Class Groups of Maximal Orders over Krull Domains.

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1. Introduction.

Noncommutative Krull rings have been studied in some recent papers, cfr. e.g. [3,12,19]. The first examples of these rings were already studied in 1968 by R. Fossum [8], namely maximal orders over Krull domains. Our main motivation for studying this class of rings comes from the next result by M. Chamarie [3].

Theorem 1.1. : (M. Chamarie) if Γ is a Noetherian (or affine) prime p.i. ring with center R and classical ring of quotients Σ, then there exists an intermediate ring $\Gamma \subset \Lambda \subset \Sigma$ which is a maximal order over the complete integral closure of R which is a Krull domain.

Besides their ringtheoretical importance, the spectra of these maximal orders seem to be the most natural noncommutative generalization of affine normal varieties, [10], so a closer investigation of them might shed some new light on "noncommutative algebraic geometry". cfr. e.g. [1,36]. One of the main problems in the theory is to find a suitable generalization of unique factorization domains and, related to this question to find a proper definition of the class group. Several possible definitions were suggested, e.g. a rather obscure K-theory class group, $W(\Lambda)$, by R. Fossum [8], the normalizing classgroup, $Cl(\Lambda)$, by M. Chamarie [3] and the central classgroup, $Cl^c(\Lambda)$, which has been studied extensively by E. Jespers and P. Wauters, cf. e.g. [11,37].

In this note we aim to survey some of our results on the normalizing and central classgroup, cfr. [15,16,17].

The second section, on the normalizing or Chamarie classgroup, is of

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a more geometrical nature. In the study of commutative Krull domains there are some important questions on class groups for which purely ring theoretical methods seem to be insufficient. To solve these one has to use some geometrical machinery, cfr. e.g. [7]. An example of such a situation is presented by some results of V.I. Danilov on the relation between Cl(R) and Cl R [[t]] for a Noetherian integrally closed domain R. First, the classgroup is expressed in terms of Picard groups of certain open subvarieties of Spec(R), then he uses the good functorial and cohomological properties of the Picard group to prove his theorems on open subvarieties and finally he pulls the obtained information back to the classgroup.

We try to generalize some of these results to maximal orders over Krull domains. To this end we introduce Weil and Cartier divisors and the corresponding class groups. Since the proofs of the classical theorems on the relation between these invariants do not generalize to the p.i.-case we had to come up with a new approach. These new proofs have the extra advantage of presenting ring theoretical interpretations (such as the type number and the genera of a maximal order)) for certain cohomology pointed sets.

Several applications are included, e.g. a characterization of those locally factorial Krull domains R for which all maximal orders in $M_n(K)$ are conjugated, a result stating that every torsion element of the Picard group of R is killed in the class group of a certain matrixing $M_n(R)$ and a proof that $Cl(\Lambda)$ is a direct summand of $Cl(\Lambda[[t]])$.

In the third section we treat the central class group. This time our approach is more ringtheoretical. We study to what extend $Cl^C(\Lambda) \cong Cl(R)$ implies that Λ is a (reflexive) Azumaya algebra over R. For tamifiable

maximal orders this result is proved, counterexamples to the general case are included and the relation between this obstruction and the universal measuring bialgebras associated with the orders is being hinted at. As an example we show that certain divisorially graded rings [18] over tamifiable maximal orders are (reflive) Azumaya algebras,providing a new approach to this class of rings.

2. Divisors and the Chamarie class group.

Throughout, we will consider the following situation. R is a Krull domain with field of fractions K and Λ is a maximal R-order in some central simple algebra Σ over K.

With $\underline{\theta}_R$ (resp. $\underline{\theta}_\Lambda$) we will denote the stucture sheaf of R (resp. Λ) over X = Spec(R). Although the Weil and Cartier divisors are defined on the noncommutative affine scheme associated with Λ over Y = Spec(Λ), $\underline{\theta}_Y^{nc}$ [36] , there is no real need to dwell upon these technicalities here, for :

Lemma 2.1. : [17] If i = Y → X is the usual (continuous) morphism, then : $i^\star(\underline{\theta}_\Lambda) \cong \underline{\theta}_\Lambda^{nc}$ and $i_\star(\underline{\theta}_\Lambda^{nc}) \cong \underline{\theta}_\Lambda$.

Proof : It follows from [3] that $\theta_{\Lambda\text{-}P}(\Lambda) = Q^{bi}_{\Lambda\text{-}P}(\Lambda) = \Lambda R_p$ where p = P ∩ R. Therefore, $i^\star(\underline{\theta}_\Lambda)$ is a subsheaf of $\underline{\theta}_\Lambda^{nc}$ and all their stalks are isomorphic, whence $i^\star(\underline{\theta}_\Lambda) \cong \underline{\theta}_\Lambda$. Similarly, $\underline{\theta}_\Lambda$ is a subsheaf of $i_\star(\underline{\theta}_\Lambda^{nc})$ and all their stalks are isomorphic, finishing the proof.

So, for p.i. maximal orders we can restrict attention to central schemes. A more general approach can be found in [17]. Let us consider the following two sheaves of not necessarily Abelian groups on X = Spec(R) :

The sheaf of units, $\underline{\theta}_\Lambda^\star$, is defined in the obvious way, i.e. $\Gamma(U,\theta_\Lambda^\star) = U(\Gamma(U,\underline{\theta}_\Lambda))$ and restriction morphisms are inclusions. It is straightforward to check $\underline{\theta}_\Lambda^\star$ is a sheaf.

The sheaf of normalizing elements, $\underline{N}_\Lambda$, which is defined by : $\Gamma(U,\underline{N}_\Lambda) = N(\Gamma(U,\underline{\theta}_\Lambda)) = \{x \in \Sigma^\star : \Gamma(U,\underline{\theta}_\Lambda).x = x.\Gamma(U,\underline{\theta}_\Lambda)\}$ and restriction morphisms are inclusions. In [15] it is shown that $\underline{N}_\Lambda$ is a sheaf of groups and the stalk of $N(\Lambda_p)$.

$\underline{N}_\Lambda$ need not be a constant sheaf. For, let $\Lambda = \mathfrak{C}[X,-]$ where - denotes the complex conjugation. Then Λ is a p.i. Dedekind ring with center $\mathbb{R}[X^2]$. In [31] it is proved that $\{X^2 + c;\ c > o\}$ is precisely the set of those prime ideals of $\mathbb{R}[X^2]$ whose valuation extends to a valuation in $\mathfrak{C}(X,-)$. this implies that for those prime ideals p, $(\underline{N}_\Lambda)_p = \mathfrak{C}(X,-)$. Now, suppose $\underline{N}_\Lambda$ were constant then $N(\Lambda) = \mathfrak{C}(X,-)$ yielding that every localization at a prime ideal is a valuation ring, a contradiction. Later on we will see that $\underline{N}_\Lambda$ is not constant if not all maximal orders over R in Σ are conjugated.

Clearly, $\underline{\theta}^\star_\Lambda$ is a normal subsheaf of $\underline{N}_\Lambda$ so we can form its quotient sheaf $\underline{b}_\Lambda = \underline{N}_\Lambda/\underline{\theta}^\star_\Lambda$ which is a sheaf of Abelian groups because $\mathcal{D}(\Lambda)$, the group of divisorial Λ-ideals [8] is commutative. In analogy with the commutative case we define :

definition 2.2. : A Cartier divisor on X is a global section of the sheaf $\underline{b}_\Lambda$. Thinking of the properties of quotient sheaves one sees that a Cartier divisor on X may be defined by giving an open covering $\{U_i; i \in I\}$ for X and for every $i \in I$ an element $n_i \in \Gamma(U_i,\underline{N}_\Lambda)$ such that for all $i,j \in I$: $n_i.n_j^{-1} \in \Gamma(U_i \cap U_j;\ \underline{\theta}^\star_\Lambda)$.

A Cartier divisor is said to be principal if it is in the image of the natural map $\Gamma(X,\underline{N}_\Lambda) \to \Gamma(X,\underline{b}_\Lambda)$. Two Cartier divisors are linearly equivalent if their quotient (which is defined locally) is principal. The Abelian group of Cartier divisor classes on X will be denoted by CaCl(X), the Cartier class group of X.

Similarly, one can define the Cartier class group of an open subvariety U of X, CaCl(U) by the exact sequence :

$$\Gamma(U,\underline{N}_\Lambda|U) \to \Gamma(U,\underline{b}_\Lambda|U) \to CaCl(U) \to 1$$

In case $\Lambda = R$, CaCl(U) is nothing but the Picard group of the open subvariety U, cfr. e.g. [10].

For the definition of non-Abelian cohomology pointed sets the reader is referred to [9] or [22].

Proposition 2.3. : If Λ is a maximal order over the Krull domain R, then :

(a) : $1 \to CaCl(X) \to H^1_{Zar}(X,\underline{\theta}^\star_\Lambda) \to H^1_{Zar}(X,\underline{N}_\Lambda)$

(b) : $1 \to CaCl(U) \to H^1_{Zar}(U,\underline{\theta}^\star_\Lambda|U) \to H^1_{Zar}(U,\underline{N}_\Lambda|U)$

Later on, we will give a ring theoretical interpretation of these cohomology pointed sets. Having defined what Cartier divisors are. Let us now look at Weil divisors : if U is an open set of X = Spec(R), then we denote with $X^{(1)}(U)$ the set $X^{(1)}(R) \cap U$, i.e. the height one prime ideals of R lying in U. With Div(U) we denote the free Abelian group generated by the height one prime ideals of Λ corresponding to $X^{(1)}(U)$ (recall that there is a one-to-one correspondence between $X^{(1)}(R)$ and $X^{(1)}(\Lambda)$, [3]). In case U = X, Div(U) is nothing but the divisor group $\mathcal{D}(\Lambda)$. The assignment U -- Div(U) defines a flabby sheaf on X which we denote by $\underline{\mathcal{D}}_\Lambda$. From [17] we retain that the following sequence of sheaves is exact :

$$1 \to \underline{\theta}^\star_\Lambda \to \underline{N}_\Lambda \to \underline{\mathcal{D}}_\Lambda$$

where the morphism $\underline{N}_\Lambda \to \underline{\mathcal{D}}_\Lambda$ is the natural one, i.e. sending a normalizing element to the divisorial ideal it generates.

definition 2.4. : A Weil divisor on U is an element of Div(U) and Cl(U) the class group of the open subvariety U, is defined by the sequence :

$$1 \to \Gamma(U,\underline{\theta}^\star_\Lambda|U) \to \Gamma(U,\underline{N}_\Lambda) \to Div(U) \to Cl(U) \to 1$$

Of course, if U = X, then Cl(U) = Cl(Λ), the Chamarie class group i.e. the quotient group of $\mathcal{D}(\Lambda)$ by $\mathcal{P}(\Lambda)$ the subgroup of those divisorial Λ-ideals which are generated by one element (which is then clearly a normalizing element!) Remark also the slight difference between our notation of similarity of Weil divisors and that of [32].

In my talk during the Noether days I tried to mimic the commutative proofs in order to ralate the Cartier class group to the Picard group Pic(Λ) = I(Λ)/$\mathcal{P}(\Lambda)$ (where I(Λ) is the group of invertible Λ-ideals) and to give a noncommutative generalization of Danilov's main tool. This approach forced me to impose a technical condition on the maximal order Λ, namely that Pic(Λ_p) = 1 for all p $\in$ X = Spec(R). The general validity of this condition depends on the following two questions on prime ideals of maximal orders which, although plausible, have not yet been proved.

(Q1) : If Λ is a maximal order over a Krull domain R and p $\in$ X = Spec(R), are there only a finite number of prime ideals of Λ lying over p? (if R is Noetherian, the answer is : yes).

(Q2) : If Λ is a maximal order over a Krull domain R and p $\in$ X = Spec(R). If P,P' $\in$ Spec(Λ) such that P $\cap$ R = P' $\cap$ R = p, is pid(Λ/P) = = pid(Λ/P') or even Λ/P $\cong$ Λ/P' ?

Since then I was able to give a more ringtheoretical proof of the result which bypassed these problems and had the extra advantage of presenting ringthoretical interpretations of the cohomology pointed sets used before.

Theorem 2.5. : [17] If Λ is a maximal order over a Krull domain R then Cl(Λ) = $\varinjlim$ CaCl(U) where the direct limit is taken over all open sets U of X containing $X^{(1)}$(R). Moreover, the following sequence of pointed sets is exact :

$$1 \to Cl(\Lambda) \to \varinjlim H^1_{Zar}(U, \underline{\theta}^{\star}_{\Lambda}|U) \to \varinjlim H^1_{Zar}(U, \underline{N}_{\Lambda}|U) \to 1$$

Let us first recall some definitions.

The genera of Λ, $G(\Lambda)$, is defined to be the set of isomorphism classes of left divisorial Λ-ideals (c-ideals in the terminology of [21]. The type number of R in Σ, $T_R(\Sigma)$ is the set of non-conjugate classes of maximal orders over R in Σ.

The determination of the type number is in particulat interesting in order to construct all maximal orders over R in Σ out of a given one, cfr. [15]

In view of Prop. 2.3., Theorem 2.5. follows from the following :

Theorem 2.6. : [17] If Λ is a maximal order over the Krull domain R, then

(a) : $1 \to Cl\Lambda \to G(\Lambda) \to T_R(\Sigma) \to 1$ is exact

(b) : $G(\Lambda) \cong \varinjlim H^1_{Zar}(U, \underline{\theta}^{\star}_{\Lambda}|U)$ where the direct limit is taken over all open subvarieties U of X containing $X^{(1)}(R)$.

(c) : $T_R(\Sigma) \cong \varinjlim H^1_{Zar}(U, \underline{N}_{\Lambda}|U)$ where the direct limit is taken over all open subvarieties U of X containing $X^{(1)}(R)$.

Proof.

(a) : The map $\varphi : Cl(\Lambda) \to G(\Lambda)$ is of course given by sending the class $[I]$ of a divisorial ideal I to the isomorphism class $< I >$ of I in $G(\Lambda)$. This map is a monomorphism of pointed sets, for if $< I > = < \Lambda >$ then one can extend the left Λ-module isomorphism $I \to \Lambda$ to an Σ-linear isomorphism : $\Sigma \to \Sigma$ showing that $I = \Lambda.n$. Because Λ is a maximal order this entails that n is a normalizing element, so $[I] = 1$. Further, the map $\psi : G(\Lambda) \to T_R(\Sigma)$ is given by sending an isomorphisms class $< A >$ of a left divisorial Λ-module A to the class of $\theta_x(A) = \{x \in \Sigma : A\,x \subset A\}$ in $T_R(\Sigma)$. Let us first check that this map is well defined. If $< A > = < B >$ then by an argument as before, $A = B\,x$ for some $x \in \Sigma^{\star}$. This

entails that $X^{-1}\theta_r(B)x \subset \theta_2(A)$. Because $\theta_r(A)$ and $\theta_r(B)$ are both maximal orders this entails that they are conjugated.

The sequence is exact in $G(\Lambda)$. For, let $\theta_r(L) = X^{-1}\Lambda x$ for some $X \in \Sigma^\star$ then, because $\Lambda = x\theta_r(L)x^{-1} \subset \theta_r(Lx^{-1})$ and Λ is a maximal order, $\theta_r(Lx^{-1}) = \Lambda$ showing that $L x^{-1}$ is a two sided divisorial Λ-ideal . So, $< L > = < Lx^{-1} >$ and $\mathrm{Ker}\ \psi \subset \mathrm{Im}\ \varphi$ and the inverse implication is trivial.

Finally we have to check that ψ is epimorphic. So, let Γ be a representant of a class in $T_R(\Sigma)$. Then $(\Lambda :_r \Gamma)$ is a divisorial R-lattice which is a left Λ-ideal and a right Γ-ideal showing that $\theta_r((\Lambda :_r \Gamma)) = \Gamma$, finishing the proof.

(b) : Let L be a left Λ-ideal which is a reflexive R-lattice, then $L^{-1} = (L :_r \Lambda)$ is a right Λ-ideal which is a reflexive R-lattice. With $\underline{\theta}_L$ (resp $\underline{\theta}_{L^{-1}}$) we will denote the structure sheaf of L (resp. L^{-1}) over Spec(R). Let $p \in X^{(1)}(R)$, then $(\underline{\theta}_L)_p = L_p = \Lambda_p.a_p$ because Λ_p is both a left and right principal ideal ring. Similarly, $(\underline{\theta}_{L^{-1}})_p = L_p^{-1} = a_p^{-1}.\Lambda_p$. Take a neighbourhood V_p of p such that $a_p \in \Gamma(V_p, \underline{\theta}_{L^{-1}})$ then it is fairly easy to see that :

$$(\underline{\theta}_L)|V_p \cong (\underline{\theta}_\Lambda|V_p).q_p$$

Let $U = u\{V_p;\ p \in X^{(1)}(R)\}$, then $\{(V_p, a_p)\}$ defines an element of $\Gamma(U, \underline{\Sigma}^\star/\underline{\theta}_\Lambda^\star)$, where $\underline{\Sigma}^\star$ denotes the constant sheaf with sections $\Sigma^\star$.

Writing out the long exact cohomology sequence of :

$$1 \to \underline{\theta}_\Lambda^\star \to \underline{\Sigma}^\star \to \underline{\Sigma}^\star/\underline{\theta}_\Lambda^\star \to 1$$

one finds :

$$\Gamma(U, \underline{\Sigma}^\star) \xrightarrow{\gamma} \Gamma(U, \underline{\Sigma}^\star/\underline{\theta}_\Lambda^\star) \to H^1(U, \underline{\theta}_\Lambda^\star) \to 1$$

In this way, one can associate every left Λ-ideal L an element of $\varinjlim H^1(U, \theta_\Lambda^\star)$. It follows from the exact sequence above that the elements associated with L and L' coincide iff $L = L'x$ for some $x \in \Gamma(U, \underline{\Sigma}^\star) = \Sigma^\star$.

Conversely, with every element of $\lim_{\rightarrow} H^1(U,\theta_\Lambda^\star)$, one can associate an isomorphism class of left Λ-ideals by choosing an element in $\Gamma(U,\Sigma^\star/\theta_\Lambda^\star)$ which generates it, say $\{(V_p,a_p)\}$ and then defining the left $\underline{\theta}_\Lambda|U$-ideal θ_L locally by $\underline{\theta}_L|V_p = (\underline{\theta}_\Lambda|V_p).a_p$ and taking its sections $\Gamma(U,\underline{\theta}_L)$.

(c) : Let Γ be any maximal R-order in Σ. With $\underline{\theta}$ (the conductor) we denote the presheaf which assigns to an open set U of Spec(R) the sections : $\Gamma(U,\underline{\theta}) = \{x \in \Sigma : \Gamma(U,\theta_r)x \subset \Gamma(U,\theta_\Lambda)\}$. An easy computation shows that $\underline{\theta}$ is actually a sheaf of left $\underline{\theta}_p$-ideals and right $\underline{\theta}_\Lambda$-ideals. Furthermore, $\underline{\theta}^{-1}$ which is defined by its sections $\Gamma(U,\theta^{-1}) = \Gamma(U,\underline{\theta})^{-1}$ is also a sheaf and a left $\underline{\theta}_\Lambda$-ideal and a right $\underline{\theta}_p$-ideal.

Now, let p be any height one prime ideal of R. Since both Λ_p and Γ_p are maximal orders over the discrete valuation ring R_p, they are conjugated i.e. $s_p^{-1}.\Gamma_p = \Lambda_p$ for some $s_p \in \Sigma^\star$. We claim that there exists a neighbourhood V(p) of p such that : $s_p^{-1}\,(\underline{\theta}_\Gamma|V(p)).s_p = \theta_\Lambda|V(p)$.

Both $\underline{\theta}$ and $\underline{\theta}^{-1}$ are sheaves, so s_p and s_p^{-1} live on a neighbourhood V(p) of p. Therefore, $s_p\,\Gamma(V(p),\underline{\theta}_\Lambda) \subset \Gamma(V(p),\underline{\theta}^{-1})$. Hence, $\Gamma(V(p),\underline{\theta}_\Lambda)s_p^{-1} \subset \Gamma(V(p),\underline{\theta}^{-1}) = \Gamma(V(p),\underline{\theta})^{-1} \subset (s_p\,\Gamma(V(p),\underline{\theta}_\Lambda)) = \Gamma(V(p),\underline{\theta}_\Lambda).s^{-1}$ yielding that $\Gamma(V(p),\underline{\theta}^{-1}) = \Gamma(V(p),\underline{\theta}_\Lambda)s_p^{-1}$ and similarly : $\Gamma(V(p),\underline{\theta}) = s_p.\Gamma(V(p),\underline{\theta}_\Lambda)$.

Thus,

$$s_p^{-1}.(\underline{\theta}_p|V(p)).s_p = \underline{\theta}_\Lambda|V(p)$$

Now, $U = U(V(p)$ is an open set containing $X^{(1)}(R)$ and $\{(V(p),s_p\}$ describes a section in $\Gamma(U,\underline{\Sigma^\star}/\underline{N}_\Lambda)$. Consider the exact sequence of sheaves of pointed sets :

$$1 \to \underline{N}_\Lambda \to \Sigma^\star \to \Sigma^\star/\underline{N}_\Lambda \to 1$$

Taking sections over U yields the exact sequence :

$$1 \to N(\Lambda) \to \Sigma^\star \to \Gamma(U,\underline{\Sigma}^\star/\underline{N}_\Lambda) \to H^1(U,\underline{N}_\Lambda) \to 1$$

Therefore, the section $\{(V(p),{}^{s}p)\}$ determines an element in $H^1(U,\underline{N}_\Lambda)$

(and so in $\lim_{\rightarrow} H^1(U,\underline{N}_\Lambda)$) which is different from the distinguished element in $H^1(U,\underline{N}_\Lambda)$ if and only if Γ is not conjugated to Λ.

Conversely, let $s \in \lim_{\rightarrow} H^1(U,\underline{N}_\Lambda)$ and choose an open set U of Spec(R) containing $X^{(1)}(R)$ and an element $s(U) \in H^1(U,\underline{N}_\Lambda)$ which represents s. Using the above exact sequence, s(U) is determined by some section in $\Gamma(U,\Sigma^*/\underline{N}_\Lambda)$. Such a section is given by a set of couples $\{(U_i,s_i)\}$ where U_i is an open cover of U, $s_i \in \Gamma(U_i,\Sigma^*)$ for every i and $s_i^{-1}.s_j \in \Gamma(U_i \cap U_j, \underline{N}_\Lambda)$ for all i and j. On U we will define the twisted sheaf of maximal orders $\underline{\theta}\,|U$ by putting $\underline{\theta}_\Gamma|U_i = s_i(\underline{\theta}_\Lambda|U_i)s_i^{-1}$. It is not hard to verify that $\Gamma = \Gamma(U,\underline{\theta}_\Gamma)$ is a maximal R-order, and this finishes the proof.

Since both $G(\Lambda)$ (cfr. [8]) and $T_R(\Sigma)$ do not depend upon the chosen maximal order Λ, $Cl(\Lambda)$ is also an invariant.

Let us give some applications :

A : conjugateness of maximal orders in matrix rings

We aim to characterize those locally factorial Krull domains (i.e. R_p is a UFD for every $p \in Spec(R)$) for which all maximal orders in $M_n(K)$ are conjugated. Remark that all regular Krull domains are locally factorial by the Auslander-Buschbaum theorem. Our result provides a large class of counterexamples to question in [21].

With $\underline{PGL}_n$ we will denote $Aut(\mathbb{P}^n_R)$, the automorphism scheme of the n-dimensional projective space over R, i.e. $\underline{PGL}_n$ is the sheafification of the presheaf which assigns $PGL_n(\Gamma(U,\theta_R))$ to an open set of Spec(R), cfr. e.g. [22].

Proposition 2.7. : [15] If R is a locally factorial Krull domain and if $\Lambda = M_n(R)$, then $H^1_{Zar}(U,\underline{N}_\Lambda|U) \cong H^1_{Zar}(U,\underline{PGL}_n|U)$ for every open set U of Spec(R).

Proof.

If we assign to an open set U of Spec(R) the group $GL_n(\Gamma(U,\underline{\theta}_R))$. $K^\star \subset GL_n(K)$, then this defines a presheaf of groups. Its sheafification will be denoted by $\underline{GL}_n.\underline{K}^\star$. This sheaf is clearly a subsheaf of $\underline{N}_\Lambda$. We will show that their stalks are isomorphic. If $p \in \mathrm{Spec}(R)$ and if $x \in N(M_n(R_p))$, then $M_n(R)x = M_n(A)$ for some divisorial R_p-ideal A. Because R_p is a UFD, $A = R_p.k$ for some $k \in K^\star$, yielding that $x \in GL_n(R_p).K^\star$ proving that $\underline{GL}_n.\underline{K}^\star \cong \underline{N}_\Lambda$.

The following sequence of sheaves of groups is exact :

$$1 \to \underline{K}^\star \to \underline{GL}_n.\underline{K}^\star \to \underline{GL}_n/\underline{GL}_n \cap \underline{K}^\star \cong \underline{PGL}_n \to 1$$

Writing out the long exact cohomology sequence yields :

$$1 = H^1_{Zar}(U,\underline{K}^\star) \to H^1_{Zar}(\underline{GL}_n.\underline{K}^\star) \to H^n_{Zar}(U,\underline{PGL}_n) \to H^2_{Zar}(U,\underline{K}^\star) = 1$$

finishing the proof.

With $\mathrm{Ref}_n(R)$ we will denote the set of isomorphism classes of reflexive R-lattices which are free of rank n at every height one prime ideals of R.

Theorem 2.8. : [15] If R is a locally factorial Krull domain, then all maximal orders in $M_n(K)$ are conjugated if and only if the map from Cl(R) to $\mathrm{Ref}_n(R)$ sending [I] to [I ⊕ ... ⊕ I] is surjective.

Proof : Writing out the long exact cohomology sequence (cfr. [9]) of the exact sheafsequence :

$$1 \to \underline{\theta}_R^\star \to \underline{GL}_n \to \underline{PGL}_n \to 1$$

and then taking direct limits over all subvarieties U of Spec(R) containing $X^{(1)}(R)$, yields :

$$(\star)\quad \varinjlim H^2(U,\underline{\theta}_R) \xrightarrow{\alpha} \varinjlim H^1(U,\underline{GL}_n) \to \varinjlim H^1(U,\underline{PGL}_n) \to \varinjlim H^2(U,\underline{\theta}_R^\star)$$

The first term in this sequence equaks Cl(R) by Danilov's result, cfr. e.g. [7, Prof V. 18.9] and the second equals $\mathrm{Ref}_n(R)$, by [22,p.134]. The

The third term equals $T_R(\Sigma)$ by Prop. 2.7. and Th.2.6. c. Let us now look at the last term. Because R is locally factorial, Cartier divisors coincide with Weil divisors, cfr. [10], i.e. the sequence :

$$1 \to \underline{\theta}_R^{\star} \to \underline{K}^{\star} \to \underline{\mathcal{D}}_R \to 1$$

is exact. Writing out the long cohomology sequence and using flabbiness of $\underline{\mathcal{D}}_R$ and the fact that $\underline{K}^{\star}$ is constant yields :

$$1 = H^1_{Zar}(U,\underline{\mathcal{D}}_R) \to H^2_{Zar}(U,\underline{\theta}_R^{\star}) \to H^2_{Zar}(U,\underline{K}^{\star}) = 1$$

Therefore, the last term in (★) vanishes. So, $T_R(M_n(K)) = 1$ if and only if α is epimorphic. Now, α is derived from the sheafmorphism $\underline{\theta}_R^{\star} \to \underline{GL}_n$ which assigns diag(u) to a unit $u \in (\underline{\theta}_R^{\star})_p$, locally. This shows that the map α is given by assigning the isomorphism class of $I \oplus \ldots \oplus$ (n times) to $[I]$.

Using Steinitz' theorem (cfr. e.g. [23]) one immediately deduces :

Corollary 2.9. : [15] If R is a Dedekind domain, then all maximal orders in $M_n(K)$ are conjugated if and only if $(-)^n : Cl(R) \to Cl(R)$ is an epimorphism.

B : Type number and the polynomial extension.

Th. 2.8. shows that the type is not necessarily preserved under taking matrix rings. Even more, the type number is not invariant under polynomial extension. From Th. 1.6. one deduces :

Corollary 2.10 : If Λ is a maximal order over R in Σ then $T_R(\Sigma) \cong T_{R[t]}(\Sigma(t))$ if and only if $G(\Lambda) \cong G(\Lambda[t])$.

Counterexamples are now easily provided. Let Δ be any p.i. skewfield and let $\Lambda = \Delta[t]$, then $G(\Lambda) = 1$ since $\Delta[t]$ is a principal left ideal domain. However, $G(\Lambda[t,s,J]) \neq 1$ since there are projective non-free left ideals in $\Delta[s,t]$, cfr. [25].

Asymptotical and Azumaya class groups

It follows from Th. 2.6. and Th. 2.8. that $\underline{N}_\Lambda$ is not necessarily a constant sheaf, so there may be non-trivial normalizing elements (i.e. $\notin GL_n(R).K^\star$) in $M_n(R)$. The easiest example of such a situation can be found in [24] :

Let $R = Z[\sqrt{-5}]$ and $I = (2, 1+\sqrt{-5})$ a non-principal ideal then $M_2(I)$ is generated by the normalizing element :

$$\begin{pmatrix} 2 & -1+\sqrt{-5} \\ 1+\sqrt{-5} & -2 \end{pmatrix}$$

This example shows that the natural map $Cl\ R \to Cl\ \Lambda$ can have a non-trivial kernel. As was suggested to me by E. Formanek the reduced norm on Λ can be used to show that this kernel consists of n-torsion elements where $n = p.i.d.(\Lambda)$.

In order to describe the non-trivial normalizing elements in matrixrings or Azumaya algebras one has to determine the relation $Cl(R)$ and the next two invariants :

(A) : the asymptotical class group, $Cl_\infty(R) = \varinjlim_{n\in\mathbb{N}} Cl(M_n(R))$

(B) : the Azumaya classgroup, $Cl_{Az}(R) = \varinjlim_{\Lambda\ az} Cl(\Lambda)$

Lemma 2.11: [17] If R is a Dedekind domain, $Cl_\infty(R) = \mathbb{Q} \otimes Cl(R)$.

Proof : First, we will show that the kernel of the natural morphism $\pi : Cl(R) \to Cl\ M_n(R)$ consists of n-torsion elements. For, suppose that $\pi(A) = M_n(A) = M_n(R).n$ for some normalizing element n, then, taking determinants yields : $A^n = R.\det(n)$. Conversely, let us prove that any torsion element is killed in the classgroup of a suitable matrixring. So, let A be a fractional ideal such that $A^n \cong R$, then by Steinitz' theorem, cfr. [23] , there is an isomorphism :

$$\psi : R \oplus \ldots \oplus R \xrightarrow{\cong} A \oplus \ldots \oplus A$$

which is represented by an $n \times n$-matrix $\alpha \in GL_n(K)$. This means that $M_n(R)\alpha = M_n(A)$ showing that $[A] \in \mathrm{Ker}(Cl(R) \to Cl(M_n(R)))$, finishing the proof.

This result will be generalized in a joint paper with M. Vanden Bergh to Krull domains of finite Krull dimension.

Theorem 2.12. : If R is a Krull domain of finite Krull dimension, then $Cl_\infty(R) = Cl_{AZ}(R) = Cl(R)/Tors(Pic(R))$.

D : Formal power series

In [20] it is proved that $\Lambda[[t]]$ is a maximal order over $R[[t]]$ if Λ is a maximal order over R. Most of the technical machinery described above was developed in order to study te relation between $Cl(\Lambda)$ and $Cl(\Lambda[[t]])$. As an easy consequence of Th. 2.5. we obtain the following noncommutative generalization of a result of Danilov, cfr. e.g. [7].

Theorem 2.13. [17] If Λ is a maximal order over a Krull domain R, then $Cl(\Lambda)$ is a direct summand of $Cl(\Lambda[[t]])$.

Proof. (sketch) Let X and Y denote respectively Spec R and Spec R[[t]] then $j : f(t) \mid\!- f(o)$ induces a closed regular immersion $X \to Y$ which identifies X with V(T). Furthermore, if $i : Y \to X$ denotes the natural morphism, then we obtain the following exact diagram :

$$\begin{array}{ccccccccc}
1 & \to & Cl(\Lambda) & \to & \varinjlim H^1(V, \underline{\theta}^\star_\Lambda) & \to & \varinjlim H^1(U, \underline{N}_\Lambda) & \to & 1 \\
 & & & & \downarrow i^\star \uparrow j^\star & & \downarrow i^\star \uparrow j^\star & & \\
1 & \to & Cl\,\Lambda[[t]] & \to & \varinjlim H^1(V, \underline{\theta}^\star_{\Lambda[[t]]}) & \to & \varinjlim H^1(U, \underline{N}_{\Lambda[[t]]}) & \to & 1
\end{array}$$

and a carefull investigation of these maps learns that $j^\star$ induces a natural splitting of the inclusion $Cl(\Lambda) \to Cl\,\Lambda[[t]]$.

3. Graded (reflexive) Azumaya algebras and the central class groups.

In the foregoing section we studied the divisorial ideal structure of a maximal order. A more difficult problem seams to be the determination of the (prime) ideal structure of an arbitrary maximal order, e.g. questions (Q 1) and (Q 2) above.

A standard trick is to study certain nice ringextensions of Λ with an easier ideal structure and pulling back the obtained information to Λ. Developing such a procedure as well as studying its obstruction is the main aim of this section. For a Krull domain R there are at least two classes of maximal orders which are reasonable understood, namely Azumaya algebras [13,6] and reflexive Azumaya algebras [26,38] . Let us briefly recall their definitions:

If Λ is a maximal order over a Krull domain R , consider the natural R-algebra morphism :

$$m : \Lambda^e = \Lambda \otimes \Lambda^{opp} \to End_R(\Lambda)$$

which is determined by $m(\Sigma a_i \otimes b_i)(\lambda) = \Sigma a_i.\lambda.b_i.\Lambda$ being a divisorial R-lattice (i.e. $\Lambda^{**} = \Lambda$), so is $End_R(\Lambda)$, [26].

This entails that m extends to a morphism :

$$m' : (\Lambda^e)^{**} = \bigcap_{p \in X^{(1)}(R)} (\Lambda \otimes \Lambda^{opp})_p \to End_R(\Lambda)$$

Now, Azumaya algebras are orders for which m is an isomorphisms. They are f.g. projective modules over R, their prime ideal structure is homeomorphic with that of R and even its Chamatie class group is suitable for description at least if $Kdim(R) < \infty$, Th. 2.12. The Brauer group Br(R), cfr. e.g. [6,13] describes the Azumaya algebras up to Morita equivalence.

Extending an idea of Yuan [38], M. Orzech defines in [26] a reflexive Azumaya algebra to be a maximal order such that m' defined above is an

isomorphism. Two refelxive Azumaya algebras Λ and Γ are said to be similar if there exist divisorial R-lattices M and N such that :

$$(\Lambda \otimes \mathrm{End}_R(M))^{**} \cong (\Gamma \otimes \mathrm{End}_R(N))^{**}$$

The similarity classes of reflexive Azumaya algebras form a group under multiplication $(- \otimes -)^{**}$, $\beta(R)$, the so called reflexive Brauer group which has been studied by M. Orzech [26]. Let us recall two basic properties :

(β_1) : $\beta(R) = \cap\{Br(R_p);\ p \in X^{(1)}(R)\}$

(β_2) : the following sequence is exact :

$$1 \to \mathrm{Pic}(R) \to Cl(R) \to B\,Cl(R) \to Br(R) \to \beta(R)$$

Here, B Cl(R) is the so called Brauer-class group. It is defined by taking the set of isomorphism classes of reflexive R-lattices M such that $\mathrm{End}_R(M)$ is a f.g. projective R-module and then taking equivalence classes for the relation :

$$M \sim N \text{ iff } (M \otimes P)^{**} \cong (N \otimes Q)^{**}$$

for some f.g. projective R-modules P and Q.

The cokernel of the morphism $Br(R) \to \beta(R)$ is not so easy to determine it may be nontrivial.

If the Krull domain R is G-graded (where G is an arbitrary, but usually ordered, group with neutral element e), then one can define the graded (reflexive) Brauer group of R, $Br^g(R)$ resp. $\beta^g(R)$, cfr. [34,35] in the obviuos way.

That is, a graded (reflexive) Azumaya algebra Λ is a G-graded order with center R such that the natural map :

$$m^{(')} : \Lambda^{e(**)} \cong \Lambda \otimes \Lambda^{opp(**)} \to \mathrm{End}_R(\Lambda)$$

is a degree preserving isomorphism.

Two graded (reflexive) Azumaya algebras Λ and Γ are said to be similar if there exist G-graded f.g. projective R-modules (resp. divisorial

R-lattices such that :

$$\Lambda \otimes \mathrm{End}_R(P)^{(\star\star)} \cong \Gamma \otimes \mathrm{End}_R(P)^{(\star\star)}$$

the isomorphisms being degree preserving. For more details on the graded Brauer group, the reader is referred to [35].

The phylosophy is that, most ungraded results have a graded counterpart if G is an ordered group and graded results sometimes suffice to describe the Brauer groups, cfr. e.g. [34,35].

Let us define the central class group, $Cl^c(\Lambda)$, of Λ. With $P^c(\Lambda)$ we denote the subgroup of $\mathcal{D}(\Lambda)$ consisting of those divisorial ideals which are gerenarted by one central element. $Cl^c(\Lambda)$ is then defined to be the quotient group $\mathcal{D}(\Lambda)/P^c(\Lambda)$.

It is easy to verify that the natural morphism $\mu : Cl(R) \to Cl^c(\Lambda)$ is injective and that $\mathrm{Coker}(\mu)$ is a finite group. For, take any element c in the Formanek center of Λ, then there are only a finite number of prime ideals P of Λ suc that $\Lambda.c \subset P$. Because the localization at the other height one primes are Azumaya algebras, $P = \Lambda.(P \cap R)$ for allmost all $P \in X^{(1)}(\Lambda)$. For tha finitely many exceptions, $P_i = (\Lambda.(P_i \cap R)^{n_i})^{\star\star}$.

So, $\mathrm{Coker}(\mu) = \bigoplus_{i=1}^{n} Z/n_i Z$.

First, we will study a graded ring extension of Λ which kills off $\mathrm{Coker}(\mu)$. So consider the $Z \oplus \ldots \oplus Z$-graded subring $\Lambda(\emptyset)$ of $\Sigma[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ which is defined by :

$$\Lambda(\emptyset)(m_1,\ldots,m_n) = (P_1^{m_1}. \ldots P_n^{m_n})^{\star\star} X_1^{m_1} \ldots X_n^{m_n}$$

where the P_i are the finitely many exceptional many exceptional height one primes. Part (a) of the next proposition was proved in a joint paper with F. Van Oystaeyen in a more general setting [18].

Proposition 3.1. : [16] If Λ is a maximal order over a Krull domain R, then with notations as above :

(a) : $\Lambda(\emptyset)$ is a maximal order over its center $R(\emptyset)$ which is a Krull domain.

(b) : $Cl\ R(\emptyset) \cong Cl\ \Lambda(\emptyset)$

Proof : (b)

By [11,IV,2.2] , the following sequence is exact :

$$1 \to Cl_g^C(\Lambda(\emptyset)) \to Cl^C\Lambda(\emptyset) \to Cl^C(\Sigma[X_1, X_1^{-1},\ldots,X_n,X_n^{-1}]) \to 1$$

where $Cl_g^C(\Lambda(\emptyset))$ is defined to be the quotient group of $\mathcal{D}_g(\Lambda(\emptyset))$, the subgroup of $\mathcal{D}(\Lambda(\emptyset))$ of the $Z \oplus \ldots \oplus$ Z-graded divisorial ideals of $\Lambda(\emptyset)$, by $P_g^C(\Lambda(\emptyset)) = \{\Lambda(\emptyset).c \mid c \in h(k[X_1,X_1^{-1},\ldots,X_n, X_n^{-1}]\}$. Now, $\Sigma[X_1, X_1^{-1},\ldots,X_n, X_n^{-1}]$ being an Azumaya algebra over a factorial domain, $Cl^C(\Sigma[X_1, X_1^{-1},\ldots,X_n,X_n^{-1}]) = 1$, whence $Cl_g^C(\Lambda(\emptyset)) \cong Cl^C(\Lambda(\emptyset))$.

Furthermore, as in [18] it is easy to verify that the sequence below is exact :

$$1 \to <[P_1],\ldots,[P_n]> \to Cl_g^C(\Lambda(\emptyset)) \to 1$$

Similarly, $Cl_g(R(\emptyset)) \cong Cl(R(\emptyset))$ and

$$1 \to <[p_1],\ldots,[p_n]> \to Cl(R) \to Cl_g(R(\emptyset)) \to 1$$

whence one finally obtains the exact diagram :

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & | & & & & \\
1 & \longrightarrow & <[p_i]> & \longrightarrow & Cl(R) & \longrightarrow & Cl(R(\emptyset)) & \longrightarrow & 1 \\
 & & | & & | & & & & \\
1 & \longrightarrow & <[P_i]> & \longrightarrow & Cl^C(\Lambda) & \longrightarrow & Cl^C(\Lambda(\emptyset)) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \bigoplus Z/n_i Z & \cong & \bigoplus Z/n_i Z & & & &
\end{array}$$

finishing the proof.

All rings $R(\emptyset)$ occuring in this way are of the following type :let R be a Krull domain, then for any (finite) set of height one prime ideals $\{p_1,\ldots,p_n\}$ and for any set of natural numbers $\{m_1,\ldots,m_n\}$ one can define the so called lepidopterous Rees ring [33] $R(p_i m_i)$ to be the $Z \oplus \ldots \oplus Z$-graded subring of $K[X_1,X_1^{-1}, \ldots,X_n,X_n^{-1}]$ defined by :

$$R(p_i,m_i)(i_1,\ldots,i_n) = (p_1^{[[\,i_1/m_1\,]]}.\ \ldots\ .\ p_n^{[[\,i_n/m_n\,]]})^{**}X^{i_1}\ldots X_n^{i_n}$$

where $[[\,a:b\,]] = \mathrm{sign}(a/b).\,[\,|a/b\,|]$ ($[\,./.\,]$ denotes the integral part of ./. . These rings are readily seen to be Krull domains. They are ordered in the following way :

$$R(p_i,m_i) \leqslant R(p'_j,\ m'_j) \text{ iff } \{p_i\} \subset \{p'_j\} \text{ and } m_i \mid m'_j$$

for the corresponding values of i and j.

If Λ is an Azumaya algebra (or more generally, a reflexive Azumaya) algebra over R, then $Cl(R) \cong Cl^c(\Lambda)$. F. Van Oystaeyen and later E. Jespers asked whether the inverse implication is also valid. Using the foregoing construction of lepidopterous Rees rings, the next theorem is not hard to prove :

Theorem 3.2. [16]

(1) : If R is a Krull domain with field of fractions K such that the Jespers-Van Oystaeyen holds for all $R(p_i,\ m_i)$; then :

$Br(K) = \varinjlim \beta^g(R(p_i,m_i))$.

(2) : If R is moreover a Dedekind domain, then $Br(K) = \varinjlim Br^g(R(p_i,m_i))$.

The key lemma in our approach to the Jespers-Van Oyesteyen conjecture is [16] :

Lemma 3.3. : If Λ is a maximal order over a discrete valuation ring R with $Cl^c(\Lambda)$, then one of the following situations occur :

(a) : Λ is an Azumaya algebra.

(b) : $Z(\Lambda/\Lambda m)$ is a purely inseparable field extension of R/Rm.

The proof of this lemma comes down to a verification that prime ideals of $\Lambda[t]$ satisfy the unique-lying-over property with respect to $R[t]$. This is perhaps the proper place to present a method for constructing maximal orders with a split-up prime spectrum. The first example

(known to the author) of such a situation was constructed by M. Ramros [27]. He gives a maximal order over a regular local ring of global dimension 2 such that there are exactly two maximal ideals lying over the central radical. The funny (?) thing about our class of examples is that the problem reduces entirely to commutative field-theory. Let Λ be any maximal order over a Krull domain R and suppose that $P \in \mathrm{Spec}(\Lambda)$ lies uniquely over R (it follows from [3] that this property is equivalent with : $b(P)$ satisfies the left and right Öre-conditions). The fiber of the extension $\Lambda \to \Lambda[t]$ in P equals Spec $Q(\Lambda/P)[t]$ whereas the central fiber in $p = P \cap R$ equals Spec $Q(R/p)[t]$. Therefore, the fiber in P does not split up over its center if and only if $Z(Q(\Lambda/P))$ is a purely inseparable fields extension of $Q(C/p)$. Split-examples are now easy to construct :

Take $\Lambda = \mathbb{C}[X,-]$ and $P = (X)$, then $\Lambda/P = \mathbb{C}$ and $R|P = \mathbb{R}$. Let $f(t)$ be any irreducible polynomial over $\mathbb{R}$ which splits over $\mathbb{C}$, e.g. $t^2 + 1 = (t + i)(t - i)$, then $(X, t + i)$ and $(X, t-i)$ are two prime ideals of $\Lambda[t]$ lying over the same central prime ideal (X^2, t^2+1).

From lemma 3.3 one deduces immediately:

Theorem 3.4 : If R is a Krull domain duch that R_p/pRp is a perfect field for every $p \in X^{(1)}(R)$ and if Λ is a maximal order over R, then :

(a) : Λ is a reflexive Azumaya algebra iff $Cl(R) \cong Cl^c(\Lambda)$.

(b) : Λ is an Azumaya algebra iff $Cl(R) \cong Cl^c(\Lambda)$ and if Λ is a flat R-module.

So, in particular, the Jespers-Van Oystaeyen conjecture holds for applications in algebraic number theory and algebraic geometry (over a field of characteristic zero).

Instead of putting restrictions on the Krull domain R, one may prefer to consider a subclass of maximal orders. From [8] we retain that a reflexive

R-order Γ is said to be tame iff Γ_p is an HNP-ring for every $p \in X^{(1)}(R)$. A maximal order Λ over R is said to be tamifiable if $\Lambda \otimes S$ is a tame order over S, where S is the integral closure of R in a separable splitting subfield of Σ.

Theorem 3.5. : [16] If Λ is a maximal order over a Krull domain R, then :

(a) : Λ is a reflexible Azumaya algebra iff $Cl^C(\Lambda) \cong Cl(R)$ and Λ is tamifiable.

(b) : Λ is an Azumaya algebra iff $Cl^C(\Lambda) \cong Cl(R)$, Λ is tamifiable and Λ is a flat R-module.

We need :

Lemma 3.6. : [16] If Λ is a tamifiable maximal order, then so is $\Lambda(\emptyset)$

Therefore, if Λ is a tamifiable maximal order, one can construct an extension $\Lambda \to \Lambda(\emptyset)$ such that $\Lambda(\emptyset)$ is a graded (reflexive) Azumaya algebra over a certain lepidopterous Rees ring $R(\emptyset)$. This approach will, in particular, be interesting if Λ is a flat R-module, for, in this case $\Lambda(\emptyset)$ will turn out to be an Azumaya algebra. Hence, its prime ideal structure is homeomorphic with that of $R(\emptyset)$, which can be expressed in terms of prime ideals of R, and the obtained information can be pulled back to Λ.

A further development of this approach will probably lead to a better understanding of tamifiable maximal orders.

Let us now look at the obstruction against this approach, i.e. do there exist maximal orders which do not satisfy the Jespers-Van Oystaeyen conjecture ? Clearly, this problem has a local nature, i.e. we may restrict attention to maximal orders over a discrete valuation ring R. By R^{sh} we will denote the strict Henselization [22] of R.

Theorem 3.7. : [16]

(a) : If Λ is a maximal order over R, then Λ is an Azumaya algebra iff $Cl^c(\Lambda) \cong Cl(R)$ and R^{sh} splits Σ.

(b) : The Jespers-Van Oystaeyen conjecture holds for maximal orders over R if and only if it holds for maximal orders over R^{sh}.

This theorem reduces the problem to the following one :
Does there exist a discrete valuation ring Λ with central uniformizing parameter m such that its center R is a strict Henselian discrete valuation ring and $\Lambda/\Lambda.m$ is a commutative purely inseparable field extension of R/R.m ?
Commutativety of $\Lambda/\Lambda.m$ follows from the fact that $\Lambda/\Lambda.m$ is a division ring over its center, but $Br(R/R.m) \to Br\ Z(\Lambda/\Lambda m)$ is epimorphic (being a purelely inseparable field extension [13]) and $Br(R/R.m) = 1$ (R being a stric Henselian valuation ring), so $\Lambda/\Lambda.m = Z(\Lambda/\Lambda.m)$.

That such a situation can occur is made clear by the following examples due to D. Saltman [28] :

equicharacteristic case : Let F be a field of characteristic p and $K = F((t))$, the field of Laurent sequences over F equipped with the natural discrete valuation and let R be the associated (complete) valuation ring. Let {a,b} be contained in a p-basis for F and let Δ be the cyclic algebra $[at^{-p}, b]$. Choose $\alpha \in \Delta$ such that $\alpha^p - \alpha = a\, t^{-p}$, then $(\alpha.t)^p - t^{p-1}(\alpha.t) = a$ whence $K(\alpha)/K$ is a field extension such that the corresponding residue fields are $F(q^{1/p})$ and F. Since $b \notin (F(a^{1/p}))^p$, one can verify that b is not a norm of $K(\alpha)/K$ yielding that Δ is a skewfield. Since any valuation on a complete field extends to a finite dimensional skewfield over it, there exists a valuation ring Λ in Δ over R with $Cl^c(\Lambda) = 1$ and one verifies that $\Lambda/\Lambda.t = F(a^{1/p}, b^{1/p})$.

General case : Let K be a field of characteritic zero and residue class field F of characteristic p. Suppose K contains a primitive p^{th} root of unity, say ω. Again, assume that {a,b} is a part of a p-basis for F. Choose preimages a',b' $\in$ K of a and b and let Δ be the cyclic algebra (a',b') defined over K. Again there exists a valuation ring Λ in Δ such that $\Lambda = \Lambda.t = F(a^{1/p},b^{1/p})$ and $Cl^c(\Lambda) = 1$.

In Saltman's approach (only for exponent one and degree p^r-extensions) the inner derivations of Λ determine the structure of the purely inseparable field extension $\Lambda/\Lambda.t \leftarrow R/R.t$. In the general case, the structure will be determined by the universal bialgebra associated with Λ, [14] .

References.

[1] M. Artin, Local structure of maximal orders on surfaces, LNM 917, 146-181, Springer Verlag, Berlin (1982).

[2] H. Bass, Algebraic K-theory, Benjamin, New York, (1968).

[3] M. Chamarie, Anneaux de Krull noncommutatifs, thèse, Université Claude-Bernard, Lyon, (1981).

[4] S. Chase, On the automorphism scheme of a purely inseparable field extension, Proceedings Park City, 75-106, Academic Press, New York, (1972).

[5] S. Chase, M.E. Sweedler, Hopf algebras and Galois theory, LNM 97, Springer Verlag, Berlin, (1969).

[6] F. De Meyer, E. Ingraham, Separable algebras over commutative rings, LNM 181, Springer Verlag, Berlin, (1971).

[7] R. Fossum, The divisor class group of a Krull domain, Ergebn. 74, Springer Verlag, Berlin, (1973).

[8] R. Fossum, Maximal orders over Krull domains, J. Algebra 10, 321-332, (1968).

[9] J. Giraud, Cohomologie non abélienne, Grundl. der Math. Wiss. 179, Springer Verlag, Berlin, (1971).

[10] R. Hartshorne, Algebraic geometry, Springer Verlag, Berlin, (1977).

[11] E. Jespers, Ω-Krull ringen, thesis K.U.L. Louvain, (1982).

[12] E. Jespers, L. Le Bruyn, P. Wauters, A. Characterization of central Ω-Krull rings, J. Algebra, to appear soon

[13] M. Knus, M. Ojanguren, Théorie de descente et algèbres d'Azumaya, LNM 389, Springer Verlag, Berlin, (1974).

[14] L. Le Bruyn, Universal bialgebras associated with orders, Comm. Alg. 10(5), 457-478, (1982).

[15] L. Le Bruyn; A note on maximal orders over Krull domains, UIA - 82.19, submitted for bublication.

[16] L. Le Bruyn, On the Jespers-Van Oystaeyen conjecture, UIA-82.32, submitted for publication.

[17] L. Le Bruyn, The Chamarie class group of a noncommutative Krull ring, in preparation.

[18] L. Le Bruyn, F. Van Oystaeyen; Generalized Rees rings and relative maximal orders satisfying polynomial identities, J. Algebra, to appear soon (1982).

[19] H. Marubayashi, Noncommutative Krull rings, Osaka J. Math. 13, 491-500, (1976).

[20] H. Marubayashi, Polynomial rings over Krull orders in simple Artinian rings, Hokkaido Math. Journ. 9, 63-78, (1980).

[21] G. Maury, J. Raynand, Ordres maximaux au sens de K. Asano, LNM 808, Springer Verlag, Berlin, (1981).

[22] J.S. Milne, Etale cohomology, Princeton Univ. Press., Princeton, (1980).

[23] J. Milnor, Introduction to algebraic K-theory, Ann. of Math. Studies 72, Princeton, (1971).

[24] S. Montgomery, Fixed rings of finite automorphism groups, LNM 818, Springer Verlag, Berlin (1981).

[25] M. Ojanguren, R. Sridharan, Cancellation of Azumaya algebras, J. Algebra 18, 501-505 (1971).

[26] M. Orzech, Brauer groups and class groups for a Krull domain, LNM 917, 68-90, Springer Verlag, Berlin, (1982).

[27] M. Ramras, Maximal orders over regular local rings of dimension two, Trans. Amer. Math. Soc. 142, 457-479 (1969).

[28] D. Saltman, Division algebras over discrete valued fields, Comm. in Algebra 8(18), 1749-1774 (1980).

[29] R. Swan, Algebraic K-theory, Springer Verlag, Berlin, LNM 76, (1968).

[30] M.E. Sweedler, Hopf algebras, Benjamin, New York, (1969).

[31] J.P. Van Deuren, J. Vna Geel, F. Van Oystaeyen, Genus and the Riemann-Roch theorem for noncommutative function fields in one variable, Sem. Dubreil, LNM 867, Springer Verlag. Berlin (1981).

[32] M. Vanden Bergh, J. Van Geel, A duality theorem for orders in central simple algebras over function fields, these proceedings.

[33] F. Van Oystaeyen, Some constructions of rings, these proceedings.

[34] F. Van Oystaeyen, Graded Azumaya algebras and Brauer Groups, LNM 825, 158-172, Springer Verlag, (1981).

[35] F. Van Oystaeyen, A. Verschoren, Relative invariants for rings, Marcel Dekker, to appear, (1983).

[36] F. Van Oystaeyen, A. Verschoren, Non-commutative Algebraic geometry, LNM 887, Springer Verlag, Berlin (1981).

[37] P. Wauters, E. Jespers; What is an Ω-Krull ring ?
These Proceedings.

[38] S. Yuan, Reflexive modules and algebra class groups over Noetherian integrally closed domains, J. Algebra 32, 405-417, (1974).