THE BASS-QUILLEN CONJECTURE AND MAXIMAL ORDERS.

by

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0. The problem.

In 1955 J.P. Serre wrote in his famous article "Faisceaux Algébriques Cohérents":

"On ignore s'il existe des $k[X_1, \ldots, X_n]$-modules projectifs de type fini (k un corps) qui ne soient pas libre".

The moral impulse behind this question is that the affine n-space $\mathbb{A}_k^n$ should behave like a contractible space in topology and hence should only have trivial vectorbundles over it.
In 1976, D. Quillen and A. Suslin independently proved the validity of Serre's conjecture. Actually they proved the stronger result:

<u>Theorem</u> (Quillen-Suslin)
If R is a commutative regular domain of Krull dimension $\leqslant 2$, then any finitely generated projective module P over $R[X_1, \ldots, X_n]$ is extended from R.
In view of this result one is led to the following natural extension of the original Serre-conjecture:

Bass-Quillen conjecture:

If R is a commutative regular domain of finite Krull dimension; then every finitely generated projective module P over $R[X_1, \dots, X_n]$ is extended from R. Using techniques such as Quillen induction and Quillen patching one can show that this conjecture is equivalent to the following one:

(BQ_R): If R is a commutative regular local domain of finite Krull dimension, then every finitely generated projective module P over R[t] is free.

If one supposes that P is a graded module with respect to the natural gradation on R[t], then P is indeed free. This result was already contained in the classical book of Cartan and Eilenberg. Another positive result about (BQ_R) was proved independently by Lindel-Lüthebohmert and Mohan Kumar. If R is a complete regular local domain of finite Krull dimension, then (BQ_R) holds. The essential point in this case is that $R \cong k[[X_1, \dots, X_n]]$ for some field k by the structure result of Cohen. To deal with the Bass-Quillen conjecture in the general (equi-characteristic) case, a natural ideal would be to go to the completion $\hat{R}$ of the regular local domain R, in vohe the validity of ($BQ_{\hat{R}}$) and then somehow try to make a descent to the original regular local ring R. This descent part, however, is no easy task and so far no one has been able to carry it out successfully.

In this note we aim to give some translations of this descent-problem in noncommutative ring-theory, in particular the theory of maximal orders and of Azumaya algebras. This approach has the extra advantedge that one can now invoke ring theoretical features (such as spectra, regularity, etc.) in order to obtain more information.

1. Descent and homogenization of maximal orders.

Let us recall some standard definitions, cfr. e.g. [5]. If R is any ring and R[t] is graded in the usual way, i.e. deg(t) = 1, then one can build out of any graded R[t]-module M the graded R[t,s]-module (where deg(s) = 1) in the obvious way. If we decompose an element $m \in M$ into homogeneous elements, say $m = m_k + \ldots + m_c + \ldots + m_1$, where $\deg(m_i) = i$; then we can associate to it an homogeneous element $m^\star \in M[s]$ given by: $m^\star = m_k s^{k+1} + \ldots + m_o s^1 + \ldots + m_1$. We say that $m^\star$ is the homogenized element of m. Conversely, if u is an homogeneous element of M[s], say $u = u_{-k} s^{k+1+r} + \ldots + u_o s^{1+p} + \ldots + u_1 s^p$ where $\deg(u_i) = i$, then $u_\star = u_{-k} + \ldots + u_o + \ldots + u_1$ is said to be the dehomogenized element of u. Now, let N be a (not necessarely graded) R[t]-submodule of M, then by $N^\star$ we mean the R[t,s]-submodule of M[s] generated by the elements $n^\star$, $n \in N$. Any homogeneous element $n \in N^\star$ is of the form $s^r, n_1^\star$ where $n_1 \in N$ and $r \geq 0$. This graded module is called the homogenized module of N. Conversely, to a graded R[t,s]-submodule L of M[s] one can associate $L_\star = \{u_\star ; u \in h(L)\}$ where h(L) denotes the set of all homogeneous elements of L. From [5] we retain that there is an exact functor:

$$E : R[t,s]\text{-gr} \to R[t]\text{-mod};\ E(M) = R[t,s]/(s-1) \quad M$$

such that $E(L) = L_\star$ for every graded R[t,s]-submodule L of some M[s]. We will consider a faithfully flat extension of regular local rings $R \subset S$ with corresponding fields of fractions $K \subset L$. By a suitable gradation on $M_n(L[t])$ we mean a gradation which extends the usual gradation of R[t], K[t], S[t] and L[t]. If Λ is an Azumaya algebra over R[t] which is contained in $M_n(K[t])$, then one can homogenize Λ as an R[t]-submodule of the graded R[t]-module $M_n(L[t])$.

A typical example of a suitable gradation is taking an element $\alpha \in GL_n(L[t])$ and defining the set of homogeneous elements of degree m of $M_n(L[t])$ to be the set $\alpha^{-1}.t^m.M_n(L).\alpha$.

Proposition 1:

If A is a factorial Krull domain, then the following two statements are equivalent

(1): every projective rank n A-module is free

(2): all endomorphism rings of projective rank n A-modules are isomorphic as A-algebras.

Proof.

(1) ⇒ (2): trivial.

(2) ⇒ (1): (cohomological proof).

It follows from a sheaf version of the Skolem-Noether theorem, cfr. e.g. [4, IV.2.3, 2.4] that the sequence:

$$1 \to \mathbb{G}_m \to \underline{GL}_n \to \underline{PGL}_n \to 1$$

is exact both in the Zariski and étale topology. We obtain:

$$(+)\quad \begin{array}{ccccccc} H^1_{Za}(X, \underline{O}_A^\star) & \to & H^1_{Za}(X, GL_n) & \to & H^1_{Za}(X, PGL_n) & \to & H^2_{Za}(X, \underline{O}_A^\star) \\ \wr\Vert & & \wr\Vert & & \downarrow & & \downarrow \\ H^1_{et}(X, \mathbb{G}_m) & \to & H^1_{et}(X, GL_n) & \to & H^1_{et}(X, PGL_n) & \to & H^2_{et}(X, \mathbb{G}_m) \end{array}$$

where the two first isomorphisms come form [4, p. 134]. A being factorial, Weil and Carter divisors coincide and therefore $H^2_{Za}(X, \underline{O}_A^\star) = 1$, yielding the exact sequence:

$$1 \to H^1_{Za}(X, \underline{PGL}_n) \to H^1_{et}(X, \underline{PGL}_n) \to Br(A).$$

Now, $H^1_{et}(X, PGL_n)$ is the set of A-algebra isomorphisms of rank n, Azumaya algebras over A, whence $H^1_{Za}(X, PGL_n)$ classifies endomorphisms in β of projective rank n A-modules upto A-algebra isomorphism.

Because A is factorial it follows from (+) that
H^1_{Za} (X, $\underline{GL}_n$) $\cong H^1_{Za}$ (X, $\underline{PGL}_n$) finishing the proof.

(ring theoretical proof):

if P is a projective rank n A-module, then it follows from $End_A(P) \cong M_n(A)$ that P has endomorphisms $\varphi_1, \ldots, \varphi_n$ such that $\varphi_i \circ \varphi_j = \delta_{ij} \varphi_i$ and $\Sigma \varphi_i = 1_p$. One can easily deduce from this that P has a decomposition

$$P = \varphi_1(P) \oplus \varphi_2(P) \oplus \ldots \oplus \varphi_n(P)$$

into reflexive A-modules of rank one. Now, A being factorial this entails that P is a free A-module.

Remark:

The cohomological proof given above has the advantedge that it gives a method to compare information about isomorphism classes (as A-modules) of projective rank n A-modules with information about isomorphism classes (as A-algebras) of their endomorphism rings for arbitrary Krull domains A. In general however this comparision will depend upon Pic(A), H^2_{Za} (X, $\underline{\Theta}^{\star}_A$) and the kernel of the natural map Br(A) $\rightarrow$ Br(Q(A)).

We will now apply prop. 1 to the case of interest to us; i.e. $R \subset S$ a faithfully flat extension of regular local rings and Λ will be an R[t]-Azumaya algebra contained in $M_n(K[t])$, which will be an endomorphism ring of a projective rank n R[t]-module since Br R[t] $\hookrightarrow$ Br K(t) because R[t] is a regular domain; cfr. [6].

Therefore, if we assume that S satisfies the Bass-Quillen conjecture, then one can find every R[t]-Azumaya algebra Λ contained in $M_n(K[t])$ an element $\alpha_\Lambda \in GL_n(L[t])$ such that $\Lambda \otimes S[t] = \alpha_\Lambda^{-1}.M_n(S[t]).\alpha_\Lambda$.

Theorem 2: (faithfully flat descent)

With notations and assumptions as above, equivalent are:

(1): every projective rank n $R[t]$-module is free.

(2): for every $R[t]$-Azumaya algebra Λ one can find a suitable gradation on $M_n(L[t])$ s.t. $\Lambda^\star$ is a maximal $R[t;s]$-order and $(\alpha_\Lambda^{-1}.M_n(S[t]).\alpha_\Lambda)^\star$ is an Azumaya algebra.

Proof.

(1) ⇒ (2): trivial.

(2) ⇒ (1): If $\Lambda^\star.K(t,s) \cong \mathrm{End}_{K(t,s)}(V)$ for some finite dimensional $K(t,s)$-vectorspace V, then it follows from [0, Prop. 4.2] that there is a f.g. reflexive $R[t,s]$-submodule E of V s.t. $\Lambda^\star = \mathrm{End}_{R[t,s]}(E)$. Because $\mathrm{End}_{R[t,s]}(E) \otimes S[t,s] \cong \Lambda^\star.S[t,s] \subset (\alpha^{-1}.M_n(S[t]).\alpha_\Lambda)^\star$ and $(\mathrm{End}_{R[t,s]}(E) \otimes S[t,s])^{bid} \cong \mathrm{End}_{S[t,s]}((E\otimes S[t,s])^{bid})$ is a maximal $S[t,s]$-order, it follows that:

$$\mathrm{End}_{[t,s]}(E \quad S[t,s]) \cong (\alpha_\Lambda^{-1}.M_n(S[t]).\alpha_\Lambda)^\star$$

Hence it is a projective $S[t,s]$-module. $S[t,s]$ being regular this implies that $E \otimes S[t,s]$ is a f.g. projective $S[t,s]$-module and by faithfully flat descent E is a f.g. projective $R[t,s]$-module. This entails that $\Lambda^\star$ is a graded Azumaya algebra which represents the trivial class in $\mathrm{Br}(R[t,s])$. It follows from the injectivity of $\mathrm{Br}^g R[t,s] \hookrightarrow \mathrm{Br}\, R[t,s]$, cfr. [2, Prop. 2] or [1] for the more general case, that $\Lambda^\star \cong \mathrm{END}_{R[t,s]}(P)$ for some f.g. graded projective $R[t,s]$-module P.

Now, $P \cong P_o \otimes R[t,s]$ by [7, Th. 4.6] yielding that $\Lambda^\star \cong \mathrm{End}_R(P_o)[t,s] \cong M_n(R[t,s])$ because R is a local ring. Applying the exact functor E(-) on both sides yields $\Lambda \cong M_n(R[t])$ and Prop. 1 finishes the proof.

2. Some possible approaches.

A first idea might be to put on $M_n(L[t])$ the gradation defined by conjugation with α_Λ since then we have that $(\alpha_\Lambda^{-1}.M_n(S[t]).\alpha_\Lambda)^\star = \alpha_\Lambda^{-1}.M_n(S[t,s]).\alpha_\Lambda$ is an Azumaya algebra. However, for arbitrary α_Λ, $\Lambda^\star$ need not be a maximal order as the following example shows:

Example 1. (Swan, Stafford)

Let R be any commutative Krull domain and suppose that S is an overring of R which contains an element x which is trancendental over R or algebraic of degree ≥ 3 so that 1, x, x^2 are linearly independant over R. Now choose $\Lambda = \mathrm{End}_{R[t]}(P)$ where P is free of rank 2, then $S[t] \otimes P \cong S[t] \oplus S[t]$ but we change the base for this by the transvection

$$\alpha = \begin{pmatrix} 1 & x.t \\ 0 & 1 \end{pmatrix}$$

getting $\Lambda \subset \alpha.M_2(S[t]).\alpha^{-1}$. For convenience we will look at the isomorphic situation: $\alpha^{-1}.\Lambda.\alpha \subset M_2(S[t])$. If

$$\lambda = \sum_{i=o}^{n} \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \quad t^i \in \Lambda$$

then we obtain that $\alpha^{-1}.\lambda.\alpha$ equals

$$\sum_{i=o}^{n} \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} t^i + \sum_{i=o}^{n} \begin{pmatrix} -c_i & a_i-d_i \\ 0 & c_i \end{pmatrix} .x.t^{i+1} - \sum_{i=o}^{n} \begin{pmatrix} 0 & c_i \\ 0 & 0 \end{pmatrix} .x^2.t^{i+2}$$

and this representation is unique since t^i, $x.t^j$, $x^2.t^k$ are linearly independent over R. In order to see that $\Lambda^\star$ is not a maximal order observe that $\Lambda^\star$ is free as a R[t,s]-module on:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} s + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x.t \quad \text{and}$$

$$C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} s^2 + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} .x.s.t - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .x^2.t^2$$

which follows from inspection from the expression for $\alpha^{-1}.\lambda.\alpha$ above.

The multiplication table is

	A	B	C
A	0	0	s.B
B	s.A	s.B	0
C	$s^2.1 - sB$	s.C	0

whence the free $R[t,s]$-module with basis 1, A, $s^{-1}.B$, C is also an order which contains $\Lambda^\star$ properly.

Another idea might be to put on $M_n(L[t])$ the usual gradation. Then we will see below that both $\Lambda^\star$ and $(\alpha_\Lambda^{-1}.M_n(S[t]).\alpha_\Lambda)^\star$ are maximal orders. However, $(\alpha_\Lambda^{-1}.M_n(S[t]).\alpha_\Lambda)^\star$ need not be an Azumaya algebra as the following example shows:

<u>Example 2.</u> (Le Bruyn, Van den Bergh)

Let a, b $\in$ S such that $b \nmid a$ and b is not a unit in S, then consider

$$\alpha = \begin{pmatrix} 1 & \frac{a}{v}t \\ 0 & 1 \end{pmatrix} .$$

We have

$$\begin{pmatrix} 1 & -\frac{a}{v}t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}\begin{pmatrix} 1 & \frac{a}{v}t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha(t) - \frac{a}{v}\gamma(t)t & \beta(t) + \frac{a}{v}t(\alpha(t)-\delta(t)) - \frac{a^2}{v^2}\gamma(t)t^2 \\ \gamma(t) & \delta(t) + \frac{a}{v}\, t\gamma(t) \end{pmatrix}$$

Using this form it is not to hard to calculate that $(\alpha^{-1}\, M_2(S[t])\alpha)^\star$ is generated as an $S[t,s]$-module by the following matrices:

$$G_1 = \begin{pmatrix} s & \frac{a}{v}t \\ 0 & 0 \end{pmatrix} ; \quad G_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; \quad G_3 = \begin{pmatrix} -\frac{a}{v} st & -\frac{a^2}{v^2} t^2 \\ s^2 & \frac{a}{v} st \end{pmatrix}$$

$$G_5 = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} ; \quad G_7 = \begin{pmatrix} 0 & 0 \\ b^2 s & 0 \end{pmatrix} ;$$

$$G_8 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } G_9 = \begin{pmatrix} 0 & 0 \\ bs & at \end{pmatrix} .$$

It is now fairly easy to check that $(\alpha^{-1} M_2(S[t])\alpha)^\star$ is not a projective $S[t]$-module.

We will now investigate when $\Lambda^\star$ is a maximal order. Altough all results remain valid for arbitrary $\mathbb{Z}$-graded Krull domains and maximal orders in graded central simple algebras, we will restrict attention heere to the case considered above.

<u>Lemma 3.</u>

$\Lambda^\star_s$, the localization of $\Lambda^\star$ at the homogeneous element s in an Azumaya algebra over $R[t;s,s^{-1}]$.

<u>Proof.</u>

Because Λ is a subring of the graded ring $M_n(L[t])$, it follows that $\Lambda^\star$ is a graded subring of $M_n(L[t,s])$. Because s is a unit of degree one in the grade ring $\Lambda^\star_s$ it is clear that $\Lambda^\star_s = (\Lambda^\star_s)_o\ [s,s^{-1}]$, where $(-)_o$ denotes the part of degree zero.

Now, $(\Lambda^\star_s)_o = \Sigma\ s^{-k}, (\Lambda^\star)_k$ and further

$(\Lambda^\star)_k = \{s^1\ \lambda^\star;\ \lambda \in \lambda,\ 1 + \deg\ (\lambda^\star) = k\}$ and therefore any element

$x \in (\Lambda^\star_s)_o$ is of the form:

$x = \lambda_{n_1}.s^{-n_1} + \ldots + \lambda_{n_k}.s^{-n_k}$ where $\deg(\lambda_{n_i}) = n_i$ and $\lambda_{n_1} + \ldots + \lambda_{n_k} \quad \Lambda$.
Define a map $\psi : \Lambda \to (\Lambda_s^\star)_o$ by sending $\lambda = \lambda_{n_1} + \ldots + \lambda_{n_k}$ to
$\psi(\lambda) = \lambda_{n_1}.s^{-n_1} + \ldots + \lambda_{n_k}.s^{-n_k}$, then ψ is a ringisomorphism whence
$(\Lambda_s^\star)_o$ is an Azumaya algebra over $\psi(R[t]) = R[t.s^{-1}]$. Finally,
$\Lambda_s^\star = (\Lambda_s^\star)_o\ [s,s^{-1}]$ is Azumaya over $R[t,s^{-1}]\,[s,s^{-1}] = R[t,s,s^{-1}]$.

From this lemma we retain that $\Lambda^\star$ is a prime p.i.-ring. $\Lambda^\star$ being Z-graded, its center is also graded. Clearly, $Z(\Lambda^\star)_\star \subset Z(\Lambda) + R[t]$ whence $R[t,s] \subset Z(\Lambda^\star) \subset Z(\Lambda^\star)_\star^\star \subset R[t,s]$, whence $Z(\Lambda^\star) = R[t,s]$. The next fact, which was brought to my attention by T. Stafford, shows that the problem of checking maximality of $\Lambda^\star$ actually reduces to the corresponding problem for $M_n(K[t])^\star$.

Proposition 4.

The following statements are equivalent:
(1): $\Lambda^\star$ is a maximal $R[t,s]$-order.
(2): $M_n(K[t])^\star$ is a maximal $K[t,s]$-order.

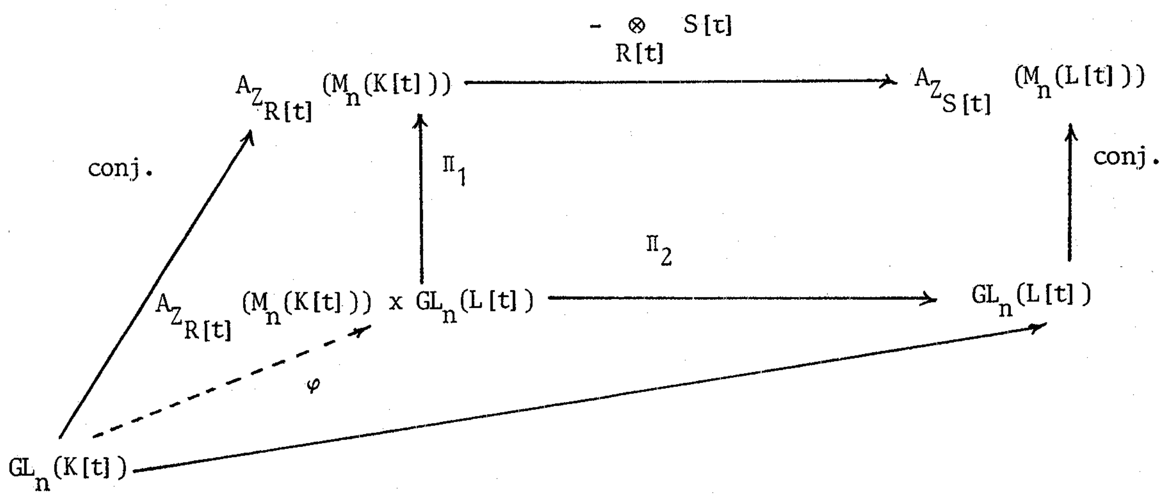
Proof.

For any nonzero element x of R it is clear that $(\Lambda^\star)_x = (\Lambda_x)^\star$, showing that $\Lambda^\star.K[t,s] = M_n(K[t])^\star$. Therefore, the implication (1) $\Rightarrow$ (2) is trivial.
(2) $\Rightarrow$ (1): Let Γ be an $R[t,s]$-order containing $\Lambda^\star$ such that $r.\Gamma \subset \Lambda^\star$ for some $r \in R[t,s]$, then $\Lambda^\star.K[t,s] \subset \Gamma.K[t,s]$ and $r.\Gamma.K[t,s] \subset \Lambda^\star.K[t,s] = M_n(K[t])^\star$. By maximality of $M_n(K[t])^\star$ this entails that $\Lambda^\star \subset \Gamma \subset M_n(K[t])^\star$.

Therefore we can define the internal homogenization $\Gamma^{(g)}$ of Γ, cfr. e.g. [5]. $\Lambda^{\star} \subset \Gamma^{(g)} \subset M_n(K[t])^{\star}$ whence $\Lambda = (\Gamma^{(g)})$ showing that $\Gamma^{(g)} \subset (\Gamma^{(g)})_{\star}{}^{\star} \subset \Lambda^{\star}$ whence $\Lambda^{\star} = \Gamma^{(g)}$ and thus finally $\Lambda^{\star} = \Gamma$.

In particular, if we take the usual gradation on $M_n(L[t])$, the $\Lambda^{\star}$ is a maximal $R[t,s]$-order since $M_n(K[t])^{\star} = M_n(K[t,s])$ and a similar argument shows that $(\alpha^{-1}.M_n(S[t]).\alpha_\Lambda)^{\star}$ is maximal.

We will now investigate which α_Λ can arise. With $\mathbb{A}_{Z_A}(\Sigma)$ we will denote the set of all A-Azumaya algebras contained in Σ. Clearly, $U(\Sigma)$ has a natural action on $A_{Z_A}(\Sigma)$ by conjugation. Now, look at the pullbach-diagram:



Therefore, we are interested in the following set:

$$BQ(S/R) = \Pi_2\ (A_{Z_{R[t]}}(M_n(K[t])) \times GL_n(L[t]) = \operatorname{Im} \varphi)$$

<u>Theorem 5.</u> (faithfully flat descent, first approach)

With notations and assumptions as above, equivalent are:

(1): every projective rank n $R[t]$-module is free.

(2): $\forall \alpha \in BQ(S/R)$: $M_n(K[t])^{\star}$ is a maximal order.

Proof.

Suppose that Λ is an endomorphism ring of a projective rank n $R[t]$-module which is not isomorphic to $M_n(R[t])$, then we may suppose that Λ is contained in $M_n(K[t])$ (upto isomorphism).
Therefore, $\Lambda \otimes S[t] = \alpha^{-1}.M_n(S[t]).\alpha$ for some $\alpha \in BQ(S/R)$.
Using Prop. 4 the homogenization of Λ with respect to the gradation on $M_n(L[t])$ defined by conjugation with α is a maximal order and therefore Th. 2 finishes the proof.

Theorem 6. (faithfully flat descent, second approach).

With notations and assumptions as above, equivalent are:

(1): every projective rank n $R[t]$-module is free.

(2): $\forall\alpha \in BQ(S/R)$: $(\alpha^{-1}.M_n(S[t]),\alpha)^{\star}$ is a flat $S[t,s]$-module.

Proof.

As in the proof of Th. 5 $\Lambda \otimes S[t] = \alpha^{-1}.M_n(S[t]).\alpha$ for some $\alpha \in BQ(S/R)$. By Prop. 4 both $\overset{\star}{\Lambda}$ and $(\alpha^{-1}.M_n(S[t]).\alpha)^{\star}$ are maximal orders where homogenization is with respect to the usual gradation on $M_n(L[t])$. Therefore, $(\alpha^{-1}\ M_n(S[t]).\alpha)^{\star}$ is a reflexive $S[t,s]$-Azumaya algebra which is Azumaya by [3, Prop. 3.1]. Th. 2 now finishes the proof.

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