Splitting by Galois Objects.

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0. Introduction.

Results due to S.U. Chase - A. Rosenberg [1] and O.E. Villamayor - D. Zelinski [12], a.o., may justify the assertion that Amitsur cohomology is the right one as far as splitting phenomena are concerned. However this cohomology lacks one pleasant property : a manageable description of the split Azumaya - algebras determined by 2 - cocycles.

In the fields case, such a description exists for Galois field extensions using group cohomology. This is the well known (classical) crossed product theorem (see e.g. [4]. For a relatively big class of field extensions (among which are the finite normal and modular extensions) one can extend this result using a cohomology theory, due to M.E. Sweedler [8], for algebras which are modules over a given Hopf algebra. The conditions we must impose on the field extension N/k are precisely the ones needed for N to be a Galois object over k, in the sense of S. Chase - M. Sweedler [2].

In the ring case, the De Meyer - Ingraham sequence [3] for Galois extensions of rings offers a satisfactory generalization of the classical crossed product theorem. Again, these Galois extensions are nothing but a special case of the more general notion of Galois object.

Hence, it became plausible that one could find a seven term exact sequence in about the same manner Sweedlers result generalizes the classical crossed product theorem. The purpose of this note is to prove this assumption. Using this sequence we will explicitly describe a set of Azumaya algebras which generate the p-component of the Brauer group of a commutative ring of characteristic p.

The main application (and indeed the main motivation for writing out this note) will be the study of splitting by modular ringextensions, in the sense of F. Van Oystaeyen - A. Holleman [5], [11]. This will draw some new light on results of F. Van Oystaeyen on the p-component of the Brauer groups of a field of characteristic p [10], and make it possible to extend some results to the ring case. In this note we contend ourselves to hint at the connection, and hope to return to this topic in a subsequent paper.

1. Galois Objects.

In [2], Chase and Sweedler introduced the notion of Galois object. Their main motivation was to extend the fundamental theorem of Galois theory for commutative rings, replacing the finite group of automorphisms in the classical theory by a Hopf algebra.

In this section we define Galois objects (our definition differs sightly from the one given in [2], in order to avoid technical difficulties in the sequel), recollect some properties as proved in [2] and give some examples. Necessary Hopf-theoretic notions can be found in [2] and [9].

Let R be a commutative ring with unit. A Hopf algebra H over R (with multiplication m, unit μ, comultiplication Δ, counit ε and antipode S) will be called finite if H is a finitely generated projective R-module. For such an Hopf algebra H we can form its dual Hopf algebra $H^{\star} = Hom_R(H,R)$ with multiplication $\Delta^{\star}$, unit $\varepsilon^{\star}$, comultiplication $m^{\star}$, counit $\mu^{\star}$ and antipode $S^{\star}$. Throughout this section H will always be a finite Hopf algebra over R.

<u>Definition 1.1.</u> : An H-object is a pair (S,α) where S is a commutative R-algebra and $\alpha : S \to S \otimes H^{\star}$ is an R-algebra morphism satisfying :

$$(\alpha \otimes 1) \circ \alpha = (1 \otimes m^{\star}) \circ \alpha : S \to S \otimes H^{\star} \otimes H^{\star}$$

$$(1 \otimes \mu^{\star}) \circ \alpha = 1_S \qquad : S \to S \otimes R \simeq R$$

If C is an R-coalgebra ; A, B are R-algebras and $\psi : C \otimes A \to B$ is an R-module morphism.

We say that C measures A to B if for all $c \in C$ and for all $a, a' \in A$:

$$\begin{cases} \psi(c \otimes aa') = \sum_{(c)} \psi(c_{(1)} \otimes a)\, \psi(c_{(2)} \otimes a') \\ \psi(c \otimes 1) = \varepsilon_c(c)\, 1_B \end{cases}$$

where we used the Sweedler - Heyneman notation : $\Delta_c(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$.

We will ow associate with every H-object (S,α) a measuring from S to S. Let us denote à la Sweedler -Heynemann $\alpha(s) = \sum_{(s)} s_{(1)} \otimes s_{(2)}$ with $s_{(1)} \in S$ and $s_{(2)} \in H^{\star}$. Now, define $\psi : H \otimes S \to S$ by $\psi(h \otimes s) =$ $= \sum_{(s)} s_{(1)} \langle s_{(2)}, h\rangle$. It is easily checked that ψ is a measuring. From now on $\psi(h \otimes s)$ wil; be abbreviated to h;s.

Definition 1.2. : Let (S,α) be an H-object. We define the R-algebra homomorphism $\gamma_S : S \otimes S \to S \otimes H^{\star}$ by the formula :

$$\gamma_S(x \otimes y) = (x \otimes 1)\, \alpha(y) = \sum_{(y)} x\, y_{(1)} \otimes y_{(2)}$$

(S,α) will be called a Galois H-object if the following conditions hold :

$$\begin{cases} S \text{ is a faithfully flat R-module} \\ \gamma_S : S \otimes S \to S \otimes H^{\star} \text{ is an isomorphism} \end{cases}$$

If S is an H-object, we define an R-algebra S # H as follows. As an R-module S # H = $S \otimes H$ except that we write s # h rather than $s \otimes h$. Multiplication in S # H is defined by the formula :

$$(s \# h)(t \# g) = \sum_{(h)} s(h_{(1)}.t) \# h_{(2)}\, g$$

S # H is an algebra, called the smash product of S and H. We define a left S # H - module structure on S by the formula :

$$(s \# h)(t) = s(h.t) \quad (s,t \in S;\ h \in H)$$

From [2] we recollect the following :

Proposition 1.3 :

The following are equivalent for any H-object S :

(1) S is a Galois H-object

(2) S is a f.g. faithful projective R-module and the mapping $\varphi : S \# H \to End_R(S)$, arising from the left $S \# H$-module structure on S is an isomorphism of algebras.

Proposition 1.4 :

Let S be a Galois H-object, $D = S \# H$ and

$J = \{h \in H \ \ \forall h' \in H : h'h = \varepsilon(h')h\}$ a right ideal of H.

Then the functors

$D\text{-mod} \to R\text{-mod} \qquad M \to JM$

$R\text{-mod} \to D\text{-mod} \qquad N \to S \otimes N$

are isomorphisms of categories. In particular, $M \cong S \otimes JM$ as left D-modules for any M in D-mod.

Proposition 1.5. :

Let S be a Galois H-object, then

$S^H = \{s \in S \mid \forall h \in H : h.s = \varepsilon(h)s\} = R$

Example 1 (cfr. [7])

Let G be a finite group and let R [G] denote the group algebra of G over R, then R [G] is a finite R-Hopf algebra. An R-algebra S is Galois R [G] - object if and only if the following conditions are satisfied :

(1) There is a representation of G as a group of automorphisms of the R-algebra S.

(2) S is a faithful R-algebra and R is the subring of G-invariant elements of S

(3) There exist elements $x_1,\dots,x_n$; $y_1,\dots,y_n$ of S such that $\sum_{i=1}^{n}\sigma(x_i)y_i = \delta_{\sigma,1}$ for $\sigma \in G$.

Thus, Galois R [G]-objects are precisely the Galois extensions in the sense of De Meyer-Ingraham [3] with Galois group G.

Example 2.

Let us recall that an R-algebra S is said to be modular over R in the sense of F. Van Oystaeyen [10], [11] and A. Holleman [5] if there is a finite abelian group G and set mappings :

$$\begin{cases} u : G \to S \quad \sigma \mapsto u_\sigma \\ f : G \times G \to U(R) \end{cases}$$

(with U(R) the multiplicative group of invertible elements) with the the following properties :

(a) the induced R-linear mapping $\tilde{u} : R[G] \to S$ is an isomorphism.

(b) $\forall \sigma, \tau \in G : u_\sigma u_\tau = f(\sigma,\tau)\, u_{\sigma\tau}$

The group G is called the basic group of S. It is uniquely determined by the algebra structure of S, [11]. It is clear that S is a free R-module with basis $\{u_\sigma, \sigma \in G\}$.

Let $R[G]^\star$ be the dual Hopf algebra of the Hopf algebra R [G]. Letting $\{v_\sigma, \sigma \in G\}$ be the dual basis for $R[G]^\star$, $\{v_\sigma, \sigma \in G\}$ is a set of pairwise orthogonal idempotent elements whose seem is the identity element of $R[G]^\star$ Counit and comultiplication are given by :

$$\Delta v_\sigma = \sum_{\tau \in G} v_\tau \otimes v_{\sigma\tau^{-1}} \; ; \; \varepsilon(v_\sigma) = \delta_{1,\sigma}$$

$$R[G]^{\star\star} = R[G] \text{ has basis } \{w_\sigma, \sigma \in G\}$$

Proposition 1.6. : If S is a modular ringextension of R, then S is a Galois $R[G]^\star$-object if G is the basic group of S

Proof.

Let us define : $\alpha : S \to S \otimes R[G]$

$$u_\sigma \mapsto u_\sigma \otimes w_\sigma$$

$(\alpha \otimes 1) \circ \alpha(u_\sigma) = u_\sigma \otimes w_\sigma \otimes w_\sigma = (1 \otimes m^\star) \circ \alpha(u_\sigma)$

$(1 \otimes \mu^\star) \circ \alpha = u_\sigma \otimes 1 = 1_S(u_\sigma)$

thus (S,α) is an $R[G]^\star$-object. Further, S is clearly faithfully flat, hence it will be sufficient to prove that the map

$\gamma : S \otimes S \to S \otimes R[G] \quad U_\sigma \otimes u_\tau \vdash u_\sigma u_\tau \otimes w_\tau$

is an isomorphism. This follows from the fact that all $f(\sigma,\tau)$ are invertible elements of R. □

Example 3

Let R be a commutative ring of characteristic p and let S be a purely inseparable extension of the form $R[x] / (x^{p^e} - a)$; $a \in R$.

H will be the cocommutative finite Hopf-algebrz with basis $\{h_i ; i = o,\dots,p^e-1\}$ and with :

$$h_i h_j = \binom{i+j}{i} h_{i+j} \; ; \; \Delta(hr) = \sum_{i=o}^{r} h_i \otimes h_{r-i}$$

$$h_o = 1 \; ; \; \varepsilon(_i) = \delta_{o,i}$$

$H^\star$ will be the Hopf-algebra with basis $\{g_i ; i=o,\dots,p^e-1\}$ and

$$\Delta^\star(g_i \otimes g_j) = g_{i+j} ; \; m^\star(g_i) = \sum_{j \le i} \binom{i}{j} v_j \otimes g_{i-j}$$

$$\varepsilon^\star(1) = g_o ; \; \mu^\star(g_j) = \delta_{o,j}$$

It is easy to check that S is a Galois H-object with :

$$\alpha : S \to H^\star \quad \alpha(x^j) = x^j \otimes g_o + \sum_{1 \le i \le j} \binom{j}{i} x^{j-i} \otimes g_i$$

2. Hopf Cohomology.

In this section we will extend the work, as originated in [8], to the ring case and to arbitrary contravariant functors from the category of commutative R-algebras to the category of Abelian groups.

Recall that, if C is an R-coalgebra and A an R-algebra, then $Hom_R(C,A)$ can be given an R-algebra structure. For $f,g \in Hom_R(C,A)$ the product

$f \star g$ is $m_A \circ (f \otimes g) \circ \Delta_C$. Thus for $c \in C$, $f \star g(c) = \sum_{(c)} f(c_{(1)}) g(c_{(2)})$. The unit element of $Hom_R(C,A)$ is $\mu_A \circ \varepsilon_C$. This product of functions is alled convolution. If C is a cocommutative R-coalgebra and A is a commutative R-algebra, then it is clear that $Hom_R(C,A)$ is a commutative R-algebra. In the sequel, H will be a cocommutative Hopf algebra over R (i.e. where the underlying coalgebra is cocommutative). A will be a commutative R-algebra and $\psi : H \otimes A \to A$ measures A to A. For $o < q$ let $\otimes^q H$ denote $H \otimes \ldots \otimes H$ q-times. $\otimes^q H$ has the coalgebra structure on the tensor product of coalgebras (cfr. e.g. [2], [9]); $\otimes^o H = R$.

The Hopf cohomology arises from the cosimplicial algebra :

$S(H,A)$: $Hom_R(\otimes^o H,A) \rightarrow Hom_R (\otimes^1 H,A) \rightrightarrows (\otimes^2 H,A) \rightrightarrows \ldots$

where S is used en hommage de M.E. Sweedler, with the commutative R-algebras. $Hom_R (\otimes^q H,A)$ as defined above and with face operators :

$\delta_i : Hom_R (\otimes^q H,A) \to Hom_R (\otimes^{q+1} H,A)$

specified by :

$\delta_o (f) = \psi(I \otimes f)$

$\delta_i (f) = f(I \otimes \ldots \otimes I \otimes m \otimes I \otimes \ldots \otimes I)$ with m at place i, for $i = 1,\ldots,q$

$\delta_{q+1} (f) = m_A(f \otimes \varepsilon)$

A contravariant functor F from the category of commutative R-algebras to the category of Abelian groups will carry the cosimplical algebra S(H,A) to a cosimplical Abelian group which becomes a cochain complex S(H,A,F) with alternating sums of face operators as boundary homomorphisms. For $n \geq o$, to the Hopf cohomology group $H^n(H,A,F)$ are the cohomology groups of the complex S(H,A,F).

<u>Some examples.</u>

The functors U and Pic from commutative R-algebras to Abelian groups are the usual ones : U(T) is the group of invertible elements of T, and Pic T is the group of isomorphism classes of invertible (projective rank one) T-modules.

Let us briefly look at $H^i(H,A,u)$ for $i = 0,1$ and $H^\circ(H,A,Pic)$.

$U(Hom_R(\otimes^\circ H,A)) = U(A)$ and if $a \in Z^\circ(H,A,u))$ then for all $h \in H$: $(h.a) =$ $= \varepsilon(h)a$ and hence $a \in A^H \cap U(A)$. Thus $H^\circ(H,A,U) = U(A^H)$. If S is a Galois h-object, then by Prop. 1.5. $H^\circ(H,S,U) = U(R)$.

If $f \mapsto S \in Z^1(H,A,U)$ then $\mu(\varepsilon \otimes \varepsilon) = \psi(I \otimes f) \star f^{-1}\, m \star f \otimes \varepsilon$ or

$fm = \psi(I \otimes f) \star (f \otimes \varepsilon)$. Thus for all $h,h' \in H$:

$f(hh') = \sum_{(h)(h')} (h_{(1)}.f(h'_{(1)}))\, f(h_{(r)})\varepsilon(h'_{(2)}) = \sum_{(h)} (h_{(1)}.f(h'))f(h_{(2)})$

hence, f can be viewed as a "crossed" homomorphism and $H^1(H,A,U)$ is the group of crossed homomorphisms modulo the subgroup of inner crossed homomorphisms, i.e. of the form $D^1(a)$ for $a \in U(A)$. For $h \in H$, $D^1(a)(h) =$ $= (h.a)^{-1}a$.

At this point it is worth mentioning that one can prove in exactly the same way as in [8] :

$H^q(R[G],A,U) \cong H^q(G,U(A))$ for all $q \geq 0$.

3. <u>Comparison with Amitsur cohomology</u>.

Let S be a commutative R-algebra. The Amitsur cohomology (cfr. [1]) arises from the cosimplicial algebra.

$C(S/R) : S \rightrightarrows \otimes^2 S \;\substack{\to\\\to\\\to}\; \otimes^3 S \;\substack{\to\\\to\\\to\\\to}\; \ldots.$

where $\otimes^h S$ denotes the k-fold tensor product with itself and the face operators $d_i : \otimes^h S \to \otimes^{h+1} S$ are given by $d_i(x_1 \otimes \ldots \otimes x_k) =$ $= x_1 \otimes \ldots \otimes x_i \otimes 1 \otimes x_{i+1} \otimes \ldots \otimes x_k$, the x_j being arbitrary elements of S and $o \leq i \leq k$.

<u>Theorem 3.1.</u> : If S is a Galois H-object, then C(S/R) and S(H,S) are isomorphic as cosimplical algebras.

Proof.

Let $\gamma : S \otimes S \to S \otimes H^{\star}$ be the algebra isomorphism as defined in 1.2, $\psi : H \otimes S \to S$ the with α corresponding measuring and we will denote $\psi(h \otimes s) = h.s$.

First we will define R-algebra isomorphisms $\pi_q : \otimes^{q+1} S \to \mathrm{Hom}(\otimes^q H, S)$ as follows :

q = 1 : π_1 is the composite $S \otimes S \overset{\gamma}{\to} S \otimes M^{\star} \overset{\theta}{\to} \mathrm{Hom}\,(H,S)$ with $\theta(s \otimes u)(h) =$ $= s <u,h>$. Clearly, π_1 is an R-module isomorphism, so it will be sufficient to check that θ is an algebra map.

Let $\star_o$ be convolution in $H^{\star}$; $\star_i$ convolution in $\mathrm{Hom}\,(\otimes^i H,S)$.

$$
\begin{aligned}
\theta\,[(s \otimes u)(t \otimes v)]\,(h) &= \theta(st \otimes u \star_o v)(h) \\
&= st < u \star_o v, h > \\
&= st \sum_{(h)} u(h_{(1)} v(h_{(z)} \\
&= \sum_{(h)} (su(h_{(1)}))(t\, v(h_{(2)})) \\
&= \sum_{(h)} \theta(s \otimes u)(h_{(1)} \theta(t \otimes v)(h_{(2)} \\
&= [\theta(s \star u) \star_1 \theta(t \otimes v)]\,(h)
\end{aligned}
$$

Thus π_1 is an algebra isomorphism.

q > 1 : Let π_q be the composite :

$$
\begin{array}{ccccc}
\otimes^{q+1} S & \xrightarrow{I^{q-1} \otimes \gamma} & (\otimes^q S) \otimes H^{\star} & \xrightarrow{\pi_{q-1} \otimes I} & \mathrm{Hom}(\otimes^{q-1} H, S) \otimes H^{\star} \\
 & \searrow & & & \downarrow \theta_q \\
 & & & & \mathrm{Hom}(\otimes^q H, S)
\end{array}
$$

$\theta_q(f \otimes g)(h \otimes h') = f(h) <g, h'>$

Again, π_q is clearly an R-module isomorphism and by induction hypothesis we may assume that π_{q-1} is an algebra map, so it is sufficient to proof that θ_q is an algebra map:

$\theta_q[(f\otimes g)(h\otimes i)](h\otimes h') =$

$\theta_q(f \star_{q-1} h \otimes g \star_o i)(h\otimes h') =$

$(f \star_{q-1} h)(h)(g \star_o i)(h') =$

$\sum_{(h)} f(h_{(1)})\ h(h_{(2)}) \sum_{(h')} g(h'_{(1)}) i(h'_{(2)}) =$

$\sum_{(h)} \sum_{(h')} f(h_{(1)}) g(h'_{(1)}) h(h_{(2')}) i(h'_{(2)}) =$

$\sum_{(h)} \sum_{(h')} \theta_q(f\otimes g)(h_{(1)}\otimes h'_{(1)})\ \theta_q(h\otimes i)(h_{(2)}\otimes h'_{(2)}) =$

$[\theta_q(f\otimes g) \star_q \theta_q\ (h\otimes i)]\ (h\otimes h')$. Done.

Next we prove :

$\pi_q(s_o\otimes \dots \otimes sq)(h_1\otimes \dots \otimes h_q) = s_oh_1.(s_1h_2.(\dots h_{q-1}(h_q.s_q)\dots)$

<u>q = 1</u> : $\pi_1(x\otimes y)(h) = \theta(\sum_{(y)} xy_{(1)}\otimes y_{(2)})(h)$

$= \sum_{(y)} x\ y_{(1)} <y_{(2)},h> = x(h.y)$

<u>q > 1</u> : $\pi_q(s_o\otimes \dots \otimes s_a)\ (h_1 \otimes \dots \otimes h_a) =$

$\theta_q \circ (\pi_{q-1}\otimes I)[\sum_{(s_q)} s_o\otimes \dots \otimes s_{q-2}\otimes s_{q-1}(s_q)_{(1)}\otimes (s_q)_{(2)}]\ (h_1\otimes \dots \otimes h_q) =$

$\sum_{(s_q)} s_oh_1.(s_1h_2.(\dots (s_{q-2}(h_{q-1}.s_{q-1}(s_q)_{(1)})\dots)<(s_q)_{(2)},hq> =$

$s_oh_1.(\dots h_{q-1}.(s_{q-1} \sum_{(s_q)} (s_q)_{(1)} <(s_q)_{(2)},h_q>)\dots) =$

$s_oh_1.(s_1h_2.(\dots h_{q-1}(s_{q-1}(h_q.s_q)\dots)$

Finally we have to check that the following diagrams commute :

$$\begin{array}{ccc} \otimes^{q+1}S & \xrightarrow{\pi_q} & Hom_R(\otimes^q H,S) \\ \downarrow d_i & & \downarrow \delta_i \\ \otimes^{q+2}S & \xrightarrow{\pi_{q+1}} & Hom_R(\otimes^{q+1} H,S) \end{array}$$

$i = 0$: $[\delta_o \circ \pi_q\ (s_o\otimes \dots \otimes s_a)]\ (h_o \otimes \dots \otimes h_q) =$

$\psi(h_o\otimes s_oh_1.(s_1h_2.(\dots (h_q.s_q)\dots) =$

$h_o.(s_oh_1.(s_1h_2.(\dots (h_q.s_q)\dots) =$

$$\pi_{q+1}(1 \otimes 1_o \otimes\ldots\otimes s_a)(h_o \otimes \ldots \otimes h_a) =$$

$$[\pi_{q+1} \circ d_o(s_o \otimes \ldots \otimes s_a)](h_o \otimes \ldots \otimes h_a) =$$

$1 \le i \le q$: $[\delta_i \circ \pi_q(s_o \otimes\ldots\otimes s_a)]\ (h_o \otimes\ldots\otimes h_a) =$

$$= \pi_q(s_o \otimes\ldots\otimes s_a)(h_o \otimes\ldots\otimes h_{i-2} \otimes h_{i-1}h_i \otimes \ldots \otimes h_a)$$

$$= s_o h_o.(s_1 h_1.(\ldots s_{i-2}h_{i-2}.(s_{i-1}h_i.(s_i h_{i+1}.(\ldots(h_q.s_a)\ldots)$$

$$= \pi_{q+1}(s_o \otimes\ldots\otimes s_{i-1} \otimes 1\ \ s_i \otimes\ldots\otimes s_q)(h_o \otimes\ldots\otimes h_q)$$

$$= [\pi_{q+1} \circ d_i(s_o \otimes\ldots\otimes s_q)]\ \ (h_o \otimes\ldots\otimes h_q)$$

$i = q+1$: $[\delta_{q+1} \circ \pi_q(s_o \otimes\ldots\otimes s_q)]\ \ (h_o \otimes\ldots\otimes h_a) =$

$$\pi_q(s_o \otimes\ldots\otimes s_q)(h_o \otimes\ldots\otimes h_{a-1})\ \varepsilon_H(h_q) =$$

$$s_o h_o.(s_1 h_1.(\ldots(h_{q-1}.s_q)\ldots)\varepsilon_H(h_a) =$$

$$s_o h_o.(s_1 h_1.(\ldots h_{q-1}.(s_q(h_{q.1}))\ldots) =$$

$$\pi_{q+1}(s_o \otimes\ldots\otimes s_q \otimes 1)(h_o \otimes\ldots\otimes h_q) =$$

$$[\pi_{q+1} \circ d_{q+1}(s_o \otimes\ldots\otimes s_q)]\ (h_o \otimes\ldots\otimes h_q)$$

This finishes the proof. □

Corollary 3.2. Let F be any contravariant functor from the category of commutative R-algebras to the category of Abelian groups and let S be a Galois H-object, then for all $q \geq o$:

$$H^q(S/R,F) \cong H^q(H,S,F).$$

4. The Main theorem.

Now we are able to prove the main theorem of this note :

Theorem 4.1. : Let S be a Galois H-object, then there is an exact sequence :

$$1 \to H^1(H,S,U) \to \mathrm{Pic}\ R \to \mathrm{Pic}\ S^H \to H^2(H,S,U) \to \mathrm{Br}(S/R) \to H(H,S,\mathrm{Pic}) \to H^3(N,S,U)$$

Proof.

If S is a Galois H-object, then S is a f.g. faithfully flat projective R-module, hence "isotrivial" (cfr. [12]). Using the foregoing this theorem is nothing but a reformulation of the fameous Chase Rosenberg sequence. □

Corollary 4.2. : Hilbert's theorem 90

If $Pic(R) = 1$, then $H^1(H,S,U) = 1$

Corollary 4.3 : Crossed Product Theorem.

If $Pic(S) = 1$, then $Br(S/R) \cong H^2(H,S,U)$

$Pic(S \otimes S) = 1$

Translating the Chase-Rosenberg morphisms via the isomorphisms of Coroll. 3.2 we get group morphisms which are similar to the ones defined in the De Meyer-Ingraham sequence [3]. We illustrate this with two examples.

$\alpha : H^1(H,S,U) \to Pic(R)$

Let $f \in Z^1(H,S,U)$, we can define a morphism :

$\theta_f : S \# M \quad S \# H$ via $\theta_f(s \# h) = \sum_{(h)} sf(h_{(1)}) \# h_{(r)}$

We can form a left $S \# H$-module S_f, which is isomorphic to S as an R-module S_f, with action : $(s \# h).s' = \theta_f(s \# h)s'$; $s,s' \in S$, $h \in H$.

By Proposition 1.4., we have a left $S \# H$-module isomorphism : $S_f \cong S \otimes JS_f$.

Now, $rank_R S = rank_R S_f$ hence $JS_f \in Pic(R)$.

Define $\alpha' : Z^1(H.S.U) \to Pic(R)$ by $\alpha'(f) = [JS_f]$.

If $f \in B'(H,S,U)$, then there exists a unit a in S with $\forall h \in H : f(h) =$ $= (h.a)a^{-1}$, hence :

$\theta_f(s \# h) = \sum_{(h)} s(h_{(1)}.a)a^{-1} \# h_{(2)} = a^{-1} (\Sigma s(h_{(1)}.a) \# h_{(2)})$

$= a^{-1} (s \# h)(a \# 1) = a^{-1}(s \# h)a.$

So, we have an R-module isomorphism $\psi : R \to JS_f$ by $\psi(r) = a^{-1}r$, therefore $\alpha'(f) = 1$.

Therefore it makes sense to define :

$\alpha : H'(H,S,U) \to PicR$ via $\alpha([f]) = [JS_f]$.

Verification of the fact that this is indeed the right morphism is left to the reader.

$\beta : H^2(H,S,U) \to Br(S/R)$

We will introduce a generalization of the classical crossed product construction to arbitrary Galois obejects. Recall in S # H multiplication is given by :

$$(s \# h)(s' \# h') = \sum_{(h)} s(h_{(1)}.s') \# h_{(2)}h'.$$

We alter this multiplication by $\sigma : H \otimes H \twoheadrightarrow S \in Z^2(H,S,U)$ as follows :

$S \#_\sigma H$ is the R-module $S \otimes H$ with multiplication :

$$(s \#_\sigma h)(s' \#_\sigma h') = \sum_{(h),(h')} s(h_{(1)}.s')\,\sigma(h_{(2)} \otimes h'_{(1)}) \#_\sigma h_{(3)}h'_{(2)}$$

using the convention : $(\Delta \otimes I) \circ \Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$

A boring but important calculation shows that $S \#_\sigma H$ has an associative multiplication if $\sigma \in Z^2(H,S,U)$.

5. The p-component of the Brauer group

In this section, R will be a commutative ring of characteristic p. We will explicitely describe a set of Azumaya algebras which generate the p-component of the Brauer group : $Br(R)_p$, thus characterizing results of Saltman [6] .

Let S be the purely inseparable extension of the form $R[x]/(X^{p^e}-a)$; $e \in \mathbb{N}$; $a \in R$ and H the corresponding Hopf-algebra, as in example 3 of section 1. For every $\sigma \in Z^2(H,S,U)$ we will denote :

$$R(a,e,\sigma) = S \#_\sigma H.$$

Theorem 5.1. : $Br(R)_p$ is generated by the $R(a,e,\sigma)$.

Proof.

Let A be an Azumaya-algebra over R of exponent p^e, then by [6, The.6.2] there are $a_1,\dots,a_m \in R$ such that A is split by $R[x_1^{p^e}-a_1,\dots,x_m^{p^e}\dots a_m) = R[a_1^{1/p^e},\dots,a_m^{1/p^e}]$.

Using a result of [1] , A is similar to an A' with A' containing

$R[a_1^{1/p^e},\dots,a_m^{1/p^e}]$ as a maximal commutative subalgebra of A' and A' is left projective over $R[a_1^{1/p^e},\dots,a_m^{1/p^e}]$. By a theorem of Yuan [13], $A' \cong A_1 \otimes \dots \otimes A_m$ where A_i contains $R[a_i^{1/p^e}]$ as a maximal commutative subalgebra and A_i is left projective over $R[a_i^{1/p^e}]$. Using the fact that the kernel of the multiplication map $R[a_i^{1/p^e}] \otimes R[a_i^{1/p^e}] \to R[a_i^{1/p^e}]$ is nilpotent, one can show that the Chase-Rosenberg morphism $\beta' : H^2(S_i/R,U) \to Br(S_i/R)$ is surjective, hence $\beta : H^2(H,S_i,U) \to Br(S_i/R)$ is also $R[x]/(x^{p^e} - a_i)$. Therefore $A_i \sim R(a_i,e,\sigma_i)$ for some $\sigma_i \in Z^2(H,S_i;U)$ and $A \sim R(a_1,e,\sigma_1) \otimes \dots \otimes R(a_m,e,\sigma_m)$, finishing the proof. □

Furthermore , as the reader may easily verify :

$R(a,e,\sigma) \cong R < x,y > /(X^{p^e}-a,\ Y^{p^e} - b,\ XY-YX+1)$.

6. Splitting by modular ringextensions .

Again, R will be a commutative ring of characteristic p, S a modular extension (see example 2 of section 1) with basic group G, which is a p-group. Since the kernel of the multiplication map $S \otimes S \to S$ is nilpotent, the map $\beta : H^2(R[G]^\star,S,U) \to Br(S/R)$ is surjective, hence every Azumaya algebra split by S is equivalent to $S \#_\sigma R[G]^\star$ for some $\sigma \in H^2(R[G]^\star,S,U)$. Thus, $S \#_\sigma R[G]^\star$ is the tensor product of a Galois algebra in the sense of Hoeschmann, $R[G]^\star$, and a group algebra $R[G]^\star$ with a skew multiplication on it, depending on the 2-cocycle $f(\alpha,\beta)$ in $H^2(G,U(R))$ describing the algebra structure of S and of the 2 cocycle σ in $H^2(R[G]^\star,S,U)$. It is possible to choose a basis of $S \#_\sigma R[G]^\star$, $\{x_\sigma,\ y_\sigma;\ \sigma \in G\}$ such that $\oplus\ R x_\sigma \cong R[G]^\star$ and :

$S \#_\sigma R[G]^\star = \oplus R x_\sigma [y_\sigma;\ y_\sigma y_\tau = g(\sigma,\tau) y_{\sigma\tau}\ \ g \in H^2_{sym}(G,U(R))]$

ow let us look at the p-component of the Brauer group of a field of characteristic p, k. If A is a central simple k-algebra in $Br(k)_p$, then one can prove as in the previous section that A is split by a purely

inseparable field extension of the form $k[x_1,\dots,x_n]/(x^{p^e}-q_1,\dots,x^{p^{e_n}}-a_n]$
In [11] F. Van Oystaeyen proved that a field extension is purely inseparable of this form if and only if it is a modular field extension with basic p-group G.

Thus $A \sim L \#_\sigma R[G]^\star \cong N[y_{\bar{\tau}} ; y_\sigma y_\tau = g(\sigma,\tau) y_{\sigma\tau} \; g \in H^2_{sym}]$ with N a Galois algebra and g a symetric 2-cocycle, consistent with [11] .

As promised in the introduction we hope to come back to this topic in a subsequent note .

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