# FAITHFULLY FLAT DESCENT AND THE BASS-QUILLEN CONJECTURE.

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## 0. Introduction

Bass [3] and Quillen [10] asked whether any f.g. projective $R[X_1, ..., X_n]$-module is extended from $R$ if $R$ is a regular domain of finite Krull dimension.Using Quillen induction, this question is readily checked to be equivalent with the following :

*(Bass-Quillen conjecture) If $R$ is a regular local domain of finite Krull dimension , then every f.g. projective $R[t]$-module is free.*

Lindel And Lütkebohmert [5] and Mohar-Kumar [7] proved this conjecture in case $R \simeq k[[X_1, ..., X_l]]$. By the structure theorem of Cohen this settles the problem for complete regular local rings of equi-characteristic (i.e. char$(R)$ = char$(R/m)$ if $m$ is the unique maximal ideal of $R$). To deal with the conjecture in the general (equicharacteristic) case, a natural idea would be to go to the completion ,invoke the result over the completion, and then somehow try to make a descent to the original regular local ring ([4,p.162]).

With this note we aim to present such a faithfully flat descent argument for the Bass-Quillen conjecture.

## 1. The tools :

### A. Homogenization :

Let us recall some definitions from [9]. If $R$ is any ring and $R[t]$ is graded in the usual way , i.e. $deg(t) = 1$ , then one can build out of any graded $R[t]$-module $M$ the graded $R[t,s]$-module ($deg(s) = 1$) $M[s]$ in the obvious way. If we decompose $m \in M$ into homogeneous elements , say $m = m_{-k} + ... + m_0 + ... + m_l$, $deg(m_i) = i$ then we can associate to it an homogeneous element $m^* \in M[s]$ given by : $m^* = m_{-k}s^{k+l} + ... + m_0 s^l + ... + m_l$. We say that $m^*$ is the homogenized element of $m$. Conversely, if $u$ is an homogeneous element of $M[s]$ , say $u = u_{-k}s^{k+l+p} + ... + u_0 s^{l+p} + ... + u_l s^p$ with $deg(u_i) = i$ , then $u_* = u_{-k} + ... + u_0 + ... + u_l$ is said to be the dehomogenized element of $u$.

Let $N$ be a (not necessarely graded) $R[t]$-submodule of $M$ then by $N^*$ we mean the $R[t,s]$-submodule of $M[s]$ generated by the elements $n^*, n \in N$. $N^*$ , which is clearly a graded $R[t,s]$- submodule of $M[s]$ , is called the homogenized module of $N$. Any element $n \in N^*$ is of the form $s^r.n_1^*, n_1 \in N$ and $r \geq 0$.

Conversely, to a graded $R[t,s]$-submodule $L$ of $M[s]$ one can associate $L_* = \{u_*; u \in h(L)\}$ where $h(L)$ denotes the set of homogeneous elements of $L$.

From [9] we retain that one can define an exact functor :

$$E : R[t,s] - gr \to R[t] - mod; E(M) = R[t,s]/(s-1) \otimes M$$

such that $E(L) = L_*$ for every graded $R[t,s]$-submodule of some $M[s]$. For more details the reader is referred to [9].

Let us consider the following situation : if $K \subset L$ is a commutative fieldextension and if $\alpha \in GL_n(L[t])$ then one can put a gradation on $M_n(L[t])$ by defining the set of homogeneous elements of degree $m$ to be the set $\alpha^{-1}.t^m.M_n(L).\alpha$. Clearly, this graded structure extends the usual gradation of $K[t]$ and $L[t]$. Now, let $R$ be a Krull domain with field of fractions $K$ and let $\Lambda$ be an Azumaya algebra over $R[t]$ which is contained in $M_n(K[t])$ , then one can homogenize $\Lambda$ as an $R[t]$-submodule of the graded $R[t]$-module $M_n(L[t])$ (with the gradation defined by $\alpha$).

**Proposition 1** : With notations as above we have :

(1) : $\Lambda^*_s$, the localization of $\Lambda^*$ at the homogeneous element $s$ is an Azumaya algebra over $R[t, s, s^{-1}]$.

(2) : $\Lambda^*$ is a maximal order over $R[t, s]$.

**Proof** :

(1) : Because $\Lambda$ is a subring of the graded ring $M_n(L[t])$, it follows from [9,p.79] that $\Lambda^*$ is a graded subring of $M_n(L[t, s])$. Because $s$ is a unit of degree one in the graded ring $\Lambda^*_s$ it is clear that $\Lambda^*_s = (\Lambda^*_s)_0[s, s^{-1}]$ , where $(-)_0$ denotes the part of degree zero. Now, $(\Lambda^*_s)_0 = \sum s^{-k}.(\Lambda^*)_k$ and further $(\Lambda^*)_k = \{s^l.\lambda^*; \lambda \in \Lambda, l + deg(\lambda^*) = k\}$ , and therefore any element $x \in (\Lambda^*_s)_0$ is of the form $x = \lambda_{n_1}.s^{-n_1}+...+\lambda_{n_k}.s^{-n_k}$ where $deg(\lambda_{n_i}) = n_i$ and $\lambda_{n_1}+...+\lambda_{n_k} \in \Lambda$. Define a map $\psi : \Lambda \to (\Lambda^*_s)_0$ by sending $\lambda = \lambda_{n_1} + ... + \lambda_{n_k}$ to $\psi(\lambda) = \lambda_{n_1}.s^{-n_1} + ... + \lambda_{n_k}.s^{-n_k}$. It is rather trivial to verify that $\psi$ is a ringisomorphism and therefore $(\Lambda^*_s)_0$ is an Azumaya algebra over its center $\psi(Z(\Lambda)) = \psi(R[t]) = R[t.s^{-1}]$. This finally entails that $\Lambda^*_s = (\Lambda^*_s)_0[s, s^{-1}]$ is an Azumaya algebra over $R[t.s^{-1}][s, s^{-1}] = R[t, s, s^{-1}]$.

(2) : From part (1) we retain that $\Lambda^*$ is a prime p.i.-ring. Since $\Lambda^*$ is $\mathbb{Z}$-graded, its center is also graded. Clearly, $Z(\Lambda^*)_* \subset Z(\Lambda) = R[t]$ whence : $R[t, s] \subset Z(\Lambda^*) \subset (Z(\Lambda^*)_*)^* \subset R[t, s]$ and therefore $Z(\Lambda^*) = R[t, s]$. Let $\Gamma$ be an $R[t, s]$-order in $\Sigma = \Lambda^*.K(t, s)$ such that $\Lambda^* \subset \Gamma$ and $r.\Gamma \subset \Lambda^*$ for some $r \in R[t, s]$. If we denote with $Q^g(\Lambda^*)$ the localization of $\Lambda^*$ at the multiplicative set of all nonzero homogeneous elements of $R[t, s]$ , then it follows from part (1) that $Q^g(\Lambda^*)$ is an Azumaya algebra over the Krull domain $Q^g(R[t, s])$, so $Q^g(\Lambda^*)$ is a maximal order. Because $Q^g(\Lambda^*) \subset Q^g(\Lambda^*).\Gamma$ and $r.Q^g(\Lambda^*).\Gamma \subset Q^g(\Lambda^*)$ , it follows that $\Gamma \subset Q^g(\Lambda^*)$ so we can define $\Gamma^{\}}$, the internal homogenization of $\Gamma$ , [9,p.36] , i.e. the $R[t, s]$-submodule of $Q^g(\Lambda^*)$ generated by the highest degree terms of elements of $\Gamma$ , them $\Gamma^{\}}$ is clearly an order in $\Sigma$ such that $\Lambda^* \subset \Gamma^{\}}$ and $r_m.\Gamma^{\}} \subset \Lambda^*$ where $r_m$ is the highest degree term of $r$. In order to obtain from this that $\Gamma^{\}} = \Lambda^*$ we have to check that for every graded ideal $I$ of $\Lambda^*$ and every homogeneous element $q \in Q^g(\Lambda^*)$ such that $I.q \subset I$ we have that $q \in \Lambda^*$ Now, $I.M_n(L[t]).q \subset I.M_n(L[t])$ whence we may assume that $q \in M_n(L[t])$. Hence, $I_*.q_* \subset I_*$ and because $I_*$ is an ideal of the maximal order $\Lambda$ it follows that $q_* \in \Lambda$ and therefore $q = s^k.(q_*)^* \in \Lambda^*$ for some $k \geq 0$. Thus, $\Gamma^{\}} \subset \Lambda^*$ which entails that $\Gamma \subset \Lambda^*$, finishing the proof.

B : Graded Brauer groups :

Graded Azumaya algebras and graded Brauer groups were introduced by F. Van Oystaeyen in [11]. Let us briefly recall the definitions. If $R$ is a $\mathbb{Z}$-graded ring then a graded Azumaya algebra $\Lambda$ is simply an Azumaya algebra over $R$ with a $\mathbb{Z}$-graded structure extending the gradation of $R$. Two graded Azumaya algebras $\Lambda$ and $\Gamma$ are said to be graded equivalent if there exist graded f.g. projective $R$-modules $P, Q$ such that there is a degree preserving algebra isomorphism $\Lambda \otimes END_R(P) \simeq \Gamma \otimes END_R(Q)$ , where the endomorphism rings of $P$ and $Q$ are graded in the natural way , cfr. e.g. [9,p.6]. The graded Brauer group of $R$ , $Br_g(R)$ , is defined to be the graded equivalence classes of graded Azumaya algebras.

Clearly, the functor which forgets the gradation defines a morphism $Br_g(R) \to Br(R)$ . Injectivity of this map for Gr-Dedekind domains was proved by F. Van Oystaeyen [11,2.11] and recently for graded Krull domains by M. Vanden Bergh (unpublished). Injectivity for arbitrary $\mathbb{Z}$-graded rings was announced by S. Caenepeel. Because for graded regular domains the proof is rather trivial , we include :

**Proposition 2** : If $R$ is a graded regular Krull domain , then the natural morphism $Br_g(R) \to Br(R)$ is injective.

**Proof** : Suppose that $\Lambda$ is a graded Azumaya algebra which is of the form $\Lambda = End_R(P)$ for some f.g. projective $R$-module $P$. With $K^g$ we will denote the localization of $R$ at the multiplicative set of all nonzero homogeneous elements of $R$ , then $K^g$ is a graded field (i.e. all homogeneous elements invertible). Because $\Lambda \otimes K^g$ is trivial in $Br(K^g)$ and $Br_g(K^g) \to Br(K^g)$ is injective , [11] we obtain :

$$\Lambda \to K^g \otimes \Lambda \simeq END_{K^g}(V)$$

for some f.g. graded projective $K^g$-module $V$ . Let $\{v_i; 0 \leq i \leq n\}$ be a basis of $V$ over $K^g$ consisting of homogeneous elements and denote $F = \sum R.v_i$. Identifying $\Lambda$ with its image in $END_{K^g}(V)$ , $E = \Lambda.F$ is a f.g. graded $R$-module containing a $K^g$-basis of $V$. Then, there are natural inclusions :

$$\Lambda \subset END_R(E) \subset END_{K^g}(V)$$

and because $\Lambda$ is a maximal R-order , $\Lambda = END_R(E)$. It follows from [2,Prop.4.1] that $END_R(E) \simeq END_R(E^b)$ (where $E^b$ denotes the bidual of $E$). Because $\Lambda$ is a f.g. projective $R$-module and $R$ is regular , $E^b$ is f.g. projective [8,Th.11.5] and graded, finishing the proof.

## 2 : The proof

For any commutative domain $R$ one denotes with $PGL_n$ the automorphism scheme of the n-dimensional projective space over $R$, i.e. $PGL_n$ is the sheafification of the presheaf which assigns $PGL_n(\Gamma(U, \mathcal{O}_R))$ to an open set $U$ of $X = Spec(R)$, cfr. [6,p.134]. As usual , $\mathcal{O}_R$ is the structure sheaf of $R$.

**Proposition 3** : If $R$ is a locally factorial Krull domain , then $H^1_{Za}(X, PGL_n)$ is the set of $R$-algebra isomorphism classes of endomorphism rings of projective rank n R-modules.

**Proof** : It follows from a sheaf version of the Skolem-Noether theorem ,cfr. e.g. [6,IV.2.3,2.4] that the sequence :

$$1 \to G_m \to GL_n \to PGL_n \to 1$$

is exact as a sequence of sheaves of groups both in the Zariski (Za) and étale (et) topology. Therefore, one obtains the exact diagram :

$$H^1_{Za}(X, \mathcal{O}^*_R) \to H^1_{Za}(X, GL_n) \to H^1_{Za}(X, PGL_n) \to H^2_{Za}(X, \mathcal{O}^*_R)$$

$$H^1_{et}(X, G_m) \to H^1_{et}(X, GL_n) \to H^1_{et}(X, PGL_n) \to H^2_{et}(X, G_m)$$

where the two first isomorphism come from [6,p.134] . $R$ being locally factorial Weil divisors coincide with Cartier divisors yielding that the sequence below is exact in the Zariski topology :

$$1 \to \mathcal{O}^*_R \to K^* \to \mathcal{D}_R \to 1$$

where $\mathcal{D}_R$ is the sheaf of Weil divisors and $K^*$ the constant sheaf with sections $K^*$ , the nonzero elements of the field of fractions $K$ of $R$. Using flabbiness of $\mathcal{D}_R$ this entails that $H^2_{Za}(X, \mathcal{O}^*_R) = 1$.So, we obtain the exact sequence :

$$1 \to H^1_{Za}(X, PGL_n) \to H^1_{et}(X, PGL_n) \to Br(R)$$

where the factorization of $H^1_{et}(X, PGL_n) \to H^1_{et}(X, G_m)$ through $Br(R)$ comes from the proof of [6,IV.2.5]. Finally, $H^1_{et}(X, PGL_n)$ is the set of $R$-algebra isomorphism classes of rank n Azumaya algebras , hence $H^1_{Za}(X, PGL_n)$ are the isomorphism classes of trivial rank n Azumaya algebras , finishing the proof.

**Corollary 4** : If $R$ is a locally factorial Krull domain such that $Pic(R) = 1$ then $H^1_{Za}(X, PGL_n) = 1$ iff all f.g. projective rank n $R$-modules are free.

**Proof** : Because $H^1_{Za}(X, \mathcal{O}^*_R) = Pic(R) = 1$ we obtain as in the foregoing proof that $H^1_{Za}(X, GL_n) \simeq H^1_{Za}(X, PGL_n)$. Finally, $H^1_{Za}(X, GL_n)$ is the set of isomorphism classes of projective rank n $R$-modules , [6,p.134] , finishing the proof.

**Theorem 5** : (faithfully flat descent for the Bass-Quillen conjecture)

If $R \subset S$ is a faithfully flat extension of regular local domains and if every f.g. projective $S[t]$-module is free, then every f.g. projective $R[t]$-module is free.

**Proof** :

In view of the Auslander-Buchsbaum theorem [1] and the foregoing results, we have to check that $End_{R[t]}(P) \simeq M_n(R[T])$ as $R[t]$-algebras for every projective rank n $R[t]$-module P. If $K$ (resp. $L$) denotes the field of fractions of $R$ (resp. of $S$) then we can replace $End_{R[t]}(P)$ by an isomorphic $R[t]$-algebra $\Lambda$ such that $\Lambda \subset M_n(K[t]) \subset M_n(L[t])$. For, $End_{R[t]}(P) \otimes K[t] \simeq End_{K[t]}(P \otimes K[t]) \simeq M_n(K[t])$ because all projective modules over $K[t]$ are free.

Because $S[t]$ is flat over $R[t]$ , $\Lambda \otimes S[t] \simeq \Lambda.S[t] \subset M_n(L[t])$ , so , $\Lambda.S[t]$ is a trivial $S[t]$ Azumaya algebra and therefore $\Lambda.S[t] \simeq M_n(S[t])$ by the assumptions on $S$ and Coroll.4 .By the Skolem-Noether theorem , cfr. e.g. [6,IV.1.4] this isomorphism comes from an inner automorphism of $M_n(L[t])$ , hence we can find an element $\alpha \in GL_n(L[t])$ such that the diagram below is commutative, all inclusions being canonical :

$$M_n(K[t]) \subset M_n(L[t])$$

$$\Lambda \subset \alpha^{-1}.M_n(S[t]).\alpha$$

On $M_n(L[t])$ we define the gradation determined by $\alpha$ as in section 1. Since $\alpha^{-1}.M_n(S[t]).\alpha$ is clearly a graded subring of $M_n(L[t])$ , homogenization yields the following commutative diagram :

$$M_n(K[t])^* \subset M_n(L[t,s])$$

$$\Lambda^* \subset \alpha^{-1}.M_n(S[t,s]),\alpha$$

It follows from Prop.1.(1) that $\Lambda^*.K(t,s) \simeq End_{K(t,s)}(V)$ for some finite dimensional $K(t,s)$-vectorspace $V$. Because $\Lambda^*$ is a maximal order in $End_{K(t,s)}(V)$ (Prop.1.(2)) , there exists by [2,Prop.4.2] a f.g. reflexive $R[t,s]$-submodule $E$ of $V$ such that $\Lambda^* = End_{R[t,s]}(E)$.

Because $End_{R[t,s]}(E) \otimes S[t,s] \simeq \Lambda^*.S[t,s] \subset \alpha^{-1}.M_n(S[t]).\alpha$ and $(End_{R[t,s]}(E) \otimes S[t,s])^b \simeq End_{S[t,s]}((E \otimes S[t,s])^b)$, cfr. [2,Prop.4.1] and [4,Prop.2.13] , is a maximal $S[t,s]$-order , it follows that :

$$End_{S[t,s]}(E \otimes S[t,s]) \simeq End_{S[t,s]}((E \otimes S[t,s])^b) \simeq \alpha^{-1}.M_n(S[t,s]).\alpha$$

Hence it is a projective $S[t,s]$-module. Because $S[t,s]$ is a regular domain this implies that $E \otimes S[t,s]$ is a f.g. projective $S[t,s]$-module , cfr. e.g. [8,Th.11.5] . $S[t,s]$ being a faithfully flat extension of $R[t,s]$ entails that $E$ is a f.g. projective $R[t,s]$-module.

This entails that $\Lambda^*$ is a graded Azumaya algebra which represents the trivial class in $Br(R[t,s])$ . It follows from Prop.2 that $\Lambda^* \simeq END_{R[t,s]}(P)$ for some f.g. graded projective $R[t,s]$-module $P$ . Now, $P \simeq P_0 \otimes R[t,s]$ by [4,Th.4.6] yielding that $\Lambda^* \simeq End_R(P_0)[t,s] \simeq M_n(R[t,s])$ because $R$ is a local ring. Applying the exact functor E(-) on both sides yields : $\Lambda \simeq M_n(R[t])$, finishing the proof.

The same argument remains valid in case $R$ and $S$ are local factorial domains such that $Br(R) \to Br(K)$ is monomorphic.

**Corollary 6** : (Bass-Quillen conjecture : equicharacteristic case)
If $R$ is an equicharacteristic regular local domain then every f.g. projective $R[t]$-module is free.

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