Quillen Patching for Maximal Orders

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Throughout this note, R will be a Krull domain with field of fractions K and Σ will be a central simple K-algebra. Two maximal R-orders in Σ, Λ and Γ, are said to be conjugated if there exists an $\alpha \in \Sigma^{\star}$ scuh that $\Lambda = \alpha^{-1}.\Gamma.\alpha$. It is clear that this defines an equivalence relation on the set of all maximal R-orders in Σ; the (pointed) set of equivalence classes will be denoted by $t_R(\Sigma)$ and is called the type number of R in Σ. In [4,5] the author presented a cohomological interpretation of this invariant and calculated it in the special case : $\Sigma = M_n(K)$ and R locally factorial. It turned out that in this case the type number contains a lot of information about (reflexive) modules over R.

This fact shows that it is an arduous tash to study the behaviour of the type number under polynomial extensions since any positive result on it will probably shed some new light on deep module-theoretic questions such as the Bass-Quillen conjecture. With this short note we aim to provide a tool which we think will be useful in such a study, namely a maximal order equivalent of the famous Quillen patching theorem for modules, cfr. [6] . Let Λ be any maximal $R[t]$-order in $\Sigma(t)$, then we define :

$I(\Lambda) = \{f \in R \mid \exists$ maximal R_f-order Γ in Σ : Λ_f is conjugated to $\Gamma[t]\}$.

Theorem 1. (Quillen patching for maximal orders)

If Λ is a maximal $R[t]$-order in $\Sigma(t)$, then $I(\Lambda)$ is an ideal of R.

Proof. Clearly $R.I(\Lambda) \subset I(\Lambda)$ and therefore it remains to show that whenever f_o and f_1 are elements of $I(\Lambda)$, so is $f_o + f_1 = f$.

After replacing R by R_f, Λ by Λ_f etc. we may therefore assume that f_o and f_1 are comaximal elements in R.

Choose maximal R_{f_i}-orders Γ_i in Σ such that $\Lambda_{f_i} \cong \Gamma_i[t]$ as $R_{f_i}[t]$ - algebras. Then, we have :

$$(\Lambda/t\Lambda)_{f_o} \cong \Gamma_o$$

$$(\Lambda/t\Lambda)_{f_1} \cong \Gamma_1$$

and hence it is no restriction to impose that $\Gamma_o = \Gamma_{f_o}$ and $\Gamma_1 = \Gamma_{f_1}$ where $\Gamma = \Lambda/t\Lambda$. Therefore, take $R_{f_i}[t]$ - algebra isomorphisms :

$$\varphi_i : \Lambda_{f_i} \to \Gamma_{f_i}[T]$$

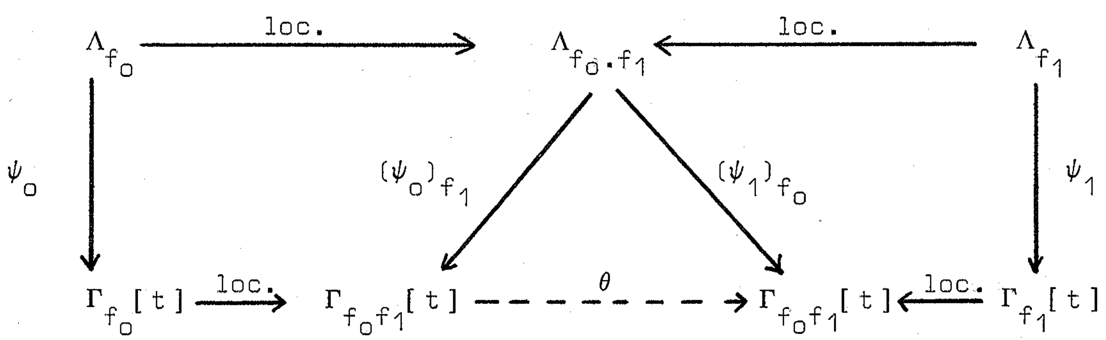
Reduction modulo t gives an R_{f_i}-algebra isomorphism :

$$\overline{\varphi}_i : \Gamma_{f_i} \to \Gamma_{f_i}$$

and hence, after composing φ_i with the extended isomorphism $\overline{\varphi}_i^{-1}[t] : \Gamma_{f_i}[t] \to \Gamma_{f_i}[t]$ we can take $R_{f_i}[t]$ - algebra isomorphism :

$$\psi_i : \Lambda_{f_i} \to \Gamma_{f_i}[t]$$

such that ψ_i reduces modulo t to the identity map of Γ_{f_i}. With these notations we have the following localization-diagram :



where $\theta = (\psi_1)_{f_o} \circ (\psi_o)_{f_1}^{-1}$ is an $R_{f_o f_i}[t]$ - algebra automorphism of $\Gamma_{f_o f_1}[t]$.

We claim that θ is given by conjugation with a unit of $\Gamma_{f_o f_1}[t]$. From [5] we retain that the following exact diagram is a commutative one :

$$\begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Outcent}(\Gamma_{f_o f_1}) & \longrightarrow & \mathrm{Cl}^c(\Gamma_{f_o f_1}) & \longrightarrow & \mathrm{Cl}(\Gamma_{f_o f_1}) & \longrightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Outcent}(\Gamma_{f_o f_1}[t]) & \longrightarrow & \mathrm{Cl}^c(\Gamma_{f_o f_1}[t]) & \longrightarrow & \mathrm{Cl}(\Gamma_{f_o f_1}[t]) & \longrightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1 & & \end{array}$$

where for every maximal order Θ we denote with $\mathrm{Cl}^c(\Theta)$ (resp. $\mathrm{Cl}(\Theta)$) the quotient group of the groups of divisorial Θ-ideals generated by a central element (resp. normalizing element, i.e. an element, s.t. $x.\theta = \Theta.x$) and Outcent $(\Theta) = \mathrm{Autcent}(\Theta)/\mathrm{Inn}(\Theta)$, cfr. e.g. [2].

Now, suppose $\theta \in$ Outcent $(\Gamma_{f_o f_1}[t])$ then by the diagram above there exists a normalizing element n of $\Gamma_{f_o f_1}$ and a unit $\gamma \in \Gamma_{f_o f_1}[t]^{\star}$ such that θ is given by conjugation with $n.\gamma$. Because θ reduces to the identity modulo t we have that $n.\gamma(o) \in K^{\star}$, so $n \quad \Gamma^{\star}_{f_o f_1}.K^{\star}$ whence θ is given by conjugation with $\gamma(o)^{-1}.\gamma = \alpha$, i.e. with a unit of $\Gamma_{f_o f_1}[t]$, finishing the proof of our claim.

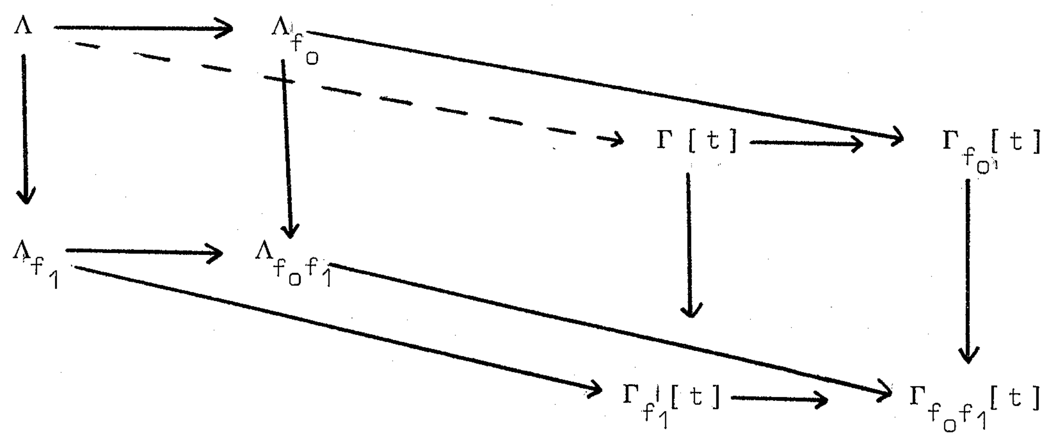
Because $\alpha(0) = 1$; there exist by [3, v.1.3] units $\alpha_o \in \Gamma_{f_o}[t]^{\star}$ such that:

$$\alpha = (\alpha_1)^{-1}_{f_o}.(\alpha_o)_{f_1}$$

If we denote by c_x conjugation with the element x and $\theta_i = c_{\alpha_i} \circ \psi_i$, then we obtain the following commutative diagram :

$$\begin{array}{ccccc} \Lambda_{f_o} & \xrightarrow{\text{loc.}} & \Lambda_{f_o f_1} & \xleftarrow{\text{loc.}} & \Lambda_{f_1} \\ \theta_o \downarrow & & \downarrow (\theta_o)_{f_1} = (\theta_1)_{f_o} & & \downarrow \theta_1 \\ \Gamma_{f_o}[t] & \xrightarrow{\text{loc}} & \Gamma_{f_o . f_1}[t] & \xleftarrow{\text{loc.}} & \Gamma_{f_1}[t] \end{array}$$

To finish the proof of the theorem, f_o and f_1 are comaximal elements of R and therefore the front and rear square of the diagram below are pullback diagrams. The existence of an $R[t]$-algebra isomorphism $\Lambda \to \Gamma[t]$ amounts essentially to the fact that the pullback construction is a functor.



In the next theorem we aim to refine this result in the special case that $\Sigma = M_n(K)$.

Theorem 2: If R is any Krull domain and Λ is a maximal $R[t]$-order in $M_n(K(t))$, then $I(\Lambda)^{**} = R$.

Proof.

By the foregoing theorem, $I(\Lambda)$ is an ideal of R and hence $X(I(\Lambda)) = \{P \in \mathrm{Spec}(R) : I(\Lambda) \not\subset P\}$ defines an open set in the Zariski topology on Spec(R). If $X^{(1)}(R)$, the set of height one prime ideals of R, is contained in $X(I(\Lambda))$, then we are done. Therefore, suppose $p \in X^{(1)}(R)$ such that $I(\Lambda) \subset p$. Because Λ is a maximal order in $M_n(K(t))$, $\Lambda = \mathrm{End}_{R[t]}(M)$ for some reflexive $R[t]$- module M, cfr. [1]. Because $R_p[t]$ is a regular ring of global dimension two, $\Lambda_p = \mathrm{End}_{R_p[t]}(M_p)$ is an Azumaya algebra since M_p is a reflexive and hence projective $R_p[t]$-module. By Seshadri's theorem, cfr. e.g. [3]. M_p is free and therefore $\Lambda_p \cong M_n(R_p[t])$.

This entails that there exists an element $\alpha(t) \in GL_n(K(t))$ such that $\Lambda_p = \alpha(t)^{-1} . M_n(R_p[t]).\alpha(t)$. From this one deduces by localizing further that :

$$\begin{cases} \Lambda_{p[t]} = \alpha(t)^{-1}.M_n(R[t]_{p[t]}).\alpha(t) \\ \Lambda_q = \alpha(t)^{-1}.M_n(R[t]_q).\alpha(t) \end{cases}$$

for all $q \in X^{(1)}(R[t])$ such that $q \cap R = 0$. By a standard argument as in the proof of [5, Th. 2.6.] one can extend these equalities to an open subvariety $X(J)$ of Spec $R[t]$ for some ideal J of $R[t]$ and such that

$$\{p[t]\} \cup \{q \in X^{(1)}(R[t]) : q \cap R = 0\} \subset X(J)$$

If $X^{(1)}(R[t]) \subset X(J)$, then $X^{(1)}(R) \subset X(J) \cap \mathrm{Spec}(R) \subset X(I(\Lambda))$.
Therefore, suppose that $J^{**} = p_1^{k_1} * \dots * p_n^{k_n}[t]$ for some $p_i \in X^{(1)}(R)$ all different from p, then $J^{**} \cap R \not\subset p$.
So, take $f \in (J^{**} \cap R) \setminus p$, then

$$\Lambda_f = \alpha(t)^{-1}.M_n(R_f[t]).\alpha(t)$$

because $X(J^{**}) \cap X^{(1)}(R[t]) = X(J) \cap X^{(1)}(R[t])$ and this entails that $\Gamma(X(J),\underline{\theta}_\Lambda) = \Gamma(X(J^{**}),\underline{\theta}_\Lambda)$ because R is a Krull domain.
Therefore, $I(\Lambda) \not\subset p$ for any $p \in X^{(1)}(R)$, finishing the proof. ∎

It follows from the theorem above that in case R is a Dedekind domain, any maximal $R[t]$-order in $M_n(K(t))$ is conjugated to some $\Gamma[t]$ where Γ is a maximal R-order in $M_n(K)$. Therefore, we recover the classical Bass-Serre theorem cfr. e.g. [3] which states that every f.g. projective $R[t]$-module is extended from R in case R is a Dedekind domain.

Theorem 2 does not hold for maximal orders in arbitrary central simple algebras, even if we restrict attention to Azumaya maximal orders.

Let k be any field and Δ a finite dimensional skewfield over it, then $\Delta[t]$ is a maximal $k[t]$-order in $\Sigma = \Delta(t)$ such that every left ideal of $\Delta[t]$ is free entailing that $t_{k[t]}(\Sigma) = 1$. However, by a result of Ojanguren and Sridharan, cfr. e.g. [3], there exists a f.g. projective left ideal of $\Delta[t,s]$ which is not free, say L. It is clear that $\Lambda = \mathcal{O}_r(L)$ is a maximal $k[t,s]$-order (even Azumaya) which is not conjugated to an extended one, and therefore $I(\Lambda)$ is a proper ideal of the Dedekind domain $k[t]$.

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