^2, Y_1^{-2}])$$

where  $\varphi(a \oplus b \epsilon) = a \oplus -b \epsilon$  and  $\alpha$  is given by

$$\begin{aligned} \alpha(1 \oplus 0) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \alpha(0 \oplus \epsilon) &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ \alpha(Y_1) &= \begin{pmatrix} 0 & 1 \\ Y_1^2 & 0 \end{pmatrix} \end{aligned}$$

Therefore,  $\Lambda[P] / \Lambda[P]m[P]$  is a  $\mathbb{Z}$ -graded central simple algebra over  $F[Y_1^2, Y_1^{-2}]$ . However, the part of degree 0 of  $\Lambda[P] / \Lambda[P]m[P]$  is semisimple ( $F \oplus F\epsilon$ ) corresponding to the fact that  $\Lambda$  is not quasi-local. Each factor corresponds to one of the two maximal ideals of  $\Lambda$  lying over  $m = (X, Y)$  :

$$M_1 = \Lambda(i, j-1) \quad M_2 = \Lambda(i, j+1)$$

Having characterized smooth maximal orders over regular domains in quaternion algebras, we will now study their Zariski local structure, i.e. the number of conjugacy classes over a regular local domain. One of the basic ingredients in this study is a result of Grothendieck [8] on descent of modules. For convenience, we state that theorem here :

Theorem [8, 2.5.8] Let  $R$  be a Noetherian (semi) local ring,  $\Lambda$  a finite  $R$ -algebra and let  $M_1, M_2$  be finite left  $\Lambda$ -modules. Let  $R \rightarrow S$  be a faithfully flat morphism with  $S$  Noetherian. If  $M_1 \otimes S \cong M_2 \otimes S$  as left  $\Lambda \otimes S$ -modules, then  $M_1 \cong M_2$  as left  $\Lambda$ -modules.

Using the above theorem, we will have to compute the conjugacy classes of the extended orders  $\Lambda \oplus_R R^{sh}$  where  $\Lambda$  is smooth over  $R$  and  $R^{sh}$  denotes the strict Henselization of  $R$  cfr. [17] or [23].

Case 0 is easy : if one maximal order  $\Lambda$  over  $R$  in  $\Sigma$  is Azumaya, then every other smooth order say  $\Gamma$  is Azumaya too. Since  $Br(R^{sh}) = 0$ ,  $\Lambda \otimes R^{sh} \cong M_2(R^{sh}) \cong \Gamma \otimes R^{sh}$  and by descent  $\Lambda \cong \Gamma$  as  $R$ -algebras, yielding that  $\Lambda$  and  $\Gamma$  are conjugated.

Before treating the other cases, let us recall the definition of the graded Brauer group as introduced by F. Van Oystaeyen in the  $\mathbb{Z}$ -graded case in [26].

If  $T$  is any  $\mathbb{Z}^{(n)}$ -graded ring, then a graded Azumaya algebra over  $T$  is an Azumaya algebra over  $T$  admitting a  $\mathbb{Z}^{(n)}$ -gradation extending the gradation of the center. Two graded algebras  $\Gamma$  and  $\Omega$  are said to be gr-equivalent if there exist finitely generated graded projective  $T$ -modules  $P$  and  $Q$  such that there exists a degree preserving isomorphism

$$\Gamma \otimes_T \text{END}_T(P) \cong \Omega \otimes_T \text{END}_T(Q)$$

where the rings  $\text{END}_T(-)$  and the tensorproducts are equipped with the natural gradation, cfr. e.g. [18]. The set of gr-equivalence classes of graded Azumaya algebras forms a group with respect to the tensor-product,  $\text{Br}^g(T)$ , called the graded Brauer group of  $T$ .

If  $T$  is a  $\mathbb{Z}$ -graded Krull domain it was shown by S. Caenepeel, M. Van den Bergh and F. Van Oystaeyen [6] that the natural (i.e. gradation-forgetting) morphism  $\text{Br}^g(T) \rightarrow \text{Br}(T)$  is monomorphic. Their argument can easily be extended to the  $\mathbb{Z}^{(n)}$ -graded case.

S. Caenepeel [5] calls a graded local ring  $R$  (i.e. having a unique maximal graded ideal) gr-Henselian if every finite graded commutative  $R$ -algebra  $B$  is graded decomposed, i.e. when it is the direct sum of graded local rings. In the  $\mathbb{Z}$ -graded case it turns out that a graded local ring is gr-Henselian iff its part of degree zero is Henselian, [5].

This result can be generalized to  $\mathbb{Z}^{(n)}$ -graded rings. Furthermore, if  $R$  is gr-Henselian with maximal graded ideal  $m$  then the natural map :

$$\text{Br}^g R \rightarrow \text{Br}^g R/m$$

is monomorphic by a similar argument as in the ungraded case.

Lemma 3 : If  $\Lambda$  is a smooth maximal order over a regular local domain  $R$  in a quaternion algebra  $\Sigma$ , then

(1) in case 1,  $R^{\text{sh}}$  splits  $\Sigma$ .

(2) in case 2,  $R^{\text{sh}}$  does not split  $\Sigma$

Proof.

$\Lambda[P] \otimes_R R^{\text{sh}}$  is a graded Azumaya algebra over  $R^{\text{sh}}[P] = R[P] \otimes_R R^{\text{sh}}$ .

$R^{\text{sh}}[P]$  is gr-Henselian because the part of degree zero (or of  $(0,0)$ ) is Henselian, with maximal graded ideal  $m^{\text{sh}}[P]$ .

(1) In this case  $R^{\text{sh}}[P] / m^{\text{sh}}[P]$  is the graded field  $R^{\text{sh}}/m^{\text{sh}}[Y_1^2, Y_1^{-2}]$  where  $Y_1^2 = pX_1^2$ . Now,

$$\bar{\Gamma} = (\Lambda[P] \otimes_R R^{\text{sh}}) / (\Lambda[P] \otimes_R R^{\text{sh}})m^{\text{sh}}[P]$$

is a graded central simple algebra of dimension 4 over  $R^{\text{sh}}/m^{\text{sh}}[Y_1^2, Y_1^{-2}]$ .

Calculating the part of degree zero of  $\bar{\Gamma}$  it turns out that  $\bar{\Gamma}_0$  need to be an algebra over  $R^{\text{sh}}/m^{\text{sh}}$  of dimension two. Because  $R^{\text{sh}}/m^{\text{sh}}$  is separably closed and  $\text{char}(R^{\text{sh}}/m^{\text{sh}}) \neq 2$ , we must have  $\bar{\Gamma} \cong R^{\text{sh}}/m^{\text{sh}} \oplus R^{\text{sh}}/m^{\text{sh}}$  as algebras. hence  $\bar{\Gamma}$  contains zero divisors and therefore

$$\bar{\Gamma} \cong M_2(R^{\text{sh}}/m^{\text{sh}}[Y_1^2, Y_1^{-2}])$$

(cfr. also example 4). Finally, using the injectivity of the map  $\text{Br}^g R^{\text{sh}}[P] \rightarrow \text{Br}^g R^{\text{sh}}[P] / m^{\text{sh}}[P]$  it follows that

$$\Lambda[P] \otimes_R R^{\text{sh}} \cong \text{END}_{R^{\text{sh}}[P]}(P)$$

for some graded f.g. projective  $R^{\text{sh}}[P]$ -module  $P$ . Calculating parts of degree zero on both side yields that  $\Lambda \otimes_R R^{\text{sh}}$  is an order in a matrixring.

(2). In the second case,  $R^{\text{sh}}[P] / m^{\text{sh}}[P]$  is a  $\mathbb{Z} \oplus \mathbb{Z}$ -graded field  $R^{\text{sh}}/m^{\text{sh}}[Y_1^2, Y_1^{-2}, Y_2^2, Y_2^{-2}]$  where  $Y_1^2 = pX_1^2$  and  $Y_2^2 = qX_2^2$ . Now,

$$\bar{\Gamma} = (\Lambda[P] \otimes_R R^{\text{sh}}) / (\Lambda[P] \otimes_R R^{\text{sh}})m^{\text{sh}}[P]$$

is a  $\mathbb{Z} \oplus \mathbb{Z}$  - graded central simple algebra of dimension 4 over  $R^{\text{sh}}/m^{\text{sh}}[Y_1^2, Y_1^{-2}, Y_2^2, Y_2^{-2}]$ . The homogeneous parts  $\bar{T}_{(0,0)}, \bar{T}_{(0,1)}, \bar{T}_{(1,0)}$  and  $\bar{T}_{(1,1)}$  are all non-zero and have therefore all dimension one. It follows that  $\bar{T}_{(0,0)} = R^{\text{sh}}/m^{\text{sh}}$  and

$$\bar{T} \cong (aY_1^2, bY_2^2)_{R^{\text{sh}}/m^{\text{sh}}[Y_1^2, Y_1^{-2}, Y_2^2, Y_2^{-2}]}$$

for some  $a, b \in (R^{\text{sh}}/m^{\text{sh}})^*$ . Because  $R^{\text{sh}}/m^{\text{sh}}$  is separably closed and  $\text{char}(R^{\text{sh}}/m^{\text{sh}}) \neq 2$ ,  $\bar{T} \cong (Y_1^2, Y_2^2)$  so  $\bar{T} \cong R^{\text{sh}}/m^{\text{sh}}[Y_1, Y_1^{-1}, Y_2, Y_2^{-1}]$  with  $Y_1 Y_2 = -Y_2 Y_1$  (cfr. example 3). Using the norm, it is easy to check that  $\bar{T}$  has no zero divisors. Therefore  $\bar{T}$  is a non-trivial element in  $\text{Br}^g(R^{\text{sh}}[P]/m^{\text{sh}}[P])$  and hence  $\Lambda \otimes R^{\text{sh}}$  cannot be an order in a matrixring, finishing the proof.

Proposition 2 : (Zariski local structure in case 1)

All smooth maximal orders over a regular local domain in a quaternion algebra are conjugated.

proof.

For any smooth maximal order  $\Lambda$  in case 1 we know by the proof of lemma 3 that

$$\Lambda[P] \otimes R^{\text{sh}} \cong \text{END}_{R^{\text{sh}}[P]}(P)$$

for some f.g. graded projective  $R^{\text{sh}}[P]$ -module  $P$ . Because  $R^{\text{sh}}[P]$  is graded local,  $P$  is graded free; i.e. of the form :

$$P \cong R^{\text{sh}}[P](\sigma_1) \oplus R^{\text{sh}}[P](\sigma_2)$$

where  $\sigma_i \in \mathbb{Z}$  and  $R^{\text{sh}}[P](\sigma_i)$  is the graded  $R^{\text{sh}}[P]$ -module determined by taking for its homogeneous part of degree  $\alpha$  :  $R^{\text{sh}}[P](\sigma_i)_\alpha = R^{\text{sh}}[P]_{\sigma_i + \alpha}$ . Therefore,

$$\Lambda[P] \otimes R^{\text{sh}} \cong M_2(R^{\text{sh}}[P](\sigma_1, \sigma_2))$$



where the part of degree  $\alpha$  of the ring on the right side is given by the formula :

$$M_2(R^{sh}[P])(\sigma_1, \sigma_2)_\alpha = \begin{pmatrix} R^{sh}[P]_\alpha & R^{sh}[P]_{\alpha+\sigma_1-\sigma_2} \\ R^{sh}[P]_{\alpha+\sigma_2-\sigma_1} & R^{sh}[P]_\alpha \end{pmatrix}$$

A straightforward computation shows that up to a graded isomorphism the  $\sigma_i$  may be chosen to be 0 or 1. Since all isomorphisms above were gradation preserving :

$$\Lambda \otimes_R R^{sh} \cong \begin{pmatrix} R^{sh} & R^{sh}[P]_{\sigma_1-\sigma_2} \\ R^{sh}[P]_{\sigma_2-\sigma_1} & R^{sh} \end{pmatrix}$$

So, we are left to show that all rings which can occur in such a way are conjugated.

The diagram is a 2x2 grid with axes labeled  $\sigma_1$  (horizontal) and  $\sigma_2$  (vertical). The diagonal elements are 0 and 1. The off-diagonal elements are matrices:

- Top-right ( $\sigma_1=1, \sigma_2=0$ ):  $\begin{pmatrix} R^{sh} & R^{sh} \\ p & R^{sh} \end{pmatrix}$
- Bottom-left ( $\sigma_1=0, \sigma_2=1$ ):  $\begin{pmatrix} R^{sh} & p \\ R^{sh} & R^{sh} \end{pmatrix}$
- Bottom-right ( $\sigma_1=1, \sigma_2=1$ ):  $M_2(R^{sh})$

Since  $\Lambda$  was supposed to be ramified, only the cases  $\sigma_1 \neq \sigma_2$  can occur and clearly :

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} R^{sh} & p \\ R^{sh} & R^{sh} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} R^{sh} & R^{sh} \\ p & R^{sh} \end{pmatrix}$$

Grothendieck descent finishes the proof.

Proposition 3. (Zariski local structure in case 2)

All smooth maximal orders over a regular local domain in a quaternion algebra are conjugated.

Proof.

By Lemma 3 we know that  $\Lambda[P]$  cannot be split by an étale extension of  $R$ . Nevertheless, since  $\Lambda[P]$  is a graded Azumaya algebra over a graded local domain,  $\Lambda[P]$  can be split by a gr-étale extension of  $R[P]$  (where gr-étale is defined in the obvious way). If we denote by  $S = R[X]/(X^2 - p)$ , then

$$\begin{array}{cccccc}
 (X^2 q) & \oplus & (X^{-1} q) & \oplus & (q) & \oplus & (qX) & \oplus & (qX^2) \\
 \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 (X^2 q) & \oplus & (X^{-1} d) & \oplus & (q) & \oplus & (qX) & \oplus & (qX^2) \\
 \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 (X^{-2}) & \oplus & (X^{-1}) & \oplus & S & \oplus & (X) & \oplus & (X^2) \\
 \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 (X^{-2}) & \oplus & (X^{-1}) & \oplus & S & \oplus & (X) & \oplus & (X^2) \\
 \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 (X^{-2} q^{-1}) & \oplus & X^{-1} q^{-1} & \oplus & (q^{-1}) & \oplus & (Xq^{-1}) & \oplus & (X^2 q^{-1})
 \end{array}$$

is such a gr-étale splitting ring of  $\Lambda[P]$ ,  $S(\Phi)$ . Again, it turns out that

$$\Lambda[P] \otimes S(\Phi) \cong M_2(S(\Phi))(\sigma_1, \sigma_2)$$

where  $\sigma_i \in \mathbb{Z} \oplus \mathbb{Z}$  can be chosen in the set  $\{(0,0), (0,1), (1,0), (1,1)\}$ .

We will first show that they are all graded isomorphic to  $M_2(S(\Phi))$  with the usual gradation.

An easy computation shows that :

$$\begin{aligned}
 M_2(S(\Phi))((0,0), (0,1)) &= \begin{pmatrix} 1 & 0 \\ 0 & X_2 \end{pmatrix} M_2(S(\Phi)) \begin{pmatrix} 1 & 0 \\ 0 & X_2^{-1} \end{pmatrix} \\
 M_2(S(\Phi))((0,0), (1,0)) &= \begin{pmatrix} 1 & 0 \\ 0 & X_1 \end{pmatrix} M_2(S(\Phi)) \begin{pmatrix} 1 & 0 \\ 0 & X_1^{-1} \end{pmatrix} \\
 M_2(S(\Phi))((0,0), (1,1)) &= \begin{pmatrix} 1 & 0 \\ 0 & X_1 X_2 \end{pmatrix} M_2(S(\Phi)) \begin{pmatrix} 1 & 0 \\ 0 & X_1^{-1} X_2^{-1} \end{pmatrix}
 \end{aligned}$$

Now, by a graded version of Grothendieck descent [15],  $\Lambda[P]$  is graded isomorphic to  $\Gamma[P]$  for any other smooth maximal R-order  $\Gamma$  in  $\Sigma$ . This isomorphism is given by conjugation with a unit  $\alpha$  in  $\Sigma[X_1, X_1^{-1}, X_2, X_2^{-1}]$  and because this graded ring is a domain  $\alpha$  is homogeneous i.e.  $\alpha = \sigma X_1^{e_1} X_2^{e_2}$  with  $\sigma \in \Sigma$ . Finally, it follows that  $\Gamma = \sigma \Lambda \sigma^{-1}$ , finishing the proof.

Corollary 1: Let  $R$  be a regular domain and  $\Sigma$  a quaternion algebra which contains a smooth maximal R-order  $\Lambda$ . If  $\Gamma$  is any maximal R-order, then  $S_\Gamma = \{p \in \text{Spec}(R) \mid \Gamma_p \text{ is smooth}\}$  is an open set.

Proof.

Let  $\theta_\Lambda$  (resp.  $\theta_\Gamma$ ) denote the structure sheaf of the R-algebra  $\Lambda$  (resp.  $\Gamma$ ) over  $\text{Spec}(R)$ . If  $p \in \text{Spec}(R)$  such that  $\Gamma_p$  is smooth, then  $\Lambda_p = \sigma \Gamma \sigma^{-1}$  for some  $\sigma \in \Sigma^\star$ . This equality carries over to a small neighbourhood of  $p$ , finishing the proof.

What can be said about the codimension of this set? Clearly, if  $p$  is an height one prime of  $R$ , then  $p \in S_\Gamma$ . In view of [20, Th.5.4] the only height two primes of  $R$  which do not lie in  $S_\Gamma$  are prime ideals  $p$  such that there are two prime ideals of  $\Lambda$  lying over  $p$ . If  $\Gamma$  is a projective R-order, then by [21, Th.2.2] this fact also holds for height three primes.

Corollary 2: If  $R$  is a regular domain and if  $\Lambda$  is a smooth maximal R-order in a quaternion algebra  $\Sigma$ , then there is a one-to-one correspondence between conjugacy classes of smooth maximal R-orders in  $\Sigma$  and elements of  $H_{\text{Zar}}^1(\text{Spec}(R), \underline{\text{Aut}}_\Lambda)$  where  $\underline{\text{Aut}}_\Lambda$  is the automorphism scheme of  $\theta_\Lambda$ .

## 2. Brauer-Severi schemes.

Let us first sketch the general problem. Let  $\Lambda$  be an order over a normal domain  $R$  in a central simple algebra  $\Sigma$  of dimension  $n^2$ , then one can define a functor  $F_\Lambda$  from the category of all commutative  $R$ -algebras to the category of sets :

$$F_\Lambda : \text{Comm Alg}_R \rightarrow \text{Sets}$$

which associates to an  $R$ -algebra  $A$  the set of all left ideals of  $\Lambda \otimes_R A$  which are split projective  $A$ -modules of rank  $n$ . The main problem is now to determine whether this functor is representable. By this we mean the following: does there exist a scheme  $\underline{BS}_\Lambda$  over  $\underline{\text{Spec}}(R)$ , called the Brauer-Severi scheme of  $\Lambda$ , such that for every commutative  $R$ -algebra  $A$  there is a natural one-to-one correspondence between elements  $L$  of  $F_\Lambda(A)$  and scheme homomorphisms  $\psi_L$  from  $\underline{\text{Spec}}(A)$  to  $\underline{BS}_\Lambda$  making the diagram below commutative :

$$\begin{array}{ccc} & & \underline{BS}_\Lambda \\ & \nearrow \psi_L & \downarrow \varphi \\ \underline{\text{Spec}}(A) & \xrightarrow{\varphi_A} & \underline{\text{Spec}}(R) \end{array}$$

where  $\varphi_A$  and  $\varphi$  are the structural morphisms. Therefore, one could view the Brauer-Severi scheme of  $\Lambda$  to be a scheme parametrizing the commutative  $R$ -algebras  $A$  which split  $\Sigma$ . A first step in this study usually consists in determining the étale local structure of this Brauer-Severi scheme, i.e. suppose that a point  $x \in \underline{\text{Spec}}(R)$  has an étale neighbourhood  $S$  which splits  $\Sigma$ , then one tries to find a representation of the functor

$$F_{\Lambda \otimes S} : \text{Comm Alg}_S \rightarrow \text{Sets}$$

This étale local structure has been determined in several cases

- Grothendieck [9] has shown that the étale local structure of the Brauer-Séveri scheme of an Azumaya algebra is  $\mathbb{P}_S^{n-1}$
- Artin and Mumford [1] calculated the Brauer-Séveri scheme of a maximal order over a smooth surface in a ramified quaternion algebra with a regular ramification divisor.
- Recently [3], Artin calculated the étale local structure of the Brauer Séveri scheme for a maximal order over a Dedekind domain.

If one restricts attention in the first case to Azumaya algebras over regular domains, it turns out that all rings for which there exist Brauer-Severi schemes in the literature are smooth maximal orders. Therefore, one could ask whether the functor  $F_\Lambda$  is representable for any smooth maximal order.

In this section we will prove a result which can be viewed as a first step towards this goal. We will represent  $F_\Lambda$  when restricted to some nice subcategory  $\underline{b}$  of all commutative  $R$ -algebras, including all étale or even smooth extensions of  $R$ .  $\underline{b}$  will be the full subcategory of  $\text{Comm Alg}_S$  consisting all all  $R$ -algebras  $A$  such that  $A[P] = R[P] \otimes_R A$  is a regular domain.

Let us start by proving the key lemma which translates everything in a graded question. Denote by  $F_\Lambda^g$  the functor

$$F_\Lambda^g : \underline{b} \otimes_R R[P] \rightarrow \text{Sets}$$

which assigns to an algebra  $A[P]$  with  $A \in \underline{b}$  the set of all graded left ideals of  $\Lambda[P] \otimes A$  which are graded split projective  $A[P]$ -modules of rank  $n$ .

Key Lemma : If  $\Lambda$  is a smooth maximal  $R$ -order and if  $A \in \underline{b}$ , then there is a natural one-to-one correspondence between elements of  $F_\Lambda(A)$  and elements of  $F_\Lambda^g(A)$ .

Proof.

We claim that the maps below establish the claimed one-to-one correspondence.

$$\begin{array}{ll} \psi_1 : F_{\Lambda}(A) \rightarrow F_{\Lambda}^g(A) & L \mapsto L \otimes_{(\Lambda \otimes A)} (\Lambda[P] \otimes A) \\ \psi_2 : F_{\Lambda}^g(A) \rightarrow F_{\Lambda}(A) & M \mapsto M_e \end{array}$$

where  $e$  is the identity element of the grading group. Because  $\Lambda[P] \otimes A$  is a graded ring,  $\psi_1$  and  $\psi_2$  clearly give a one-to-one correspondence between left ideals of  $\Lambda \otimes A$  and graded left ideals of  $\Lambda[P] \otimes A$ .

Because  $\Lambda[P] \otimes A$  is a graded Azumaya algebra over the graded regular domain  $A[P]$ ,  $\text{gr.gldim}(\Lambda[P] \otimes A) < \infty$  whence  $\text{gldim}(\Lambda \otimes A) < \infty$

using the equivalence of categories between  $(\Lambda \otimes A)\text{-mod}$  and  $(\Lambda[P] \otimes A)\text{-gr}$ .

It follows that  $\Lambda \otimes A$  is a regular order over  $A$ .

Because splitting and projectivity are local conditions, we may assume from now on that  $A$  is regular local.

Now, let  $L \in F_{\Lambda}(A)$ . We claim that  $L$  is split as a left  $\Lambda \otimes A$ -module.

Consider the exact sequence of left  $\Lambda \otimes A$ -modules :

$$0 \rightarrow L \rightarrow \Lambda \otimes A \rightarrow (\Lambda \otimes A) / L \rightarrow 0.$$

Because  $L \in F_{\Lambda}(A)$ , this sequence splits as a sequence of  $A$ -modules.

Therefore,  $(\Lambda \otimes A)/L$  is a left  $(\Lambda \otimes A)$ -module which is free as an  $A$ -module. Using regularity of the  $A$ -order  $\Lambda \otimes A$ , this entails by [20, Prop. 3.5] that  $(\Lambda \otimes A)/L$  is projective as a left  $(\Lambda \otimes A)$ -module, finishing the proof of our claim.

But then it is clear that  $\psi_1(L)$  is a graded split projective  $\Lambda[P] \otimes A$ -module. Finally, an easy localization argument shows that  $\psi_1(L)$  has rank  $n$ .

The proof that  $\psi_2$  maps elements of  $F_{\Lambda}^g(A)$  to  $F_{\Lambda}(A)$  is easy and is left to the reader.

Our strategy in order to represent the functor  $F_{\Lambda}$  is now easy to grasp. First, we will represent to functor  $F_{\Lambda}^g$  by a graded scheme. Graded schemes are defined formally in [14], but any intelligent reader can only come up with one possible definition of them, so we will skip it here. We believe that the proofs and examples given below give a better idea what graded schemes are than any formal definition. Representing  $F_{\Lambda}^g$  by a graded scheme is relatively easy, because  $\Lambda[P]$  is a graded Azumaya algebra so we have to mimic Grothendieck's arguments [9] in the graded case.

Afterwards, we will form out of this graded scheme a usual scheme which represents  $F_{\Lambda}$ .

All graded schemes which appear in this paper have the pleasant property that their part of degree  $e$  is a usual scheme.

Example 5 : Let  $R[P]$  be the  $\mathbb{Z}^{(n)}$ -graded ring defined in § 1 associated to a set of height one primes  $\{p_1, \dots, p_n\}$  and ramification indices  $\{e_1, \dots, e_n\}$ . Denote by  $G = \mathbb{Z}^{(n)}$  and  $H = e_1 \mathbb{Z} \oplus \dots \oplus e_n \mathbb{Z}$ . There is a natural one-to-one correspondence between  $\text{Spec}(R)$  and  $\text{Spec}_g(R[P])$ , the set of all  $\mathbb{Z}^{(n)}$ -graded prime ideals of  $R[P]$  with the induced Zariski topology.

$$\begin{aligned} \varphi_1: \text{Spec}(R) &\rightarrow \text{Spec}_g(R[P]); m \mapsto \sum_{\sigma \in G} m R[P]_{\sigma} + \sum_{\tau \in G \setminus M} R[P]_{\tau} = m[P] \\ \varphi_2: \text{Spec}_g(R[P]) &\rightarrow \text{Spec}(R); M \mapsto M_e \end{aligned}$$

where  $e = (0, \dots, 0)$ . The maps  $\varphi_i$  actually define an homeomorphism. Moreover, for any  $m \in \text{Spec}(R)$  we have :

$$(R[P]_m^g[P]_e) \cong R_m$$

It follows that the part of degree  $e$  of the affine graded spectrum of  $R[P]$  is isomorphic to  $\text{Spec}(R)$ .

A : a graded representation of  $F_{\Lambda}^g$

From now on we assume that  $\Lambda$  is a smooth maximal order over a regular domain  $R$  in a quaternion-algebra  $\Sigma$ . We will treat the two ramified cases separately:

CASE\_1 :  $P = \{p\}$  and  $\dim_{R/m}(\Lambda/\Lambda m + P) = 2$

It follows from lemma 3 that there exists an étale extension  $S$  of  $R$  which splits  $\Sigma$ . We first represent the functor :

$$F_{\Lambda \otimes S}^g : \underline{b} \otimes S[P] \rightarrow \text{Sets.}$$

by a graded scheme over  $\text{SPEC}^g(S[P])$ , i.e. we will define a graded scheme  $\underline{X}^g$  such that for any  $S$ -algebra  $A$  in  $\underline{b}$  there is a natural one-to-one correspondence between elements of  $F_{\Lambda \otimes S}^g(A)$  and graded scheme homomorphisms

$$\begin{array}{ccc} \text{SPEC}^g(A[P]) & \xrightarrow{\quad \quad \quad} & \underline{X}^g \\ \varphi_A \searrow & & \swarrow \varphi \\ & \text{SPEC}^g(S[P]) & \end{array}$$

where  $\varphi$  and  $\varphi_A$  are the structural morphisms.

In § 2 we calculated the structure of  $\Lambda[P] \otimes S$  :

$$\Lambda[P] \otimes S \cong M_2(S[P](0,1))$$

Therefore, if  $A$  is any  $\mathbb{Z}$ -graded  $S[P]$ -algebra if we denote  $\Lambda[P] \otimes S$  by  $\Gamma[P]$ , then

$$\Gamma[P] \otimes A \cong M_2(A)(0,1)$$

hence our aim is to represent the functor

$$G : \text{gr Comm Alg}_S[P] \rightarrow \text{Sets}$$

which assigns to a  $\mathbb{Z}$ -graded algebra  $A$  the set of all split projective graded left ideals of  $M_2(A)(0,1)$  of rank 2.



Take such a graded left ideal  $L \in G(A)$ , then

$$L = e_{11}L \oplus e_{22}L$$

and because all matrixelements  $e_{ij}$  are homogeneous, it follows that  $e_{11}L$  is a graded submodule of  $A \oplus A(-1)$  which is split projective of rank 1 because

$$e_{ji} e_{ii} L = e_{jj} L$$

Conversely, if  $M$  is a graded split projective submodule of rank 1 of  $A \oplus A(-1)$ , then

$$L = M \oplus e_{21} M$$

is a graded split projective left ideal of rank 2 of  $M_2(A)(0,1)$ . Therefore it will be sufficient to represent the functor

$$\text{Grass}_1^g(0,-1): \text{gr-Comm Alg}_S[P] \rightarrow \text{Sets.}$$

which assigns to a  $\mathbb{Z}$ -graded  $R[P]$  algebra  $A$  the set of all graded split projective rank one submodules of  $A \oplus A(-1)$ . As in the ungraded case we will do this by representing the subfunctor  $\text{Gr}_i^g$ ,  $i = 0,1$ , of  $\text{Grass}_1^g(0,-1)$  which assigns to a  $\mathbb{Z}$ -graded  $R[P]$ -algebra  $A$  the set of all the elements  $M \in \text{Grass}_1^g(0,-1)$  such that the composite morphism (which is gradation preserving)

$$A(-i) \xrightarrow{\varphi_i} A \oplus A(-1) \xrightarrow{u} M$$

is an isomorphism. Here  $\varphi_0 : A \rightarrow A \oplus A(-1)$  and  $\varphi_1 : A(-1) \rightarrow A \oplus A(-1)$  are the natural gradation preserving injections and  $u$  is the uniquely determined gradation preserving splitting map for  $M$ . Suppose we have a situation :

$$\begin{array}{ccc} & A \oplus A(-1) & \\ \varphi_i \nearrow & & \searrow u \\ A(-i) & \xleftarrow{w} & M \end{array}$$

$v$

such that  $v \circ \varphi_i$  is an isomorphism and let  $w$  be its inverse and  $v = w \circ u$  which satisfies  $v \circ \varphi_i = 1_{A(-i)}$ .

Conversely, suppose we have a gradation preserving morphism  $v$  which satisfies  $v \circ \varphi_i = 1_{A(-i)}$ , then it is clear that

$$M = A \oplus A(-1)/\text{Ker}(v)$$

is an element of  $\text{Gr}_1^g(A)$ . One can therefore identify  $\text{Gr}_1^g(A)$  with the set of gradation preserving split morphisms of  $\varphi_i$ . Therefore, if one defines mappings

$$\alpha_i : \text{Hom}_A(A \oplus A(-1), A(-i))_0 \rightarrow \text{Hom}_A(A(-i), A(-i))_0; \alpha_i(v) = v \circ \varphi_i$$

$$\beta_i : \text{Hom}_A(A \oplus A(-1), A(-i))_0 \rightarrow \text{Hom}_A(A(-i), A(-i))_0; \beta_i(v) = 1_{A(-i)}$$

then  $\text{Gr}_1^g(A)$  can be viewed as the kernel of the couple  $(\alpha_i, \beta_i)$ .

We claim that the functors :

$$A_i : \text{gr Comm Alg}_S[P] \rightarrow \text{Sets}; A_i(A) = \text{Hom}_A(A \oplus A(-1), A(-i))_0$$

$$B_i : \text{gr Comm Alg}_S[P] \rightarrow \text{Sets}; B_i(A) = \text{Hom}_A(A(-i), A(-i))_0$$

are representable by graded schemes.

In order to prove this claim, let us pause a moment and define graded vector fibres.

#### Interludium : graded vector fibres.

Let  $A$  be a  $\mathbb{Z}^{(n)}$ -graded commutative ring and let  $E$  be a graded  $A$ -module.

The tensor algebra  $\mathbb{T}(E) = \bigoplus_{i=1}^{\infty} (\otimes^i E)$  is given the natural  $\mathbb{Z}^{(n)}$ -gradation, i.e.

$$(\otimes^m E)_{\gamma} = \sum_{\sigma_1 + \dots + \sigma_m = \gamma} E_{\sigma_1} \otimes \dots \otimes E_{\sigma_m}$$

The symmetric algebra over  $E$ ,  $S(E)$  is obtained  $\mathbb{T}(E)$  by dividing out the homogeneous (!) two-sided ideal generated by the elements  $x \otimes y - y \otimes x$ ,  $x$  and  $y$  in  $E$ . Therefore,  $S(E)$  admits a natural  $\mathbb{Z}^{(n)}$ -gradation.

By the universal property of  $S(E)$  it is now fairly trivial to check that

every gradation preserving A-linear morphism  $E \rightarrow B$ , B being a  $\mathbb{Z}^{(n)}$ -graded commutative A-algebra factorizes uniquely in

$$E \xrightarrow{i} S(E) \xrightarrow{g} B$$

where  $i$  is the natural map and  $g$  is a graded A-algebra morphism. Further, one verifies easily that for graded A-modules E and F, there is an isomorphism of graded A-algebras between  $S(E \oplus F)$  and  $S(E) \otimes S(F)$  where the tensorproduct is graded as usual.

The graded vector fibre  $V^g(E)$  of the  $\mathbb{Z}^{(n)}$ -graded A-module E is then defined to be the graded affine spectrum  $\text{SPEC}^g(S(E))$ . Note that  $V^g(E)$  is a graded  $\text{SPEC}^g(A)$ -scheme which represents the functor  $\text{HOM}_A(E \otimes_A -, -)_e$ ,  $e$  being the identity element of  $\mathbb{Z}^{(n)}$ . Let us give a typical example :

Example 6 :

Let  $\sigma \in \mathbb{Z}^{(n)}$ , then there is an isomorphism of graded A-algebras between  $S(A(\sigma))$  and  $A[t]$  where  $t$  is an indeterminate with degree  $-\sigma$ .

More generally, let  $\sigma_1, \dots, \sigma_n \in \mathbb{Z}^{(n)}$ , then

$$S(A(\sigma_1) \oplus \dots \oplus A(\sigma_n)) \cong A[X_1, \dots, X_n]$$

where  $\deg(X_i) = -\sigma_i$ .

Clearly, we are now in a position to represent the functors  $A_i$  and  $B_i$  defined above. The functor  $A_i$  is represented by the graded scheme  $V^g(S[P](i) \oplus S[P](i-1))$  whereas the functor  $B_i$  is represented by  $V^g(S[P])$ . More specific :

$$\begin{array}{ll} A_0 \rightsquigarrow \text{SPEC}^g(S[P][X, Y]) & \deg(X) = 0 \quad \deg(Y) = 1 \\ A_1 \rightsquigarrow \text{SPEC}^g(S[P][X, Y]) & \deg(X) = -1 \quad \deg(Y) = 0 \\ B_0 \rightsquigarrow \text{SPEC}^g(S[P][X]) & \deg(X) = 0 \\ B_1 \rightsquigarrow \text{SPEC}^g(S[P][Y]) & \deg(Y) = 0 \end{array}$$

The maps  $\alpha_i$  then correspond to the graded scheme morphisms :

$$\alpha_0 \rightarrow f_0 : \text{SPEC}^g S[P][X,Y] \rightarrow \text{SPEC}^g S[P][X]$$

$$\alpha_1 \rightarrow f_1 : \text{SPEC}^g S[P][X,Y] \rightarrow \text{SPEC}^g S[P][Y]$$

arising from the natural algebra inclusions, whereas the maps  $\beta_i$  correspond to the morphisms

$$\beta_0 \rightarrow g_0 : \text{SPEC}^g S[P][X,Y] \rightarrow \text{SPEC}^g S[P][X]$$

$$\beta_1 \rightarrow g_1 : \text{SPEC}^g S[P][X,Y] \rightarrow \text{SPEC}^g S[P][Y]$$

coming from the graded algebra map sending  $X$  to 1 (resp.  $Y$  to 1).

Therefore, the functor  $\text{Gr}_0^g$  is represented by the kernel of the following diagram of graded scheme morphisms

$$\begin{array}{ccc} \text{SPEC}^g S[P][X,Y] & \xrightarrow{f_0} & \text{SPEC}^g S[P][X] \\ & & \uparrow g_0 \\ & & \text{SPEC}^g S[P][X,Y] \end{array}$$

It is fairly easy to verify that this kernel equals  $V^g(X-1) \cong \text{SPEC}^g S[P][Y]$

where  $\deg(Y) = 1$ . Similarly, the functor  $\text{Gr}_1^g$  is represented by the

kernel of the diagram :

$$\begin{array}{ccc} \text{SPEC}^g S[P][X,Y] & \xrightarrow{f_1} & \text{SPEC}^g S[P][Y] \\ & & \uparrow g_1 \\ & & \text{SPEC}^g S[P][X,Y] \end{array}$$

which is equal to  $V^g(Y-1) \cong \text{SPEC}^g S[P][X]$  where  $\deg(X) = -1$ . Having

that the subfunctors  $\text{Gr}_i^g$  are representable by graded schemes over  $\text{SPEC}^g S[P]$ , we will now glue them together in order to represent  $\text{Grass}_1^g(0,-1)$ .

First, we aim to compute the fundamental modules for the subfunctors

$\text{Gr}_1^g$ . That is an element  $M_0 \in \text{Grass}_1^g(0,-1)(S[P][Y])$  and

$M_1 \in \text{Grass}_1^g(0,-1)(S[P][X])$  such that for every graded commutative

$S[P]$  algebra  $A$  the natural one-to-one correspondences between  $\text{HOM}(\text{SPEC}^g(A), \text{SPEC}^g(S[P][Y]))$  and  $\text{Gr}_0^g(A)$  and between  $\text{HOM}(\text{SPEC}^g(A), \text{SPEC}^g(S[P][X]))$  and  $\text{Gr}_1^g(A)$  are given by assigning to a scheme morphism  $\psi, \Gamma(\text{SPEC}^g(A), \psi^*(\tilde{M}_0))$  resp.  $\Gamma(\text{SPEC}^g(A), \psi^*(M_1))$ .  
An easy computation shows that :

$$M_0 \cong S[P][Y](0) = (S[P][Y] \oplus S[P][Y](-1))/(-Y, 1)S[P][Y]$$

$$M_1 \cong S[P][X](-1) = ((S[P][X](1) \oplus S[P][X])/(1, -X)S[P][X])(-1)$$

see [14] for a detailed computation.

The open set of  $\text{SPEC}^g S[P][Y]$  over which we have to glue  $\text{SPEC}^g S[P][Y]$  with  $\text{SPEC}^g S[P][X]$  is the set for which the composed map  $\gamma$  is an isomorphism.

$$\begin{array}{ccc} & S[P][Y](0) \oplus S[P][Y](-1) & \\ \nearrow y \mapsto 0 \oplus y & & \searrow ./(-Y, 1)S[P][Y] \\ S[P][Y](-1) & \xrightarrow{\gamma} & M_0 \end{array}$$

Now,  $\gamma(S[P][Y](-1)) = Y S[P][Y](0)$ , thus  $X^g(Y)$  is the desired open set.

Similarly,  $X^g(X)$  is the open set of  $\text{SPEC}^g S[P][X]$  for which the composed morphism  $\gamma'$

$$\begin{array}{ccc} & S[P][Y](0) \oplus S[P][Y](-1) & \\ \nearrow x \mapsto x \oplus 0 & & \searrow \\ S[P][Y](0) & \xrightarrow{\gamma'} & M_1 \end{array}$$

is an isomorphism. Concluding :

**Proposition 4 :** The functor  $F_{\Lambda \oplus S}^g$  is represented by a graded scheme  $\text{GRASS}_1^g(0, -1)$  over  $\text{SPEC}^g S[P]$  which is obtained by gluing  $\text{SPEC}^g S[P][Z]$ ,  $\deg(Z) = 1$ , together with  $\text{SPEC}^g S[P][Z^{-1}]$  over  $\text{SPEC}^g S[P][Z, Z^{-1}]$ .

The scheme  $\text{GRASS}_1^g(0, -1)$  can be interpreted as the graded one-dimensional projective space over  $S[P]$ .

Above, we mentioned that the part of degree zero (or  $e$ ) of a graded scheme is often a usual scheme. In particular, the part of degree zero of the graded scheme  $\text{GRASS}_1^g(0, -1)$  is the  $S$ -scheme which is obtained by gluing  $\text{Spec}(S[P]_{\geq 0})$  with  $\text{Spec}(S[P]_{\leq 0})$  over  $\text{Spec}(S[P])$ . This scheme is never regular !

Example 7 : Let  $\Lambda = \mathbb{C}[X, -]$  then  $S$  can be taken to be  $\mathbb{C}[t], t = X^2$ .

In this case, the part of degree zero of  $\text{GRASS}_1^g(0, -1)$  is the scheme obtained by gluing two affine cones (i.e.  $\text{Spec } \mathbb{C}[x, y, t] / (x^2 - y^t)$ ) over the complement of a ruling.

CASE 2 :  $P = \{p, q\}$  and  $\dim_{R:m} (\Lambda / \Lambda_{m+P+Q}) = 1$ .

It follows from lemma 3 that there is no étale extension of  $R$  which splits  $\Sigma$ . However, one can find an étale extension  $R_1$  of  $R$  such that  $\Lambda \otimes R_1 \cong R_1 \oplus R_1 i \oplus R_1 j \oplus R_1 ij$  with  $i^2 = p$  and  $j^2 = q$ . Moreover, there exists an extension  $S = R_1[X] / (X^2 - p)$  which splits  $\Sigma$  and such that the ring  $S(\Phi)$  defined in the proof of proposition 3 is a graded étale (and even Galois) extension of  $R_1[P]$ , in particular  $S(\Phi)$  is graded regular. This entails that

$$(\Lambda \otimes R_1)[P] \otimes S(\Phi) \cong M_2(S(\Phi))(\sigma_1, \sigma_2)$$

Further, a small computation show that the  $\sigma_i$ 's may be chosen to be  $e$  or  $(0, 1)$ . The case that  $\sigma_1 = \sigma_2 = e$  cannot occur since this would entail that

$$\Lambda \otimes R_1[P] \cong M_2(S(\Phi))(e, e)^G = M_2(R_1[P])(e, e)$$

Therefore we may assume that  $\sigma_1 = e$  and  $\sigma_2 = \sigma = (0, 1)$ .

Our first objective will be to represent the functor :

$$F_S^g : \text{gr Comm Alg}_{S(\Phi)} \rightarrow \text{Sets}$$

which assigns to any graded commutative  $S(\Phi)$ -algebra  $A$  the set of all split projective graded left ideals of rank two of  $M_2(A)(e, \sigma)$ . As in case 1 it is readily verified that this problem is equivalent to finding a representation of

$$\text{Grass}_1^g(e, -\sigma) : \text{gr Comm Alg}_{S(\Phi)} \rightarrow \text{Sets}$$

Mimicing the arguments of case 1 in the  $\mathbb{Z} \oplus \mathbb{Z}$ -graded case one can prove.

Proposition 5 : The functor  $F_S^g$  (or  $\text{Grass}_1^g(e, \sigma)$ ) is represented by a graded scheme  $\text{GRASS}_1^g(e, \sigma)$  over  $\text{SPEC}^g S(\Phi)$  which is obtained by gluing  $\text{SPEC}^g S(\Phi) [Z]$ ,  $\deg(Z) = \sigma$ , together with  $\text{SPEC}^g S(\Phi) [Z^{-1}]$  over  $\text{SPEC}^g S(\Phi) [Z, Z^{-1}]$ .

By graded Galois descent, one can then find the graded scheme over  $\text{SPEC}^g R_1 [P]$  which represents the functor

$$f^g : \text{gr Comm Alg}_{R_1 [P]} \rightarrow \text{Sets}$$

which assigns to a graded  $R [P]$ -algebra  $A$  the set of all graded projective split left ideals of  $\Lambda_1 [P] \otimes A$  of rank two. A concrete computation of this graded scheme can be found in [14].

B. a representation of  $F_\Lambda$ .

We will restrict attention here to case 1. If one has an explicit form of the graded scheme over  $\text{SPEC}^g R_1 [P]$  representing the functor  $f^g$  in case 2 one can easily mimic the argument below.

If  $A \in \underline{h}_S$ , then it follows from the key lemma that there is a natural one-to-one correspondence between elements of  $F_{\Lambda \oplus S}(A)$  and graded  $\text{SPEC}^g S [P]$ -scheme homomorphisms from  $\text{SPEC}^g A [P]$  to  $\text{GRASS}_1^g(0, -1)$ .

Now, any such graded scheme morphism

$$\varphi : \text{SPEC}^g A[P] \rightarrow \text{GRASS}_1^g(0, -1)$$

is determined by graded  $S[P]$ -algebra morphisms :

$$\varphi_1 : S[P][Z] \rightarrow A[P]$$

$$\varphi_2 : S[P][Z^{-1}] \rightarrow A[P]$$

such that their localizations  $(\varphi_1)_Z$  and  $(\varphi_2)_{Z^{-1}}$  yield the same graded  $S[P]$ -algebra morphism

$$\varphi_{12} : S[P][Z, Z^{-1}] \rightarrow A[P]$$

Clearly,  $\varphi_1$  is completely determined by  $\varphi_1(Z) \in A[P]_1 = A$ .p. Therefore there is a natural one-to-one correspondence between  $S[P]$ -algebra morphisms from  $S[P][Z]$  to  $A[P]$  and  $S$ -algebra morphisms from  $S[X, Y] / (X-pY)$  to  $A$ .

Similarly,  $\varphi_2$  is completely determined by  $\varphi_2(Z^{-1}) \in A[P]_{-1} = A$ .

Therefore there is a one-to-one correspondence between graded  $S[P]$ -algebra morphisms from  $S[P][Z^{-1}]$  to  $A[P]$  and algebra morphisms from  $S[X^{-1}]$  to  $A$ .

Finally, there is a one-to-one correspondence between graded  $S[P]$ -algebra homomorphisms from  $S[P][Z, Z^{-1}]$  to  $A[P]$  and  $S$ -algebra morphism from  $S[X, X^{-1}, Y] / (X-pY)$  to  $A$ . Therefore, if  $X$  denotes the  $S$ -scheme obtained by gluing  $\text{Spec } S[X, Y] / (X-pY)$  with  $\text{Spec } S[X^{-1}]$  over  $\text{Spec } S[X, X^{-1}, Y] / (X-pY)$ , then there is a natural one-to-one correspondence between graded  $\text{SPEC}^g S[P]$ -scheme morphisms from  $\text{SPEC}^g A[P]$  to  $\text{GRASS}_1^g(0, -1)$  and  $\text{Spec}(S)$ -scheme morphisms from  $\text{Spec}(A)$  to  $X$ .

Theorem 1 :

In case 1 the étale local structure of the scheme representing the

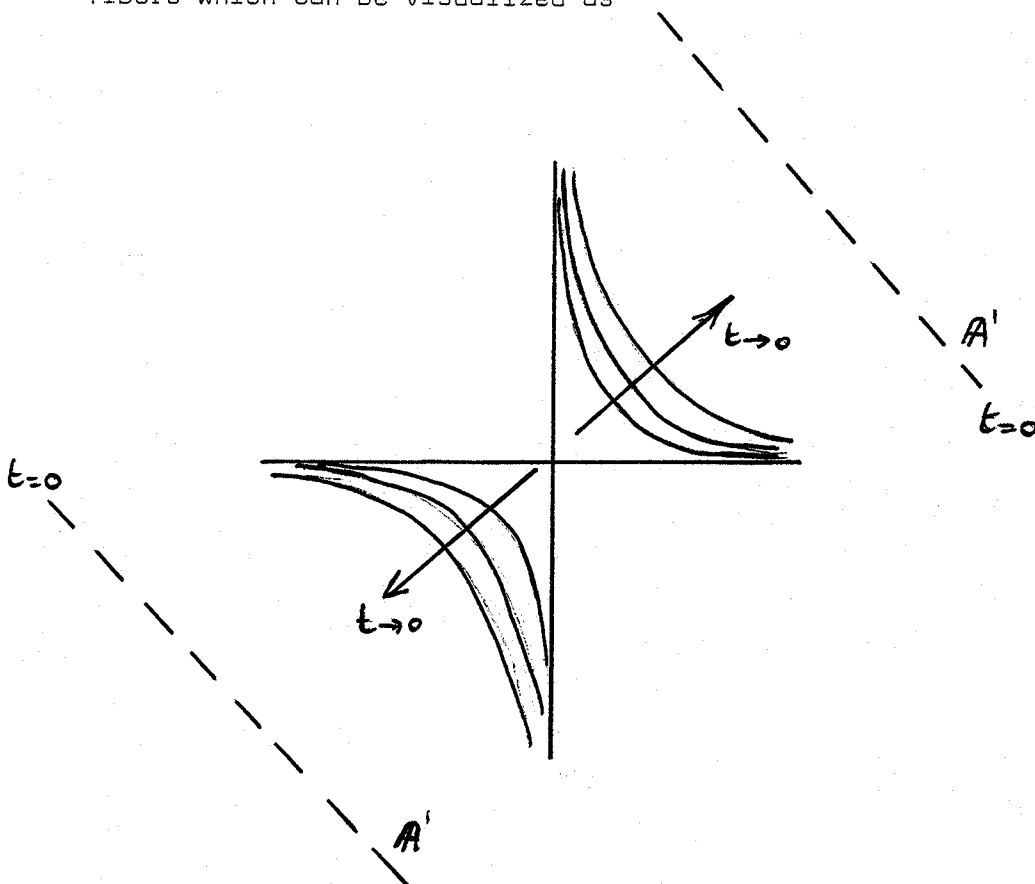


functor  $F_{\Lambda}$  is the scheme obtained by gluing  $\text{Spec } S[Y]$  with  $\text{Spec } S[Z]$  over  $\text{Spec } S[Y, Z] / (pYZ-1)$ .

The scheme  $BS'_{\Lambda}$  over  $\text{Spec}(R)$  which is obtained by étale descent from the  $S$ -scheme  $X$  is called the (weak) Brauer-Severi scheme of  $\Lambda$  and it represents the functor  $F_{\Lambda}$ .

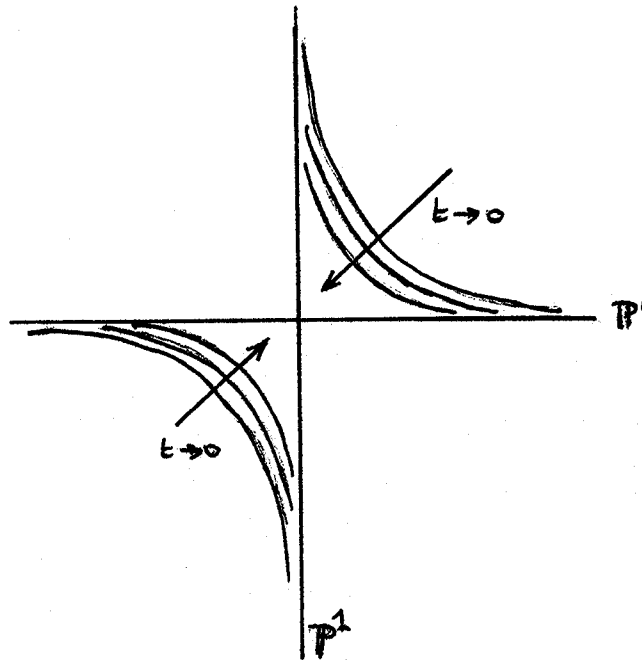
If one can compute the full Brauer-Severi scheme of  $\Lambda$ ,  $BS_{\Lambda}$ , we conjecture that there is always an open immersion  $BS'_{\Lambda} \rightarrow BS_{\Lambda}$ . Let us conclude this paper with an example :

Example 8 : (cfr. ex 2 and ex. 7). The scheme  $X$  over  $\text{Spec } \mathbb{C}[t]$  has fibers which can be visualized as



a family of conics, degenerating to a pair of distinct affine lines.

The étale local structure of the full Brauer-Severi scheme was computed by Artin. Its fibers can be visualized as a family of conics, degenerating to a pair of projective lines meeting transversally at one point.



It is fairly easy to give an open immersion of  $X$  in  $BS_{\Lambda} \otimes A_C^1$ . The weak Brauer-Severi scheme misses one point corresponding to the left ideal of rank two  $L$  in

$$\Lambda \otimes \mathbb{C}[t]/(t) \cong \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ (t)/(t^2) & \mathbb{C} \end{pmatrix}$$

where

$$L = \begin{pmatrix} 0 & \mathbb{C} \\ (t)/t^2 & 0 \end{pmatrix}$$

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