A NOTE ON MAXIMAL ORDERS OVER KRULL DOMAINS

Lieven LE BRUYN (*)
University of Antwerp , UIA , Belgium

May 1982 82-19

(*) The author is supported by an NFWO/FNRS - grant

0. Introduction

The classical Mori-Nagata theorem (stating that the integral closure of a Noetherian domain is a Krull domain) is recently generalized to rings satisfying a polynomial identity in the following result by M. Chamarie :

Theorem 0.1 : [1] If Λ is a Noetherian prime p.i.-ring with center R and ring of quotients Σ , then there exists an intermediate ring $\Lambda \subset \Lambda' \subset \Sigma$ which is a maximal order with center $\tilde{R}$ (the complete integral closure of R) which is a Krull domain.

Unlike in the commutative case, this 'integral closure' is by no means unique. This difficulty prompts the following question :

QUESTION A : If Λ is a maximal order over a Krull domain R, with ring of quotients Σ (which is a central simple algebra over K, the field of fractions of R), is it possible to describe all other maximal R-orders in Σ by means of 'invariants' of Λ ?

In this paper we provide a positive answer to this question using cohomology of the sheaf of normalizing elements of Λ (introduced in [3]).Furthermore, we will apply this result in section 3 in order to solve :

QUESTION B : If R is a locally factorial Krull domain with field of fractions K, give necessary and sufficient conditions on R such that all maximal R-orders in $M_n(K)$ are conjugated.

1. Preliminaries

Throughout, we will consider the following situation. R is a Krull domain with field of fractions K and Λ is a maximal R-order in some central simple algebra Σ over K.

With $\underline{O}_R$ (resp. $\underline{O}_\Lambda$) we will denote the structure sheaf of R (resp. Λ) over Spec(R). Our first objective is the introduction of the sheaf of normalizing elements of Λ , $\underline{N}_\Lambda$. It is defined by assigning to an open set U of the Zariski topology on Spec(R) the sections $\Gamma(U,\underline{N}_\Lambda) = N(\Gamma(U,\underline{O}_\Lambda)) = \{ x \in \Sigma^\star : x\Gamma(U,\underline{O}_\Lambda) = \Gamma(U,\underline{O}_\Lambda)x \}$.

<u>Proposition 1.1</u> : $\underline{N}_\Lambda$ is a sheaf of groups and the stalk in a prime p of Spec(R) equals $N(R_p)$.

<u>proof</u>

Let us first check that $\underline{N}_\Lambda$ with inclusions as restriction morphisms is a presheaf. A typical open set of Spec(R) is of the form $X(I) = \{ p \in \mathrm{Spec}(R) : I \not\subset p \}$ for some ideal I of R and it is well known that $\Gamma(X(I),\underline{O}_\Lambda) = Q_I(\Lambda) = \{x \in \Sigma: \; L \in \mathcal{L}(I) : Lx \subset \Lambda\}$ where $\mathcal{L}(I) = \{ L \in R : I \subset \mathrm{rad}(L) \}$. So, if $X(J) \subset X(I)$, then $\mathcal{L}(I) \subset \mathcal{L}(J)$ and we have to prove that $N(Q_I(\Lambda)) \subset N(Q_J(\Lambda))$. It follows from some results of Chamarie [1] that each $Q_I(\Lambda)$ is again a maximal order over its center which is a Krull domain and that the localization map $Q_I(.)$ defines a groupepimorphism from $\mathrm{Div}(\Lambda)$ onto $\mathrm{Div}(Q_I(\Lambda))$, where Div(.) is the group of divisorial ideals, cfr. e.g. [1].

Thus, if $x \in N(Q_I(\Lambda))$, then there exists a divisorial Λ-ideal A such that $Q_I(A) = Q_I(\Lambda)x$. Therefore, it will be sufficient to prove that $Q_J(A) = Q_J(\Lambda)x$. So, let $y \in Q_J(A)$, then there exists an ideal $K \in \mathcal{L}(J)$ such that $Ky \subset A \subset Q_I(A) = Q_I(\Lambda)x$, whence $Kyx^{-1} \subset Q_I(\Lambda) \subset Q_J(\Lambda)$ and thus $yx^{-1} \in Q_J(\Lambda)$ because every symmetric localization of Λ is idempotent, so $y \in Q_J(\Lambda)x$. Conversely, if $y \in Q_J(\Lambda)$ then $Ky \subset \Lambda$ for some $K \in \mathcal{L}(J)$, whence $Kyx \subset \Lambda x \subset Q_I(A)$. Thus, for every $k \in K$, we can find an ideal $L \in \mathcal{L}(I) \subset \mathcal{L}(J)$ such that $Lkyx \subset A$ whence $kyx \in Q_J(A)$ and thus $Kyx \subset Q_J(A)$, yielding that $yx \in Q_J(A)$. Thus, $Q_J(A) = Q_J(\Lambda)x$ finishing the proof that $\underline{N}_\Lambda$ is a presheaf, which is clearly separated. Therefore we are left to prove the gluing property. So, let $U_i : i \in I$ be an open covering of U and let $x \in \Gamma(U_i,\underline{N}_\Lambda)$ for every $i \in I$. Then, $x\Gamma(U,\underline{O}_\Lambda) = x(\cap \Gamma(U_i,\underline{O}_\Lambda)) = \cap \Gamma(U_i,\underline{O}_\Lambda)x = \Gamma(U,\underline{O}_\Lambda)x$ whence $x \in \Gamma(U,\underline{N}_\Lambda)$.

Finally, let us calculate the stalks of $\underline{N}_\Lambda$ at the point $p \in \mathrm{Spec}(R)$. Clearly, $(\underline{N}_\Lambda)_p \subset N(R_p)$. Conversely, if $x \in N(R_p)$, then there exists a divisorial Λ-ideal A such that $A_p = \Lambda_p x$. Thus, $(\underline{O}_A)_p = \Lambda_p x$ and likewise $(\underline{O}_{A^{-1}})_p = \Lambda_p x^{-1}$, where $\underline{O}_A$ (resp. $\underline{O}_{A^{-1}}$) is the structure sheaf of A (resp. A^{-1}). Now, we can choose a neighborhood V of p such that $x \in \Gamma(V,\underline{O}_A)$ and $x^{-1} \in \Gamma(V,\underline{O}_{A^{-1}})$. Then, $x^{-1}\Gamma(V,\underline{O}_\Lambda)x \subset x^{-1}\Gamma(V,\underline{O}_A) \subset \Gamma(V,\underline{O}_\Lambda)$ whence $\Gamma(V,\underline{O}_\Lambda)x \subset x\Gamma(V,\underline{O}_\Lambda)$ and likewise one can prove the other inclusion yielding that $x \quad \Gamma(V,\underline{N}_\Lambda)$, finishing the proof.

The sheaf $\underline{N}_\Lambda$ is not necessarely a constant sheaf, as the following example shows :

Example 1.2 : Let $\Lambda = \mathbb{C}[X,-]$ where - denotes the complex conjugation , then is a maximal order with center $\mathbb{R}[X^2]$. In [6] it is proved that $\{ X^2 + c \; ; \; c > 0 \}$ is precisely the set of the prime ideals of $\mathbb{R}[X^2]$ whose valuation extends to a valuation in $\mathbb{C}(X,-)$. If $\underline{N}_\Lambda$ were constant , $N(R) = \mathbb{C}(X,-)$ yielding that every localization of Λ at a prime ideal is a valuationring, a contradiction.

2. The main theorem

In this section we aim to solve question A , i.e. we will show how one can construct all maximal R-orders in a central simple algebra Σ over K from a given maximal order Λ . From [1] we retain that all maximal R-orders are equivalent. Of course, being conjugated defines an equivalence relation on the set of all maximal R-orders, so our study splits up in two cases :

I : The study of those maximal orders which are conjugated to Λ . They are of course classified by the set $\Sigma^{\star}/N(\Lambda)$.

II : A description of the equivalence classes of nonconjugate maximal orders.

The next theorem provides such a description by means of cohomology pointed sets , cfr. e.g. [2,5].

Theorem 2.1 : There is a one-to-one correspondence between :

(a) : equivalence classes of nonconjugate maximal orders

(b) : elements of the pointed set $\varinjlim H^1_{Zar}(U,\underline{N}_{\Lambda})$, where the direct limit is taken over all open sets U of Spec(R) containing $X^1(R)$, the set of all height one prime ideals of R .

proof

Let Λ' be any maximal R-order in Σ. By $\underline{O}$ (resp. $\underline{O}'$) we denote the structure sheaf of Λ(resp. Λ') over Spec(R) . $\underline{\Upsilon}$ (the conductor) is defined by assigning to an open set U of Spec(R) the sections :

$\Gamma(U,\underline{\Upsilon}) = \{x \in \Sigma : \Gamma(U,\underline{\Lambda}')x \subset \Gamma(U,\underline{\Lambda})\}$. First, we check that $\underline{\Upsilon}$ is a sheaf . We claim that inclusions are well defined restriction morphisms. For, let $X(J) \subset X(I)$ be open sets of the Zariski topology of Spec(R) and let $y \in \Gamma(X(I),\underline{\Upsilon})$,

$x \in \Gamma(X(J),\underline{O}')$, then $Lx \subset \Lambda'$ for some $L \in \mathcal{L}(J)$ whence $Lxy \subset \Gamma(X(I),\underline{O}) \subset \Gamma(X(J),\underline{O})$ entailing that $xy \in \Gamma(X(J),\underline{O})$ so $y \in \Gamma(X(J),\underline{\Upsilon})$ finishing the proof of our claim. So, $\underline{\Upsilon}$ is a presheaf.

Furthermore, if U_i is an open covering of U and if $y \in \cap\, \Gamma(U_i,\underline{\Upsilon})$, then $\Gamma(U,\underline{O}')y = \cap\, \Gamma(U_i,\underline{O}')y \subset \cap\, \Gamma(U_i,\underline{O}) = \Gamma(U,\underline{O})$ proving that $y \in \Gamma(U,\underline{\Upsilon})$ and therefore $\underline{\Upsilon}$ is a sheaf.

For every open set U of $\mathrm{Spec}(R)$, $\Gamma(U,\underline{O})$ and $\Gamma(U,\underline{O}')$ are both maximal $\Gamma(U,\underline{O}_R)$-orders, hence they are equivalent. By a local application of lemma VII.1.3 of [4] it follows that $\underline{\Upsilon}$ is a c-$\underline{O}'$-$\underline{O}$-ideal contained both in $\underline{O}$ and in $\underline{O}'$. By this we mean that for every open set U , $\Gamma(U,\underline{\Upsilon})$ is a left fractional $\Gamma(U,\underline{O}')$-ideal and a right fractional $\Gamma(U,\underline{O})$-ideal such that $(\Gamma(U,\underline{\Upsilon})^{-1})^{-1} = \Gamma(U,\underline{\Upsilon})$, where $\Gamma(U,\underline{\Upsilon})^{-1} = \{ x \in \Sigma : \Gamma(U,\underline{\Upsilon})x \subset \Gamma(U,\underline{O}) \} = \{ x \in \Sigma : x\Gamma(U,\underline{\Upsilon}) \subset \Gamma(U,\underline{O}') \}$. It is readily verified that $\underline{\Upsilon}^{-1}$ which is defined by taking for its sections $\Gamma(U,\underline{\Upsilon}^{-1}) = \Gamma(U,\underline{\Upsilon})^{-1}$ is also a sheaf and a c-$\underline{O}$-$\underline{O}'$-ideal.

Now, let p be any height one prime ideal of R. It is well known that Λ_p and Λ'_p are both principal left and right ideal rings. Therefore, there exists an invertible element s_p of Σ such that $(\underline{\Upsilon})_p = s_p\Lambda'_p$. Furthermore, $(\underline{\Upsilon}^{-1})_p\,(\underline{\Upsilon})_p = \Lambda_p$ entailing that $\Lambda_p s_p^{-1}\Lambda'_p s_p \Lambda_p = \Lambda_p$ whence $s_p^{-1}\Lambda'_p s_p \subset \Lambda_p$. By maximality of $s_p^{-1}\Lambda s_p$ this entails that $s_p^{-1}\Lambda'_p s_p = \Lambda_p$.

We claim that there is a neighborhood $V(p)$ of p such that :

$$s_p^{-1}(\underline{O}' \mid V(p))s_p = \underline{O} \mid V(p).$$

Since both $\underline{\Upsilon}$ and $\underline{\Upsilon}^{-1}$ are sheaves, s_p and s_p^{-1} live on a neighborhood $V(p)$ of p. Therefore, $s_p\Gamma(V(p),\underline{O}) \subset \Gamma(V(p),\underline{\Upsilon})$ and $\Gamma(V(p),\underline{O})s_p^{-1} \subset \Gamma(V(p),\underline{\Upsilon}^{-1})$. Hence, $\Gamma(V(p),\underline{O})s_p^{-1} \subset \Gamma(V(p),\underline{\Upsilon}^{-1}) = \Gamma(V(p),\underline{\Upsilon})^{-1} \subset (s_p\Gamma(V(p),\underline{O}))^{-1} = \Gamma(V(p),\underline{O})s_p^{-1}$ and therefore $\Gamma(V(p),\underline{\Upsilon}^{-1}) = \Gamma(V(p),\underline{O})s_p^{-1}$ and likewise, $\Gamma(V(p),\underline{\Upsilon}) = s_p\Gamma(V(p),\underline{O})$. This then entails that :

$$s_p^{-1}\,(\underline{O}' \mid V(p))s_p = \underline{O} \mid V(p) \;.$$

Thus, $\cup V(p)$ is an open set containing $X^1(R)$. Now, $X^1(R)$ equipped with the induced Zariski topology is a Noetherian space and therefore we can find a finite number amond these $V(p)$, say $V(p_1)$ $\ldots,V(p_n)$ such that $U = \cup V(p_i)$ contains $X^1(R)$.

For any $i,j \in 1,\ldots,n$ we have that :

$s_{p_i}(\underline{O} \mid V(p_i) \cap V(p_j))s_{p_i}^{-1} = s_{p_j}(\underline{O} \mid V(p_i) \cap V(p_j))s_{p_j}^{-1}$

and this entails that $s_{p_i}^{-1}s_{p_j} \in \Gamma(V(p_i) \cap V(p_j),\underline{N}_\Lambda)$.

Therefore $\{V(p_i),s_{p_i}\}$ describes a section of $\Gamma(U,\underline{\Sigma}^\star/\underline{N}_\Lambda)$.

Now consider the exact sequence of sheaves of pointed sets :

$1 \longrightarrow \underline{N}_\Lambda \rightarrow \underline{\Sigma}^\star \longrightarrow \underline{\Sigma}^\star/\underline{N}_\Lambda \longrightarrow 1$. Taking sections over U yields the exact sequence of pointed sets :

$$1 \longrightarrow N(\Lambda) \longrightarrow \Sigma^\star \longrightarrow \Gamma(U,\underline{\Sigma}^\star/\underline{N}_\Lambda) \longrightarrow H^1_{Zar}(U,\underline{N}_\Lambda) \longrightarrow 1$$

Therefore, the section $\{V(p_i),s_{p_i}\}$ determines an element in $H^1_{Zar}(U,\underline{N}_\Lambda)$ (and thus also in $\lim H^1_{Zar}(U,\underline{N}_\Lambda)$) which differs from the distinguished element in $H^1_{Zar}(U,\underline{N}_\Lambda)$ if and only if Λ' is not conjugated to Λ .

Conversely, let $s \in \lim H^1_{Zar}(U,\underline{N}_\Lambda)$ and choose an open set U of Spec(R) containing $X^1(R)$ and an element $s(U) \in H^1_{Zar}(U,\underline{N}_\Lambda)$ which represents s. Using the above exact sequence, s(U) is determined by some section in $\Gamma(U,\underline{\Sigma}^\star/\underline{N}_\Lambda)$. Such a section is given by a set of couples $\{(U_i,s_i)\}$ where U_i is an open covering of U, $s_i \in \Gamma(U_i,\underline{\Sigma}^\star)$ for every i and for all i and j we have that $s_i^{-1}s_j \in \Gamma(U_i \cap U_j,\underline{N}_\Lambda)$. On U we will define the twisted sheaf of maximal orders $\underline{O}' \mid U$ by putting $\underline{O}' \mid U_i = s_i(\underline{O} \mid U_i)s_i^{-1}$. Using the fact that $s_i^{-1}s_j \in \Gamma(U_i \cap U_j,\underline{N}_\Lambda)$ it is easily verified that this is indeed a sheaf. We claim that $\Lambda' = \Gamma(U,\underline{O}' \mid U)$ is a maximal R-order.

Firstly we will show that there exists an open refinement $\{W_k\}$ of $\{U_i\}$ and sections $t_k \in \Gamma(W_k,\underline{\Sigma}^\star)$ such that $t_k^{-1}t_l \in \Gamma(W_k \cap W_l,\underline{O}^\star)$

and with the property that the twisted sheaf of maximal orders determined by (W_k, t_k) coincides with $\underline{O}'$ on $\cup W_k$. Because $X^1(R)$ is a Noetherian space, there are a finite number among the U_i, say $U_1, \dots, U_n$ such that $U' = \cup U_i \supset X^1(R)$. For any i,j among 1,...,n , $Z(i,j) = \{ p \in U_i \cap U_j : s_i^{-1} s_j \notin \Lambda_p \}$ is a finite set, because $\mathrm{Div}(\Gamma(U,\underline{O}))$ is the free abelian group generated by $X^1(R) \cap U$ for any open set U. So, $Z(1) = Z(1,2) \cup Z(1,3) \cup \dots \cup Z(1,n)$ is a finite set. Now because the Zariski topology induced on $X^1(R)$ is the cofinite topology , there exists an open V in Spec(R) such that $V \cap X^1(R) = X^1(R) / Z(1)$. Take $W_1 = U_1 \cap V$, $W_i = U_i$, for $i \neq 1$, $t_1 = s_1 | W_1$ and $t_i = s_i$ for $i \neq 1$, then $t_1^{-1}.t_j \in \Gamma(W_1 \cap W_j, \underline{O}^{\star})$. Continuing in this manner we will eventually find (W_k, t_k) satisfying the requirements, in particular, if $W = \cup W_k$, then $\underline{O}' | W$ coincides with the twisted sheaf of maximal orders determined by the t_k.

Next, we define a sheaf $\underline{\Upsilon} | W$ by $\underline{\Upsilon} | W_k = t_k(\underline{\Upsilon} | W_k)$. Clearly, $\underline{\Upsilon} | W$ is a right $\underline{O}$ - ideal and $((\underline{\Upsilon} | W)^{-1})^{-1} = \underline{\Upsilon} | W$, this yields that for every open $V \subset W$, $\Gamma(V, \underline{O})$ is a right fractional c-$\Gamma(V, \underline{Q})$-ideal. This implies that $O_l(\Gamma(V, \underline{\Upsilon})) = \Gamma(V, \underline{O}' | W)$ is a maximal order.

In particular, $\Gamma(W, \underline{O}' | W) = \Gamma(U, \underline{O}' | \bar{U})$ is a maximal order.

Finally, the reader may check that the constructions above do not depend on the choices made.

<u>Corollary 2.2.</u> : If R is a Dedekind domain, there is a one-to-one correspondence between :

(a) equivalence classes of non-conjugate maximal orders

(b) elements of $H^1_{Zar}(X, \underline{N}_{\Lambda})$

3. Application : maximal orders in matrixrings

In this section we aim to characterize those locally factorial (i.e. R_p is a UFD for every $p \in Spec(R)$) Krull domains for which all maximal orders in $M_n(K)$ are conjugated. In this situation we are able to compute $H^1_{Zar}(U,\underline{N}_\Lambda)$ for $\Lambda = M_n(R)$.

With $\underline{PGL}_n$ we will denote $\underline{Aut}(P^n_R)$, the automorphism scheme of the n-dimensional projective space over R, i.e. $\underline{PGL}_n$ is the sheafification of the presheaf which assigns $PGL_n(\Gamma(U,\underline{O}_R))$ to any open set of Spec(R), cfr. e.g. [5].

Proposition 3.1. : If R is a locally factorial Krull domain and if $\Lambda = M_n(R)$, then $H^1_{Zar}(U,\underline{N}_\Lambda) = H^1_{Zar}(U,\underline{PGL}_n)$ for every open set U of Spec(R).

proof

If we assign to an open set U of Spec(R) the group $GL_n(\Gamma(U,\underline{O}_R)).K^\star \subset GL_n(K)$, then this defines a presheaf of groups. Its sheafification will be denoted by $\underline{GL}_n.\underline{K}^\star$ .This sheaf is clearly a subsheaf of $\underline{N}_\Lambda$. We will show that their stalks are isomorphic. If $p \in Spec(R)$ and if $x \in N(M_n(R_p))$, then $M_n(R)x = M_n(A)$ for some divisorial R_p-ideal A. Because R_p is a UFD , $A = R_p.k$ for some $k \in K^\star$, yielding that $x \in GL_n(R_p).K^\star$ proving that $\underline{GL}_n.\underline{K}^\star = \underline{N}_\Lambda$.

The following sequence of sheaves of groups is exact :

$$1 \longrightarrow \underline{K}^\star \longrightarrow \underline{GL}_n.\underline{K}^\star \longrightarrow \underline{PGL}_n \longrightarrow 1$$

where $\underline{K}^\star$ denotes the constant sheaf associated with $K^\star$.

Taking sections over U yields the following long exact cohomology sequence :

$$1 \longrightarrow \Gamma(U,\underline{K}^\star) \longrightarrow \Gamma(U,\underline{N}_\Lambda) \longrightarrow \Gamma(U,\underline{PGL}_n) \longrightarrow$$
$$\longrightarrow 1 \longrightarrow H^1_{Zar}(U,\underline{N}_\Lambda) \to H^1_{Zar}(U,\underline{PGL}_n) \longrightarrow 1$$

finishing the proof.

A : Dedekind domains

Proposition 3.2 : If R is a Dedekind domain, then all maximal R-orders in $M_n(K)$ are conjugated if and only if $(-)^n$: $Cl(R) \to Cl(R)$ sending $[A]$ to $[A^n]$ is an epimorphism.

proof

In view of Coroll.2.2 and Prop. 3.1 we have to find an equivalent condition for $H^1_{Zar}(X,\underline{PGL}_n) = 1$. Writing out the long exact cohomology sequence of the following exact sequence of sheaves of groups :

$$1 \longrightarrow \underline{O}^{\star}_R \longrightarrow \underline{GL}_n \longrightarrow \underline{PGL}_n \longrightarrow 1$$

entails :

$$H^1_{Zar}(X,\underline{O}^{\star}_R) \xrightarrow{\delta} H^1_{Zar}(X,\underline{GL}_n) \longrightarrow H^1_{Zar}(X,\underline{PGL}_n) \longrightarrow H^2_{Zar}(X,\underline{O}^{\star}_R)$$

Because R is a Dedekind domain (Krull dimension = 1) $H^2_{Zar}(X,\underline{O}^{\star}_R) = 1$. Furthermore, $H^1_{Zar}(X,\underline{GL}_n)$ is the set of isomorphism classes of projective rank n R-modules, which we denote by $Proj_n(R)$. By Steinitz' result any projective rank n module is isomorphic to $J_1 \oplus \dots \oplus J_n$ for some fractional R-ideals J_i and δ is epimorphic if and only if there exists a fractional R-ideal I such that $J_1 \oplus \dots \oplus J_n \cong I \oplus \dots \oplus I$ yielding that $J_1 \dots J_n \cong I^n$, finishing the proof.

Remark 3.3 : F. Van Oystaeyen suggested a more ringtheoretical proof of this result in the following way. Because all maximal R-orders in $M_n(K)$ are Morita equivalent and $M_n(R)$ is Azumaya, they are all Azumaya algebras. Furthermore Br(R) Br(K) whence any maximal order is of the form $End_R(P)$ where $P \in Proj_n(R)$.Applying again Steinitz' theorem to the condition $End_R(P) \cong M_n(R)$ yields the same condition on Cl(R).

B : Regular local domains

We recover the classical result of M. Ramras for matrixrings :

Proposition 3.4 : If R is a regular local ring of $\text{gldim}(R) \leq 2$, then all maximal orders in $M_n(K)$ are conjugated.

proof

We have to check that $H^1_{Zar}(U,\underline{PGL}_n) = 1$ where $U = X(m)$, m being the maximal ideal of R. Again consider the exact sequence :

$$H^1_{Zar}(U,\underline{O}^*_R) \longrightarrow H^1_{Zar}(U,\underline{GL}_n) \longrightarrow H^1_{Zar}(U,\underline{PGL}_n) \longrightarrow H^2_{Zar}(U,\underline{O}^*_R)$$

Now, $H^1_{Zar}(U,\underline{GL}_n)$ is the set of isomorphism classes of reflexive R-modules which are free of rank n at every height one prime ideal of R , $\text{Ref}_n(R)$. Because $\text{gldim}(R) \leq 2$, reflexive modules are projective whence $\text{Ref}_n(R) = \text{Proj}_n(R)$ and $\text{Ref}_1(R) = \text{Pic}(R)$. Finally, R being local $\text{Pic}(R) = \text{Proj}_n(R) = 1$ and therefore all cohomology pointed sets above are trivial exept perhaps $H^1_{Zar}(U,\underline{PGL}_n)$ but exactness of the sequence finishes the proof.

C : Locally factorial Krull domains

Theorem 3.5 : If R is a locally factorial Krull domain then all maximal orders in $M_n(K)$ are conjugated if and only if the map from $\text{Cl}(R)$ to $\text{Ref}_n(R)$ sending $[\, I \,]$ to $[\, I \oplus \dots \oplus I \,]$ is surjective.

proof

Consider the exact sequence :

$$\lim H^1(U,\underline{O}^*_R) \longrightarrow \lim H^1(U,\underline{GL}_n) \longrightarrow \lim H^1(U,\underline{PGL}_n) \longrightarrow \lim H^2(U,\underline{O}^*_R)$$

where the direct limit is taken over all opens U containing $X^1(R)$

Because R is locally factorial, Cartier divisors coincide with Weil divisors showing that the sequence :

$1 \longrightarrow \underline{O}_R^\star \longrightarrow \underline{K} \longrightarrow \underline{Div} \longrightarrow 1$ is exact. Because the sheaf of Weil divisors , $\underline{Div}$, is flabby $H^2_{Zar}(U,\underline{O}_R^\star) = 1$ for any open set U showing that the last term in the sequence vanishes.

So, by Th. 2.1 and Prop. 3.1 all maximal orders in $M_n(K)$ are conjugated iff the map from $\lim H^1(U,\underline{O}_R^\star) = Cl(R)$ to $\lim H^1(U,\underline{GL}_n) = Ref_n(R)$ which is defined by sending a class of a divisorial ideal $[I]$ to $[I \oplus \ldots \oplus I]$ is surjective.

Finally, it is a pleasure to thank F. Van Oystaeyen, J. Van Geel and M. Vanden Bergh for many stimulating conversations.

References

[1] : M. Chamarie ; Anneaux de Krull non-commutatifs , thèse , Université Claude-Bernard , Lyon , 1981

[2] : J. Giraud ; Cohomologie non abélienne , Grundl. der Math. Wiss. 179 , Springer Verlag , Berlin , 1971

[3] : L. Le Bruyn ; Class groups and noncommutative algebraic geometry , Proceedings of the Emmy Noether conference '82 , Antwerp

[4] : G. Maury , J. Raynaud ; Ordres maximaux au sens de K.Asano LNM 808 , Springer Verlag , Berlin , 1981

[5] : J.S. Milne ; Etale cohomology , Princeton Univ. Press , Princeton , 1980

[6] : J.P. Van Deuren , J. Van Geel , F. Van Oystaeyen ; Genus and a Riemann-Roch theorem for non-commutative function fields in one variable , Séminaire d'Algèbre P. Dubreil and M.P. Malliavin , LNM 867 , Springer Berlin , 1981