Arithmetical Rings I

a unified approach to arithmetical theories

L. Le Bruyn (*)

University of Antwerp, UIA, Belgium.

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Abstract

We introduce maximal orders relative to torsion couples as a tool to study virtually all existing non commutative generalizations of Dedekind and Krull domains. Further on we restrict ourselves to the Dedekind case, the so called arithmetical rings. In the third section we give some examples how this theory can be applied as well as some open problems. In the last section we introduce and relate Picard and Class groups for arithmtical rings. Their K-theoretic interpretation will enable us in part II of this paper to study their behaviour under ringextensions.

1. Maximal orders relative to torsion couples

Let us recall some torsion-theoretic notions (cfr. [7,21]. An endofunctor σ in R-mod is said to be a kernel functor if it is left exact subfunctor of the identity in R-mod ; σ is said to be idempotent if $\sigma(M/\sigma(M))=0$ for any left R-module M. To any kernel functor σ the fitter of left ideals $\mathcal{L}(\sigma) = \{L <_1 R : \sigma(R/L) = R/L) = R/L\}$ is associated and to any filter $\mathcal{L}$ of left ideals satisfying :

(K1) : if I and J $\in \mathcal{L}$, then $I \cap J \in \mathcal{L}$

(K2) : if $I \in \mathcal{L}$ and $J <_1 J$, then $J \in \mathcal{L}$

(K3) : if $I \in \mathcal{L}$ $x \in R$, then $(x:I) = \{r \in R : r x \in I\} \in \mathcal{L}$

one can associate the kernel functor $\sigma_{\mathcal{L}}$ defined by $\sigma_{\mathcal{L}}(M) = \{m \in M \mid \exists I \in \mathcal{L} : Im\ 0\}$.

$\sigma_{\mathcal{L}}$ will be idempotent if and only if $\mathcal{L}$ satisfies also :

(K4) : if $I \in \mathcal{L}$ and $J <_1 R$ such that $(x : J) \in \mathcal{L}$ for any $x \in I$, then $J \in \mathcal{L}$.

A kernel functor σ is called bilateral if its associated filter $\mathcal{L}(\sigma)$ has a cofinal set consisting of ideals, σ is said to be symmetric if it is both bilateral and idempotent. One associates to any idempotent kernel functor σ a left exact localization functor $Q_\sigma(.)$ in R-mod.

If R is σ-torsion free (i.e. $\sigma(R) = 0$), $j_\sigma : R \to Q_\sigma(R) = \varinjlim \mathrm{Hom}_R\,(I,R)$ is the canonical embedding, where the direct limit is taken over all $I \in \mathcal{L}(\sigma)$.

An idempotent kernel functor σ is said to be a T-functor if it satisfies (T): for $I \in \mathcal{L}(\sigma) : Q_\sigma(R)\, j_\sigma\,(I) = Q_\sigma(R)$.

σ will be called geometrical if it has property T and satisfies :

(G) : for any ideal I of R, $Q_\sigma(R) j_\sigma(I)$ is an ideal of $Q_\sigma(R)$.

E.g. if σ idempotent kernelfunctor, i.e. if $\mathcal{L}(\sigma)$ has a cofinal set consisting of centrally generated ideals, then σ is geometrical whenever it is a T-functor. Likewise, one can define all these concepts in Mod-R. If $\mathcal{L}^2(\sigma)$ is a multiplicatively closet set of nonzero ideals of R, we will denote with $\mathcal{L}^l(\sigma)$ (resp. $\mathcal{L}^r(\sigma)$) the filter of left (resp. right)ideals of R generated by $\mathcal{L}^2(\sigma)$. If $\mathcal{L}^l(\sigma)$ (resp. $\mathcal{L}^r(\sigma)$) is idempotent, we will denite by $Q^l_\sigma(.)$ (resp. $Q^r_\sigma(.)$) the localization functor in R-mod (resp. in mod-R) associated with $\mathcal{L}^l(\sigma)$ (resp. with $\mathcal{L}^r(\sigma)$). Finally, we will denote by $\mathcal{F}^l(R)$ (resp. $\mathcal{F}^r(R)$) the set of all idempotent kernel functors in R-mod (resp. in mod-R).

<u>Definition 1.1.</u> if R is any ring, a torsion couple (σ,τ) consists of a filter of left ideals $\mathcal{L}(\sigma)$ and a filter of right ideals $\mathcal{L}(\tau)$ such that $\mathcal{L}(\sigma)\,\mathcal{L}(\tau) \subset \mathcal{L}^2(\sigma,\tau)$ where $\mathcal{L}^2(\sigma,\tau) = \{I \triangleleft R \mid I \in \mathcal{L}(\sigma) \text{ and } I \in \mathcal{L}(\tau)\}$. Throughout this note, S will be a fixed ring and all rings considered are subrings of S.

<u>Definition 1.2.</u> : if R is a subring of S and (σ,τ) is a torsion couple of R, R is said to be a (σ,τ)-order if $\{r \in R \mid \exists I \in \mathcal{L}^2(\sigma,\tau) : Ir = 0 \text{ or } rI = 0\} = 0$. A (σ,τ)-order R in S is said to be maximal if there exists no proper overring T of R in S such that $I\,T\,J \in \mathcal{L}^2(\sigma,\tau)$ for some $I \in \mathcal{L}(\sigma), J \in \mathcal{L}(\tau)$.

<u>The mother example</u> : let R be a prime Goldie ring and let S = Q be its

classical ring of quotients. If σ is the left Goldie torsion, i.e. $\mathcal{L}(\sigma) = \{I <_l R \mid I \text{ contains a regular element}\}$,and τ is the right Goldie torsion theory, then R is readily seen to be a (σ,τ)-order in Q. Further R is a maximal order (in the sense of [13]) if and only if R is maximal as (σ,τ)-order.

Definition 1.3. : let R be a (σ,τ)- order in S. A left (right) (fractional) R-ideal A is a left (right) R-submodule of S such that $I \subset A$ for some $I \in \mathcal{L}(\sigma)$ $(I \in \mathcal{L}(\tau))$ and $A\,J \in \mathcal{L}^2(\sigma,\tau)$ $(JA \in \mathcal{L}^2(\sigma,\tau))$ for some $J \in \mathcal{L}(\tau)$ $(J \in \mathcal{L}(\sigma))$. Of course, A is an R-ideal if A is both a left and right R-ideal.

The following proposition generalizes some properties of "classical" maximal orders (cfr. e.g.[13]).

Note that if A, $B \subset S$ we denote $(A \dot{:}_l B) = \{s \in S : sA \subset B\}$ and $(A \dot{:}_r B) = \{s \in S : As \subset B\}$.

Proposition 1.4. : if R is a (σ,τ)-order in S, equivalent are :

(1) : R is a maximal (σ,τ)-order.

(2) : if A is a left R-ideal such that $A^2 \subset A$ and $R \subset A$, then R = A; if B is a right R-ideal such that $B^2 \subset B$, $R \subset B$ then R = B

(3) : if A is a left R-ideal, then $(A \dot{:}_l A) = R$, if B is a right R-ideal, then $(B \dot{:}_r B) = R$.

(4) : if $I \in \mathcal{L}^2(\sigma,\tau)$, then $(I \dot{:}_l I) = (I \dot{:}_r I) = R$

(5) : if A is an R-ideal, then $(A \dot{:} A) = (A \dot{:}_r A) = R$.

Proof

(1) $\Rightarrow$ (2) : Clearly, $R \subset (A \dot{:}_l A) \subset S$. There exists an $I \in \mathcal{L}(\tau)$ such that $A\,I \in \mathcal{L}^2(\sigma,\tau)$, hence $(A \dot{:}_l A)\,A\,i = A\,I \in \mathcal{L}^2(\sigma,\tau)$ and because $(A \dot{:} A)$ is an overring of R, $(A \dot{:}_l A) = R$ whence $A \subset R$ because $A \subset (A \dot{:}_l A)$.

(2) ⇒ (3) : if A is a left R-ideal, so is $(A \underset{l}{:} A)$. Further, $(A \underset{l}{:} A)^2 \subset (A \underset{l}{:} A)$ and $R \subset (A \underset{l}{:} A)$ whence $(A \underset{l}{:} A) = R$.

(3) ⇒ (4) : trivial

(4) ⇒ (5) : there exists an $I \in \mathcal{L}(\tau)$ such that $A\, I \in \mathcal{L}^2(\sigma,\tau)$ whence $(AI \underset{l}{:} AI) = R$. Now, $R \subset (A \underset{l}{:} A) \subset (AI \underset{l}{:} AI) = R$.

(5) ⇒ (1) : let T be an overring of R in S such that $I\,T\,J \in \mathcal{L}^2(\sigma,\tau)$ for some $I \in \mathcal{L}(\sigma)$, $J \in \mathcal{L}(\tau)$. Thus $IR \supset I\,T\,J\,I\,R \in \mathcal{L}^2(\sigma,\tau)$ whence $T\,J\,I\,R \subset (IR \underset{r}{:} IR) = R$, thus $T\,J\,I\,R \in \mathcal{L}^2(\sigma,\tau)$ because $ITJIR \subset TJIR$. Therefore, T is a left fractional R-ideal. Similarly, one proves that T is also a right fractional R-ideal. Therefore, $T \subset (T \underset{l}{:} T) = R$ finishing the proof.

<u>Observation 1.5.</u> If R is a (σ,τ)-order in S, then ${}_{\sigma}\mathbb{F}_{\tau}(R)$, the set of all fractional R-ideals, is closed under multiplication.

<u>Proof</u>

Let $A, B \in {}_{\sigma}\mathbb{F}_{\tau}(R)$ and let $I, K, M, P \in \mathcal{L}(\sigma)$; $J, L, N, Q \in \mathcal{L}(\tau)$ such that $IA, KB, AJ, BL \in \mathcal{L}^2(\sigma,\tau)$, $M \subset A$, $P \subset B$, $N \subset A$, $Q \subset B$ then, of course, $MP \subset AB$, $NQ \subset AB$ and : $R \supset KB \supset KIAB \supset K(IA)(BL) \in \mathcal{L}^2(\sigma,\tau)$ whence $(K,I)\,AB \in \mathcal{L}^2(\sigma,\tau)$, $R \supset AJ \supset ABLJ \supset (IA)(BL)\,J \in \mathcal{L}^2(\sigma,\tau)$ whence $AB(LJ) \in \mathcal{L}^2(\sigma,\tau)$.

Let us recall the construction of the group of E. Artin. Suppose that T is an ordered set with a multiplication law subject to the following conditions.

(1) : T is a semigroup with unit element e

(2) : T is a lattice (i.e. $a \vee b = \sup(a,b)$ and $a \wedge b = \inf(a,b)$ exist for any elements $a, b \in T$).

(3) : $a \le b$ implies $ac \le bc$ and $ca \le cb$ for all $a,b,c \in T$

(4) : $c(a \vee b) = ca \vee cb$ and $(a \vee b)c = ac \vee bc$ for all $a,b,c \in T$.

(5) for every bounded subset $\{a_i;\ i \in \Lambda\}$ of elements of T and every $c \in T$: $(\vee a_i)c = \vee(a_i c)$, $c(\vee a_i) = \vee(ca_i)$

(6) : There exists a mapping $(-)^{-1} : T \to T$ such that $a.a^{-1}.a \leq a$ for every $a \in T$ and if $a.x.a \leq a$ then $x \leq a^{-1}$.

(7) : if $a^2 \leq a$ and $e \leq a$, then $e = a$.

Two elements are said to be quasi-equal iff $a^{-1} = b^{-1}$. This defines an equivalence relation on T and the set of equivalence classes equipped with the multiplication $[a].[b] = [((a.b)^{-1})^{-1}]$ is a commutative (!) group (cfr. e.g. [13] .

Proposition 1.6. if R is a maximal (σ,τ)-order in S, and if ${}_{\sigma}\mathbb{F}_{\tau}(R) =$ $= T$,$\cdot \subset$. is $\leq$. , $R = e$, ${}_{\sigma}\mathbb{F}_{\tau}(R)$ satisfies (1) to (7).

Proof

(1) : cfr observation 1.5.

(2) : let $A,B \in {}_{\sigma}\mathbb{F}_{\tau}(R)$, then $A \cap B \in {}_{\sigma}\mathbb{F}_{\tau}(R)$, for, let $I, K \in \mathcal{L}(\sigma)$; $J, L \in \mathcal{L}(\sigma)$; $J, L \in \mathcal{L}(\tau)$ such that $IA, KB, AJ, BL \in \mathcal{L}^2(\sigma,\tau)$ and $M, P \in \mathcal{L}(\sigma)$; $N, Q \in \mathcal{L}(\tau)$ such that $M, N \subset A$ and $P, Q \subset B$, then $M \cap P \subset A \cap B$ and $N \cap B$. (or : $MP \subset A \cap B$ if we do not impose that $\mathcal{L}(\sigma)$ and $\mathcal{L}(\tau)$ have the intersection property).

Further, $IK(AJKB) \subset IK(A \cap B) \subset IKA \cap IKB \subset R$ and $IK(AJKB) \in \mathcal{L}^2(\sigma,\tau)$ hence so is $IK(A \cap B)$. Likewise one proves that $(A \cap B)JL \in \mathcal{L}^2(\sigma,\tau)$.

Also, $A + B \in {}_{\sigma}\mathbb{F}_{\tau}(R)$, for, $M + P \subset A + B$; $N + Q \subset A + B$; and : $KI(A + B) \subset IA + KB \subset$ and $KI(A + B)L$: $KIA\,L + KIB\,L \in \mathcal{L}^2(\sigma,\tau)$.

(3) and (4) are trivial.

(5) : let $A_i \subseteq \Sigma A_i \subseteq B \in {}_{\sigma}\mathbb{F}_{\tau}(R)$ such that I_iA_i, A_iJ_i, IB and BJ are elements of $\mathcal{L}^2(\sigma,\tau)$. Then

$IA_iJ_i \subset IA_i \subset I(\Sigma A_i) \subset R$ and $I(A_iJ_i) \in \mathcal{L}^2(\sigma,\tau)$. Then

$IA_iJ_i \subset IA_i \subset I(\Sigma A_i) \subset R$ and $I(A_iJ_i) \in \mathcal{L}^2(\sigma,\tau)$, thus $I(\Sigma A_i) \in \mathcal{L}^2(\sigma,\tau)$.

Likewise, $(\Sigma A_i) J \in \mathcal{L}^2 (\sigma,\tau)$. If $K_i \subset A_i$ and $L_i \subset A_i$ where $K_i \in \mathcal{L} (\sigma)$, $L_i \in \mathcal{L}(\tau)$, then, of course, $K_i \subset \Sigma A_i$, $L_i \subset \Sigma A_i$.

(6) : if $A \in {}_\sigma\mathbb{F}_\tau (R)$, it is easy to show that $(A \dot{:}_l R)$ is a left R-ideal and that $(A \dot{:}_l R)$ is a right R-ideal. Now, $A(A \dot{:}_l R)A \subset A$ whence $A(A \dot{:}_l A) = R$ and therefore $(A \dot{:}_l R) \subset (A \dot{:}_r R)$. Similarly, $(A \dot{:}_r R) \subset (A \dot{:}_l R)$ and thus : $(A:R) = (A \dot{:}_l R) = (A \dot{:}_r R)$ is an R-ideal satisfying the requirements of $(-)^{-1}$.

(7) : follows from proposition 1.4.

Definition 1.7. : an R ideal A is said to be a divisioral R-ideal iff $(A:R):R = A$. ${}_\sigma\mathbb{D}_\tau(R)$ will be the set of all divisorial R-ideals.

The next proposition follows from the arguments above :

Proposition 1.8. : if R is a maximal (σ,τ)-order, ${}_\sigma\mathbb{D}_\tau(R)$ equipped with the multiplication $A \star B = (AB:R):R$ is a commutative group.

definition 1.9. R is said to be a (σ,τ)-Krull order in S if

(1) R is a maximal (σ,τ)-order in S

(2) R satisfies the acc on divisorial R-ideals contained in R.

Proposition 1.10 : if R is a (σ,τ)-Krull order in S, ${}_\sigma\mathbb{D}_\tau(R) \cong \mathbb{Z}^{(\Lambda)}$ for some index set Λ and this isomorphism is order preserving.

Proof

Reverse the ordening on ${}_\sigma\mathbb{D}_\tau(R)$, i.e. $A \leq B$ iff $B \subset A$. It is readily verified that every finite subset of ${}_\sigma\mathbb{D}_\tau(R)$ has a supremum $A_1 \cap \ldots \cap A_n$ and an infimum $(A_1 + \ldots A_n:R):R$.

Moreover, any nonempty subset of positive elements of ${}_\sigma\mathbb{D}_\tau(R)$ (i.e. divisorial R-ideals contained in R) has a minimal element. A well known theorem on commutative ordered groups satisfying these properties (cfr[3]) yields that ${}_\sigma\mathbb{D}_\tau(R) \cong \mathbb{Z}^{(\Lambda)}$ for some index set Λ and the isomorphism is order preserving. Of course, the order relation on $\mathbb{Z}^{(\Lambda)}$ is defined by $(\alpha_\lambda)_\lambda \leq (\beta_\lambda)_\lambda \leq (\beta_\lambda)_\lambda$ iff $\alpha_\lambda \leq \beta_\lambda$ all $\lambda \in \lambda$.

Remark 1.11 : it is easy to verify that the maximal (with respect to inclusion) divisorial R-ideals contained in R form a set of generators of ${}_\sigma\mathbb{D}_\tau(R)$. We claim that these generators are prime ideals of R. For, in any Artin setting (i.e. a system satisfying the conditions (1) to (7) we have for any a, b, $\in$ T :

$(ab)^\star = (a^\star b)^\star = (a\ b^\star)^\star = (a^\star b^\star)^\star$ where $a^\star = (a^{-1})^{-1}$ (cfr.[13]). Now, let P be a maximal divisorial R-ideal and suppose that I and J are ideals of R such that $P \subsetneqq I$, $P \subsetneqq J$ and $I\,J \subset P$, then $(IJ)^\star \subset P^\star = P$ and $(IJ)^\star = (I^\star J^\star)^\star = R^\star = R$ a contradiction.

If P is a minimal prime ideal of $\mathcal{L}^2(\sigma,\tau)$ which is a divisorial R-ideal, then P is a generator. It is not known to the author whether every minimal prime ideal of $\mathcal{L}^2(\sigma,\tau)$ is divisorial.

Now, we aim to relate the arithmatical theory of a(σ,τ)-order R with that of its center Z(R).

Let D be a commutaive ring, contained in S and let $\mathcal{L}(\rho)$ be a multiplicatively closed filter of nonzero ideals of D. D is said to be ρ-completely integrally closed if it is ρ-torsion free and satisfies the following condition : every element $s \in S$ such that there exists an element $I \in \mathcal{L}(\rho)$: $I\,s^n \subset D$ for every $n \in \mathbb{N}$ belongs to D. The proof of the following lemma is easy and left to the reader.

Lemma 1.12. equivalent are :

1. D is a maximal ρ-order in S
2. D is ρ-completely integrally closed

If R is a (σ,τ)-order in S with center Z(R), $\mathcal{L}(\rho(\sigma,\tau)) =$

$= \{I < Z(R) \mid RI \in \mathcal{L}^2(\sigma,\tau)\}$ is a multiplicatively closed filter of nonzero ideals of Z(R).

Proposition 1.13 :

1. If R is a maximal (σ,τ)-order in S, Z(R) is $\rho(\sigma,\tau)$completely integrally closed.

2. If R is a (σ,τ)-Krull order in S, Z(R) is a $\rho(\sigma,\tau)$-Krull order.

Proof

(1): suppose there exists an element x such that for some $I \in \mathcal{L}(\rho(\sigma,\tau))$, $I x^n \subset Z(R)$ for every $n \in \mathbb{N}$. Then, R [x] is an overring of R in S such that $R I R[x] \subset R$ whence $x \in R \cap K = Z(R)$ where $K = \{\exists I \in \mathcal{L}(\rho(\sigma,\tau)) : Is \subset Z(R)\}$

(2) : in view of part (1) we are left to prove that Z(R) satisfies the ACC on divisorial Z(R)-ideals. For any $I \in \mathcal{L}(\rho(\sigma,\tau))$ we denote $(RI)^{-1} =$ $= \{s \in S \mid s R I \subset R\}$, $I^{-1} = \{k \in K \mid k I \subset Z(R)\}$, $\overline{RI} = \{s \in S \mid (RI)^{-1} s \subset R\}$, $\overline{I} = \{k \in K \mid I^{-1} q \subset Z(R)\}$. Clearly, $\overline{RI} \cap Z(R) \supset I$ if I is divisorial.

Further,

$(\overline{RI} \cap Z(R))\, I^{-1} \subset \overline{RI}(RI)^{-1} \cap K \subset R \cap K = Z(R)$ whence $I = \overline{RI} \cap Z(R)$ finishing the proof.

Remark 1.14 : If R is a (σ,τ)-Krull preorder such that $\mathcal{L}^2(\sigma_c) =$ $= \{I \triangleleft R \mid I \cap Z(R) \neq 0\} \subset \mathcal{L}^2(\sigma,\tau)_1$ Z(R) is a Krull domain? We will return to this in section 2.

Some examples

Marubayashi-Krull rings (cfr.([14, 15]) and Chamarie-Krull rings (cfr.[4]) are (σ,τ)-Krull orders where σ (resp. τ) is the left (resp. right) Goldie torsion theory. Ω-Krull rings (cfr. [9,10]) are (σ_c^l, σ_c^r)-Krull orders where $\mathcal{L}^2(\sigma_c) = \{I \triangleleft R : I \cap Z(R) \neq 0\}$.

Marubayashi's Krull-HNP-rings (cfr [16]) are (ρ^l,ρ^r)-Krull orders where $\mathcal{L}^2(\rho) = \cap\{\mathcal{L}^2(R-P);\ P \in \text{Spec } R,\ (P:R):R \neq R\}$.

Details are left to the reader.

2. Arithmetical rings

In this section we will limit ourselves to maximal (σ,τ)-orders which are of Dedekind type, namely Arithmetical rings.

Definition 2.1. A (σ,τ)-order R in S said to be an arithmetical (σ,τ)-ring iff ${}_\sigma\mathbb{F}_\tau(R)$ is a group.

Proposition 2.2. : if R is an arithmetical (σ,τ)-ring, then

1. any R-ideal is a f.g. projective left and right R-module
2. R is a (σ,τ)-Krull order in S.

Proof

1. Let A be any R-ideal, then $A^{-1}A = R = AA^{-1}$ for some R-ideal A^{-1}, whence $A^{-1} \subset (A\dot{:}_l R)$ and $A^{-1} \subset (A\dot{:}_r R)$. Further, $\Sigma f_i a_i = 1$ for some $f_i \in A^{-1}$ and $a_i \in A$. Thus for any $a \in A$, $\Sigma(af_i)a_i = a$ and $a f_i \in R$, yielding that $A = Ra_1 + \ldots + Ra_n$. Finally, $(A\dot{:}_r R)$ is contained in $Hom_R(A,R)$ the set of all left R-module morphisms from A to B and the dual basis theorem implies that A is a f.g. projective left R-module. A similar argument shows that A is a f.g. projective right R-module.

2. First, we will show that $(I\dot{:}_l R) = (I\dot{:}_r R) = I^{-1}$ for any $I \in \mathcal{L}^2(\sigma,\tau)$. To this end it suffices to prove that $(I\dot{:}_l R)$ and $(I\dot{:}_r R)$ are elements of ${}_\sigma\mathbb{F}_\tau(R)$. Because $(I\dot{:}_l R)I = R \in \mathcal{L}^2(\sigma,\tau)$ yielding that $I(I\dot{:}_l R)I = I$ whence $I(I\dot{:}_l R) = I(I\dot{:}_l R)II^{-1} = II^{-1} = R$, thus $(I\dot{:}_l R) = (I\dot{:}_r R)$, done. Also, $I^{-1}(I\dot{:}_l I)I = R$ yields that $I^{-1}(I\dot{:}_l I) \subset I^{-1}$ whence $(I\dot{:}_l I) = R$. Similarly, $(I\dot{:}_r I) = R$ and by proposition 1.4., R is a maximal (σ,τ)-order ACC on divisorial ideals contained in R follows from part 1.

Proposition 2.3. if R is an arithmetical (σ,τ)-ring, then ${}_\sigma\mathbb{F}_\tau(R)$ is the free Abelian group generated by the prime ideals of R contained in $\mathcal{L}^2(\sigma,\tau)$, which set we will denote with $\mathcal{P}(\sigma,\tau)$.

Proof

Beacause $A^{-1} = (A:R)$ for any R-ideal A, ${}_{\sigma}\mathbb{F}_{\tau}(R) = {}_{\sigma}\mathbb{D}_{\tau}(R)$ and by prop.2.2. and prop. 1.10., there exists an order preserving isomorphism $\psi: {}_{\sigma}\mathbb{F}_{\tau}(R) \to \mathbb{Z}^{(\Lambda)}$ for some index set Λ. Put $1_i = (\delta_{ij})_{j\in\Lambda}$ and let $P_i = \psi^{-1}(1_i)$. Thus, any R-ideal can be written uniquely as $A = P_1^{n_1}\dots P_k^{n_k}$. We claim that any prime ideal $P \in P(\sigma,\tau)$ equals P_i for some i. For, if P is a maximal ideal contained in $P(\sigma,\tau)$, then $P = P_1^{n_1}\dots P_k^{n_k} \subset P_i$ whence $P = P_i$. Now, let $P \in P(\sigma,\tau)$ and let Q be any maximal ideal containing P, then $\psi(Q) \leq \psi(P)$ whence $P = Q^n P_1^{1_1}\dots P_k^{1_k}$ where $n > 0$. Thus, either $P = Q$ or $P_i \subset P \subset Q = P_j$ for some $1 \leq i \leq k$, a contradiction because $\psi(P_j) \nleq \psi(P_i)$ whenever $i \neq j$. This completes the proof of the claim. Finally, Let P_i be a generator of ${}_{\sigma}\mathbb{F}_{\tau}(R)$ and suppose P_j is a maximal ideal containing P_i, then $P_i = P_j$ whence P_i is a prime ideal contained in $P(\sigma,\tau)$ finishing the proof.

It follows from the above proof that any $P \in P(\sigma,\tau)$ is a maximal ideal of R. Further, if R is an arithmetical (σ,τ)-ring, $\mathcal{L}^2(\sigma,\tau) = \{P_1^{n_1}\dots P_k^{n_k} : P_i \in P(\sigma,\tau)\}$.

Proposition 2.4. : If R is an arithmetical (σ,τ)-ring and let $\mathcal{L}(\rho^l) = \{I <_l R : \exists K \in \mathcal{L}^2(\sigma,\tau); K \subset I\}$, $\mathcal{L}(\rho^r) = \{J <_r R : \exists K \in \mathcal{L}^2(\sigma,\tau), K \subset J\}$, then ρ^l and ρ^r are idempotent T-functors.

Proof

This proof and the following depends heavily on some techniques developed by F. Van Oystaeyen, in the left Noetherian case (cfr.[21, 22, 23]). First, we will check that ρ^l and ρ^r are idempotent. In view of prop. 2.2. $\mathcal{L}(\rho^l)$ has a cofinal set consisting of f.g. R-ideals. Now let J be a left ideal of R such that $\rho^l(I/J) = I/J$ for some $I \in \mathcal{L}(\rho^l)$ which we may suppose to be an ideal f.g. as a left R-module, say $I = Ri_1 + \dots + Ri_k$. Hence, there exists an ideal K in $\mathcal{L}^2(\rho)$ such that $Ki_j \subset J$ for all $1 \leq i \leq k$. Finally, $K I \subset J$ and because $\mathcal{L}^2(\rho)$ is multiplicatively closed,

$J \in \mathcal{L}(\rho^l)$. Because ρ^l is idempotent, $Q^l_\rho(R) = \varinjlim \mathrm{Hom}_R(I,R)$ where the direct limit is taken over all $I \in \mathcal{L}^2(\sigma,\tau)$. It follows that $Q^l_\rho(R) =$ $= \{s \in S \mid \exists\, I \in \mathcal{L}^2(\sigma,\tau) : I\, s \subset R\}$. Likewise, ρ^r is idempotent and $Q^r_\rho = \{s \in S \mid \exists\, I \in \mathcal{L}^2\,(\sigma,\tau) : s\, I \subset R\}$. Clearly $Q^l_\rho(R) = Q^r_\rho(R)$ because $(I{:}_l R) = (I{:}_r R)$ for any ideal $I \in \mathcal{L}^2\,(\sigma,\tau)$.

Now, if I is an ideal contained in $\mathcal{L}^2(\rho)$, then $I\,I^{-1} = R$ where $I^{-1} \subset Q^l_\rho(R)$ whence $Q^l_\rho(R)j_\rho(I) = Q^l_\rho(R)$ and therefore, ρ^l is a T-functor.

If $\sigma,\tau \in F^l(R)$, σ is said to be τ-geometrical if $Q_\tau(R)j_\tau(I)$ is an ideal of $Q_\tau(R)$ for every $I \in \mathcal{L}^2(\sigma)$. A symmetric kernel functor ω is said to be (σ,τ)-geometrical if $\sigma \in F^l(R)$, $\tau \in F^r(R)$ and $Q^l_{\omega^l}(R)j_{\omega^l}(I)$ and $j_{\omega^r}(I)\; Q^r_{\omega^r}(R)$ are ideals of $Q^l_{\omega^l}(R)$ resp. $Q^r_{\omega^r}(R)$ for every $I \in \mathcal{L}^2(\sigma,\tau)$.

<u>Proposition 2.5.</u> : If R is an arithmetical (σ,τ)-ring and if ω is a bilateral kernel functor such that $\mathcal{L}^2(\omega)$ is contained in $\mathcal{L}^2(\sigma,\tau)$ and $\mathcal{L}^2(\omega)$ is multiplicatively closed, then :

1. ω is a (σ,τ)-geometrical T-functor.
2. R is an arithmetical (ω^l, ω^r)-ring.

<u>Proof</u>

1. The fact that ω is an idempotent T-functor is proved as in prop. 2.4. Let $A = R\,a_1+\ldots+Ra_n$ be an ideal contained in $\mathcal{L}^2(\sigma,\tau)$ and let $q \in Q^l_\omega(R)$ such that $I\,q \subset R$ for some $I \in \mathcal{L}^2(\omega) \subset \mathcal{L}^2(\sigma,\tau)$. Because ${}_\sigma\mathbb{F}_\tau(R)$ is a commutative group we have $I(Aq) = AIq \subset AR = A$. Therefore, $Q^l_\omega\,(A)\; I(Aq)$ is contained in $Q^l_\omega(R)A$ whence $Aq \subset Q^l_\omega(R)A$ finishing the proof.

2. Let $A \in {}_{\omega^l}\mathbb{F}_{\omega^r}(R)$, then $A \in {}_\sigma\mathbb{F}_\tau$ because $Q_\omega(R) \subset Q_\rho(R)$ and A is invertible in $Q_\sigma(R)$. There exists an ideal $I \in \mathcal{L}^2(\omega)$ $I\,A \in \mathcal{L}^2(\omega)$ whence $(IA)I^{-1} = I \subset R$ yieling that $A^{-1} \subset Q_\omega(R)$. Therefore, any element of ${}_{\omega^l}\mathbb{F}_{\omega^r}(R)$ is invertible in $Q_\omega(R)$.

<u>Proposition 2.6.</u> : If R is an arithmetical (σ,τ)- ring and if ω is a bilateral kernel functor such that $\mathcal{L}^2(\omega)$ is multiplicatively closed and $\mathcal{L}^2(\omega) \subset \mathcal{L}^2(\sigma,\tau)$, then :

$Q_\omega(R)$ is an arithmetical (σ', τ')-ring, where

$\mathcal{L}(\sigma')$ is generated by $\{Q_\omega(R)I;\ I \in \mathcal{L}(\sigma)\}$ and

$\mathcal{L}(\tau')$ is generated by $\{J\ Q_\omega(R);\ J \in \mathcal{L}(\tau)\}$

<u>Proof</u>

Follows easily from proposition 2.4. and compatibility of kernel functors (cfr. [23]).

Torsion couples can be ordered naturally in the following way $(\sigma,\tau) \leq (\sigma',\tau')$ iff $\sigma \leq \sigma'$ and $\tau \leq \tau'$ (i.e. $\mathcal{L}(\sigma) \subset \mathcal{L}(\sigma')$ and $\mathcal{L}(\tau) \subset \mathcal{L}(\tau')$). Whereas the largest torsion couple (σ,τ)-ring is usually difficult to compute, the largest symmetric kernel functor ρ such that R is an arithmetical (ρ^l,ρ^r)-ring can be described nicely in the following way.

Let R be any ring and let $\mathcal{T}$ be the set of all ideals I of R such that $\{r \in R:\ Ir = o \text{ or } rI = 0\} = 0$ and let S be its Martindale ring of quotients (i.e. $S = \varinjlim \mathrm{Hom}_R(I,R)$ where the direct limit is taken over all $I \in \mathcal{T}$).

Let $\mathcal{A}$ be the set of all ideals of R which are not invertible in S. We claim that $\mathcal{A}$ is inductive under inclusion. For let $\{A_i;\ i \in \Lambda\}$ be an inductive subset of $\mathcal{A}$ and suppose that $\cup A_i = A$ is invertible in S. Then A is a finitely generated left R-module, say, $A = R\,a_1 + \ldots + Ra_n$, but then $A = A_i$ for some $i \in \Lambda$, a contradiction. By Zorn's lemma, $\mathcal{A}$ contains maximal elements. We claim that any such maximal element P is a prime ideal of R. For, let $P \subsetneqq I$, $P \subsetneqq J$ and $I\,J \subset P$, then I and J are invertible whence $I \subset P\,J^{-1} \subset R$ and $J \subset I^{-1}\,P \subset R$, whence $P\,J^{-1}$ and $I^{-1}\,P$ are invertible yielding that P has a left and right inverse in S whence P is invertible, a contradiction.

We will denote with $\mathcal{A}(R)$ the set of all maximal elements of $\mathcal{A}$. For any prime ideal P of R, $\mathcal{L}^2(R-P)$ will be the multiplicatively closed set of all ideals of R not contained in R. Now, let $\mathcal{L}^2(\rho) = \cap\{\mathcal{L}^2(R-P)\,|\,P \in \mathcal{A}(R)\}$, then $\mathcal{L}^2(\rho)$ consists of invertible ideals and therefore is the associated kernel functor ρ an idempotent T-functor. Further, it is easily verified that ρ is the largest symmetric kernel functor such that R is an arithmetical (ρ^1,ρ^r)-ring.

The following proposition generalizes results of G. Bergman- P. Cohn (the center of a 2-fir, [2]) and L. Lesieur (the center of an ipli-ring [11]). It is a special case of proposition 1.13 above but it gives some more insight. With σ_c we will denote the symmetric kernel functor associated to the filter generated by the centrally generated ideals of R.

Proposition 2.7. : The center Z(R) of an arithmetical (σ,τ)-ring R is either a field or a Krull domain if $\mathcal{L}^2(\sigma_c) \subset \mathcal{L}^2(\sigma,\tau)$.

Proof

Because $\mathcal{L}^2(\sigma_c) \subset \mathcal{L}^2(\sigma,\tau)$, any ideal of the form Rc, where c is a nonzero central element, can be written uniquely as $Rc = P_1^{n_1} \ldots P_k^{n_k}$ where $P_i \in \mathcal{P}(\sigma,\tau)$, $n_i \in \mathbb{N}$. The prime ideals occuring in such a decomposition, for c running through Z(R), form a subset $\Lambda(\sigma,\tau)$ of $\mathcal{P}(\sigma,\tau)$ consisting of exactly those $P \in \mathcal{P}(\sigma,\tau)$ satisfying Formanek's condition, i.e. $P \cap Z(R) \neq 0$.

If $\Lambda(\sigma,\tau) = \emptyset$, then there exists no proper ideal Rc , Z(R) is a field.

If $\Lambda(\sigma,\tau) \neq \emptyset$, we will associate to every $P_i \in \Lambda(\sigma,\tau)$ a discrete valuation function v_i on K, the field of fractions of Z(R) by $v_i(c) = n_i$ for all $c \in Z(R)$. The following properties of v_i are easily verified.

(1) $v_i(1) = 0$, (2) $v_i\ (cc') = v_i(c) + v_i(c')$

As for $v_i(c+c')$, $R(c+c') = Rc + Rc' = P_1^{\min(n_1,n'_1)} \dots P_k^{\min(n_k,n'k)}$
yielding that $R(c+c') = P_1^{\min(n_1,n'_1)} \dots P_k^{\min(n_k,n'_k)}\ Q_1^{m_1} \dots Q_l^{m_l}$

Therefore, $v_i(c + c') \geq \min\{v_i(c), v_i(c')\}$. Of course, these valuations can be extended in a unique way to K. They satisfy the finite character property because there are only a finite number of prime ideals occuring in a decomposition of Rk, $k \in K$. Finally
$Z = \{k \in K \mid v_i(k) \geq 0\ \forall P_i \in \Lambda(\sigma,\tau)\} = Z(R)$, for if $k \in Z$, $Rk \subset R$ whence $k \in R \cap K = Z(R)$.

Remarks

(1) Prop. 2.7. is the best result one can hope for, in fact Bergman and Cohn [2] showed that every Krull domain can appear as the center of a left and right principel ideal domains, which are N-ring, cfr. section 3.
(2) The condition $\mathcal{L}^2(\sigma_c) \subset \mathcal{L}^2(\sigma,\tau)$ cannot be dropped, e.g. the coordinate ring of a singular affine curve.
(3) Arithmetical rings satisfying $\mathcal{L}^2(\sigma_c) \subset \mathcal{L}^2(\sigma,\tau)$ have the lying-over property for minimal prime ideals of its center.

3. Some examples

A. Asano orders

Let R be a prime Goldie ring and let σ^l(resp σ^r) the left (resp. right) Goldie torsion theory, i.e. $\mathcal{L}(\sigma^l)$ (resp.$\mathcal{L}(\sigma^r)$) consists of the essential left (resp. right) ideals of R. Then, $Q_\sigma^l(R) = Q_\sigma^r(R) = Q(R)$ is the classical Artinian ring of quotients and the R-ideals are precisely the usual fractical R-ideals (cfr. e.g. [13]). Thus, R is an arithmetical (σ^l,σ^r)-ring if and only if R is an Asano-order. In this setting, Prop. 2.3. is nothing but the classical result that $\mathbb{F}(R)$, the set of fractional R-ideals, is the free Abelian group generated by the non-zero prime ideals of R.

Clearly, $\mathcal{L}^2(\sigma_c) \subset \mathcal{L}^2(\sigma^l,\sigma^r)$ therefore the center of any Asano order is either a field or a Krull domain. Although we did not find any reference of this fact in the literature, we believe it is well known among specialists.

B. HNP-rings

Probably the nicest nontrivial class of arithmetical rings (and indeed the main motivation for their introduction) is the class of HNP-rings. Recall that a left and right Noetherian prime ring is said to be a HNP-ring if every left (right) ideal is a left (right) projective module. Any HNP-ring having an invertible maximal ideal is an arithmetical ring. The idea of arithmetical rings is to study those subsets of the fractional ideals which admit a group structure whereas the complementary part depends on the maximal idempotent ideals and this part can be described nicely by other methods, e.g. the state space of the K_0-theory of Goodearl (cfr.[8]). Probably the arithmetical theory of HNP-rings can be discribed entirely by a combination of these methods. Another intriguing problem in this setting is the connection between the maximal symmetrical kernel functor σ such that R is an arithmetical (σ^l,σ^r)-ring and the open set of birationality. Let us recall some definitions (cfr. [24,25]).

Let A be any ring, X = Spec A its prime spectrum equipped with the Zariski topology. A zariski open set is equal to some $X(I) = \{P \in X | I \not\subset P\}$ where I is an ideal of A. A ringhomomorphism f: A → B is said to be an extension if $B = f(A)\, Z_B(A)$ where $Z_B(A) = \{b \in B \mid \forall a \in A: b\, f(a) = = f(a)\, b\}$. In this case, $f^{-1}(P) \in$ Spec A for any P ∈ Spec B and φ : Spec B → Spec A, $\varphi(P) = f^{-1}(P)$ is a continuous mapping (cfr. [26]). A monomorphic extension f : A → B is said to be a birational extension if there exist nonempty Zariski open sets $Y(I) \subset$ Spec B and $X(J) \subset$ Spec A such that the restriction of φ yields a homomorphism between Y(I) and X(J), the birational extension property is equivalent to $\varphi(Y(I)) = X(J)$ and H = rad B(H ∩ A) for every radical ideal $H \subset$ rad I. A Zariski extension is a birational extension such that $\varphi(I)$ = Spec B. If B is a Zariski extension of its center, B is said to be Zariski central. In [18], E. Nauwelaerts and F. Van Oystaeyen proved that whenever R is an HNP-ring which is a birational extension of its center and $P \in Y(I)$, the open set of birationality, then P is an invertible maximal ideal. As a consequence of this, a birational HNP-ring has only a finite number of idempotent ideals. This result leads us to the following question.

Question 1 : Let R be an HNP-ring birational over its center and let σ be the maximal symmetric kernel functor such that R is an arithmetical (σ^l, σ^r)-ring. Is $P(\sigma^l,\sigma^r) = Y(I)$ where Y(I) is the maximal open set of birationality ? ($Y(I) \subset P(\sigma^l,\sigma^r)$ follows from the result of Nauwelaerts - Van Oystaeyen).

When R is a p.i. HNP-ring, question 1 can be answered affirmatively using some results of B. Mueller ([17]) and the fact that invertible prime ideals have the left and right Ore-condition.

C: N-rings, a question of L. Lesieur

R is an N-ring if it is an arithmetical (σ_N^l, σ_N^r)-ring where $\mathcal{L}^2(\sigma_N)$ = $\{I \lhd R \mid I$ contains a normalizing element$\}$. For any overring S of R, $N_R(S)$ will denote the set of all R-normalizing elements $\{s \in S \mid s R = R s\}$. In [11] , L. Lesieur posed the following question : if R is a prime left principal iedal ring, Z(R) its center, k = Q(Z(R)) the field of fractions of Z(R) and K = Z(Q(R)) the center of the simple Artinian ring of quotients, is k = K ?

He proved the following result : (cfr. [11])

Theorem (L. Lesieur) If R is a prime left principal ideal ring, k is algebraically closed in K.

We aim to generalize this result to arbitrary N-rings. Let us start with defining so called pseudo-valuations on the R-normalizing elements of $Q_N(R)$. Let $\{P_i;\ i \in \Lambda\} = \mathcal{P}(\sigma_N^1, \sigma_N^r)$. For any element n in $N_R(Q_N(R))$ we have $Rn = P_1^{n_1}\ldots P_k^{n_k}$ for some $i \in \Lambda$, $n_i \in \mathbb{Z}$. Now, let us define :

$v_i = N_R(Q_N(R)) \to \mathbb{Z} : v_i(n) = n_i$

It is straightforward to check that $v_i(1) = 0$ and $v_i(n.n') = v_i(n)+v_i(n')$. Further, if $n_1,\ldots,n_k$ are R-normalinzing elements in $Q_N(R)$ such that $n_1+\ldots+n_k$ is R-normalizing, then $v_i(n_1+\ldots+n_k) \geq \min\{1 \leq j \leq k,\ v_i(n_j)\}$

Proposition 3.1. If R is an N-ring and $x \in Z(Q_N(R))$ such that x is integral over Z(R), then $x \in Z(R)$.

Proof

Clearly, x is an R-normalizing element of $Q_N(R)$, whence $Rx = P_1^{n_1}\ldots P_k^{n_k}$ for some $P_i \in \mathcal{P}(\sigma_N^1, \sigma_N^r)$ and some integers $n_i \in \mathbb{Z}$. If every $n_i \geq 0$, then $x \in R \cap Z(Q_N(R)) = Z(R)$. Therefore, let us assume that $v_i(x) < 0$ and that $x^n + c_1x^{n-1}+\ldots+c_n = 0$ where $c_i \in Z(R)$. Because $c_ix^{n-i} \in N_R(Q_N(R))$ and $x^n \in N_R(Q_N(R))$, it follows that $v_i(x^n) = n\, v_i(x) \geq \min\{v_i(c_jx^{n-j});\ 1 \leq j \leq n\} = \min\{v_i(c_j) + (n-j)v_i(x);\ 1 \leq j \leq n\}$, a contradiction because $v_i(x) < 0$ and $v_i(c_j) \geq 0$.

Proposition 3.2. : If R is an N-ring, then $k = Q(Z(R))$ is algebraically closed in $K = (Z(Q_N(R))$.

Proof

It is easy to check that $Z(Q_N(R))$ is a field. Let $x \in K\setminus k$ and suppose that x is algebraic over k, say $x^n+k_1 x^{n-1}+\ldots+k_n = 0$ where $k_i \in k$,

$1 \le i \le n$. Clearing the denominators yields : $Z_o x^n + Z_1 x^{n-1} + \ldots + Z_n = 0$ where $Z_i \in Z(R)$ and $Z_o \neq 0$. Now let $1 = Z_o x$, then : $t^n + Z_o Z_1 t^{n-1} + \ldots + Z_o^{n-1} Z_n = 0$. By Prop. 3.1., $t \in Z(R)$ whence $x = tZ_o^{-1} \in k$ finishing the proof. □

If R is a prime left principal ideal ring, either R contains no normalizing non-unit elements (in which case Z(R) is a field), or R is an N ring (follows from [11, Propriété 6]). Hence, Prop. 3.2. extends Lesieur's result whereas Prop. 2.6. generalizes [11, Th. 3] because clearly $\sigma_c \le \sigma_N$.

Question 2: If R is an N-ring, is $Q(Z(R)) = Z(Q_N(R))$?

We will extend this question to one about arbitrary arithmetrical rings, the solution of which is of some importance in order to study class groups (cfr. section 4 and part II of this paper). Let R be an arithmetical (σ,τ)-ring. In order to study the relation between Z(R) and $Z(Q_\sigma(R))$ it is natural to consider the so called (σ,τ) -normal closure of R in $Q_\sigma(R)$. $N_R(Q_\sigma(R))$ is the set of R-normalizing elements of $Q_\sigma(R)$, and $N_R^{\sigma,\tau}(Q_\sigma(R))$ will be the subset consisting of those elements q such that $Rq \in {}_\sigma\mathbb{F}_\tau(R)$. The normal closure of R in $Q_\sigma(R)$ is the set $R_N = RN_R(Q_\sigma(R))$ whereas the (σ,τ)-normal closure in the set $R_N^{\sigma,\tau} = R\, N_R^{\sigma,\tau}(Q_\sigma(R))$. It is easy to check that both R_N and $R_N^{\sigma,\tau}$ are prime rings. Furthermore, $Z(R_N) = Z(Q_\sigma(R))$ whereas $Z(R) \subset Z(Q_\sigma(R))$. For, let $z \in Z(R_N^{\sigma,\tau})$ and $x \in Q_\sigma(R)$ then $I\,x \subset R$ for some $I \in \mathcal{L}(\sigma)$ whence $i\,Z\,x = Z\,i\,x = i\,x\,Z$ for every $i \in I$ yielding that $I(Zx-xZ) = 0$ whence $Zx - xZ = 0$.

Question 3 : What is the relation between $Q(Z(R))$, $Q(Z(R_N^{\sigma,\tau})$ and $Q(Z(Q_\sigma(R)))$? In particular, if $\mathcal{L}^2(\sigma_c) \subset \mathcal{L}^2(\sigma,\tau)$, is $Q(Z(R)) = Q(Z(R_N^{\sigma,\tau}))$?

D : Ω-Rings

An S-ring is an arithmetical $(\sigma^l_{R-o}, \sigma^r_{R-o})$-ring where $\mathcal{L}^2(\sigma_{R-o})$ is the set of all nonzero ideals of R. (E.g. a Goldie S-ring is nothing but an

Asano-order). An S-ring R is said to be an Ω-ring if R satisfies Formanek's condition, i.e. if every nonzero ideal has nontrivial central intersection. In this case, we conjecture that there is a strong relation between R and its center. In particular, we pose the following questions.

Question 4 : If R is an Ω-ring, is Z(R) a Dedekind domain ?

Question 5 : If R is an Ω-ring such that Z(R) is a Dedekind domain. Is R Zariski central ?

Remarks

(1) : If R is an Ω-ring satisfying a polynomial identity, then R is p.i. Asano whence Dedekind and a result of Robson's [19] yields that Z(R) is a Dedekind domain. Zariski centrality follows from some results of E. Nauwelaerts- F. Van Oystaeyen [18]. The non-p.i. cases of questions 4 and 5 remain open.

(2) : Question 5 is a generalization of the Nauwelaerts - Van Oystaeyen conjecture ([18]) asking whether a bounded Dedekind prime ring satisfying Formanek's condition is Zariski central. But even the validity of this conjecture remains obscure because it would imply that a noncommutative discrete valuation ring (cfr. [20]) in a skewfield D with center K having nontrivial valuation on K is invariant under K-automorphisms of D. Of course, when D satisfies a polynomial identity this is obvious.

4. Picard and class groups of arithmetical rings

If R is a maximal (σ,τ)-order in S, then ${}_\sigma\mathbb{D}_\tau(R)$ the set of all divisorial R-ideals is a commutative group under $*$-multiplication. ${}_\sigma\mathbf{P}_\tau(R)$ will be the subgroup of ${}_\sigma\mathbb{D}_\tau(R)$ of divisorial R-ideals Rn where $n \in N^{\sigma,\tau}(R) =$
$= \{ s \in S \mid Rs = sR \text{ and } Rs \in {}_\sigma\mathbf{F}_\tau(R)\}$.

<u>Definition 4.1. :</u> The class group, ${}_\sigma Cl_\tau(R)$, of a maximal (σ,τ)- order R in S is the group ${}_\sigma\mathbb{D}_\tau(R)/{}_\sigma\mathbb{P}_\tau(R)$.

<u>Remark</u> : this definition coincides with the usual notion of a class group for completely integrally closed commutative domains (e.g. Krull domains), with the Chamarie class group for nonommutative Krull orders (cfr. [4])and with the normalizing class group for Ω-Krull rings.

If R is an arithmetical (σ,τ)-ring, we give a K-theoretic interpretation of the class group which will enable us in part II of this paper to deduce Maeyer-Vietoris sequences relating class groups in ringextensions.

Let us recall some definitions of abstract K-theory (cfr.[1]) :

A product $\perp$ on a category $\underline{\underline{A}}$ is a functor :

$\perp : \underline{\underline{A}} \times \underline{\underline{A}} \to \underline{\underline{A}}$

which is coherently associative and commutative in the sense of MacLane (cfr.[12]. This means that there are natural isomorphisms :

$\perp \circ (1_{\underline{\underline{A}}} \times \perp) \simeq \perp \circ (\perp \times 1_{\underline{\underline{A}}}) : \underline{\underline{A}} \times \underline{\underline{A}} \times \underline{\underline{A}} \to \underline{\underline{A}}$, and $\perp \circ t \simeq \perp : \underline{\underline{A}} \times \underline{\underline{A}} \to \underline{\underline{A}}$

where t denotes the transposition of $\underline{\underline{A}} \times \underline{\underline{A}}$. The "coherence" of these isomorphisms requires that isomorphisms of products of several factors, obtained from the above by succession of three fold reassociation and twofold permutation, are all the same.

If $\underline{\underline{A}}$ is a category with product, its Grothendieck group is an Abelian group $K_0(\underline{\underline{A}})$ supplied with a map : $[\ ]_{\underline{\underline{A}}} : Ob(\underline{\underline{A}}) \to K_0(\underline{\underline{A}})$.

which is universal for maps into an Abelian group satisfying :

(Ka) : if $A \cong B$, then $[A]_{\underline{\underline{A}}} = [B]_{\underline{\underline{A}}}$

(Kb) : $[A \perp B]_{\underline{\underline{A}}} = [A]_{\underline{\underline{A}}} \perp [B]_{\underline{\underline{A}}}$

This means that any map $f : Ob(\underline{\underline{A}}) \to G$, where G is an Abelian group, satisfying the analogues of (Ka) and (Kb) is of the form $f(A) = f_0([A]_{\underline{\underline{A}}})$

for a unique homomorphism $f_0 : K_0(\underline{\underline{A}}) \to G$.
To construct $K_0(\underline{\underline{A}})$ we form the free Abelian group with the isomorphism classes of $Ob(\underline{\underline{A}})$ as a basis, and then factor out the subgroup generated by relations corresponding to (Kb).

Now, let $\mathcal{F}(\sigma,\tau)$ be the category with objects $Ob(\mathcal{F}(\sigma,\tau)) = {}_\sigma\mathbb{F}_\tau(R)$ and morphisms $[A,B]_{\mathcal{F}} = Hom_R(A,B)$, the left R-module morphisms from A to B. Remark that any left R-module morphism $f : A \to B$ can be extended uniquely to a left $Q_\rho(R)$-module morphism.
$\bar{f} : Q_\rho(R) = Q_\rho(R)\, A \to Q_\rho(R) = Q_\rho(R)B$
where $Q_\rho(R)$ is the localization at the symmetric T-functor (prop. 2.4.) determined by the filter generated by $\mathcal{L}^2(\sigma,\tau)$. Hence, $[A,B]_{\mathcal{F}} = (A\underset{r}{:}B)\rho$ $= \{ s \in Q_\rho(R) | A s \subset B\}$. Let us define a product on $\mathcal{F}(\sigma,\tau)$ in the following way : $A, B \in Ob\ (\mathcal{F}(\sigma,\tau)) = {}_\sigma\mathbb{F}_\tau(R)$, then $A \perp B = AB$ and if $q \in [A,B]_{\mathcal{F}} =$ $= (A\underset{l}{:}B)\rho$ and $q' \in [C,D]_{\mathcal{F}} = (C\underset{r}{:}D)\rho$, we claim that $q \perp q' = qq' \in [A \perp D]_{\mathcal{F}} =$ $= (AC\underset{r}{:}BD)_\rho$, for, $AC\, qq' = CA\, qq' \subset CBq' = BCq' \subset BD$.
Remark that $.\perp.$ is not necessarily a commutative product because it may be non-commutative on the morphism level. Nevertheless, it is possible to define an Abelian Grothendieck group $K_0(\mathcal{F}(\sigma,\tau))$ as above because $.\perp.$ is commutative on the object level. Similarly, one could define a category $\mathcal{F}'(\sigma,\tau)$ with $Ob\ (\mathcal{F}(\sigma,\tau)) = Ob(\mathcal{F}'(\sigma,\tau))$ and with morphisms $[A,B]_{\mathcal{F}'} = {}_R Hom(A,B)$, the right R-module morphisms from A to B. It follows from the proposition below that $K_0(\mathcal{F}(\sigma,\tau)) = K_0(\mathcal{F}'(\sigma,\tau))$.

<u>Proposition 4.2. :</u> if R is an arithmetical (σ,τ)-ring, then ${}_\sigma Cl_\tau(R) \cong$ $K_0(\mathcal{F}\ (\sigma,\tau))$.

<u>Proof</u>

Because ${}_\sigma\mathbb{F}_\tau\ (R)$ is an abelian group, $K_0(\mathcal{F}(\sigma,\tau)) \cong {}_\sigma\mathbb{F}_\tau(R)/N$ where N is the subgroup of those R-ideals I which are isomorphic to R, i.e. there exist elements g and h in S such that $Ig \subset R$, $Rh \subset I$ and

$igh = i$, $rhg = r$ for all $i \in I$ and $r \in R$ yielding that $gh = hg = 1$ and thus $Ig = R$ and $Rh = I$. Clearly, $g \in I^{-1}$ and $\Sigma g_k i_k = 1$ for some $g_k \in I^{-1}$, $i_k \in I$, whence $g = \Sigma g_k (i_k g) = \Sigma g_k j_k$ where $j_k \in R$. Thus, $g r = \Sigma g_k (j_k r) = \Sigma g_k 1_k g \in Rg$ because $1_k \in I$. Therefore, $g R \subset R g$ and similarly $h R \subset R h$. Thus, $R = h g R \subset h R g \subset R h g = R$ whence $h R g = R$ and $R g = g R$, $R h = h R$ yielding that g and h are R - normalizing. Thus, $I \in {}_\sigma\mathbb{P}_\tau(R)$ yielding that $K_0(F(\sigma,\tau)) \cong$ $\cong {}_\sigma\mathbb{F}_\tau(R)/{}_\sigma\mathbb{P}_\tau(R) = {}_\sigma Cl_\tau(R)$.

If D is a commutative Dedekind domain, there is a natural isomorphism between Cl(D) and Pic(D), the Picard group of D. This fact may for example be proved by establishing an isomorphism between the Weil divisors and the Cartier divisors on the affine scheme Spec D. Although one can study Cartier divisors on more general noncommutative schemes, we restrict ourselves in this note to introducing Cartier divisors on a suitable sheaf of rings associated with an arithmetical (σ,τ)-ring (which, of course, coincides with the usual affine scheme if R is a commutative Dedekind domain). The Cartier divisor classes will be denoted by ${}_\sigma Pic_\tau(R)$ and we will examine its relation to ${}_\sigma Cl_\tau(R)$, (note that there is no obvious relation between ${}_\sigma Pic_\tau(R)$ and the usual Picard group of R in terms of invertible R-R-bimodules (cfr.[5])).

Let us fix some notation : throughout R will be an arithmetical (σ,τ)-ring, $Q = Q_\rho(R)$ where $\mathcal{L}^2(\rho) = \mathcal{L}^2(\sigma,\tau)$, cfr. prop. 2.5. For any symmetric kernel functor ν, $Q_\nu^\rho(R)$ will be the left (or right) localization of R at the kernel functor assiciated with $\mathcal{L}^2(\rho) \cap \mathcal{L}^2(\nu)$. It follows from prop 2.7. that $Q_\nu^\rho(R)$ is an arithmetical (σ_ν,τ_ν)-ring, where $\mathcal{L}(\sigma_\nu) =$ $= \{Q_\nu^\rho(R)\, I \mid I \in \mathcal{L}(\sigma)\}$ and $\mathcal{L}(\tau_\nu) = \{J\, Q_\nu^\rho(R) \mid J \in \mathcal{L}(\tau)\}$.

We make $X = \{x\} \cup P(\sigma,\tau)$ (where x is some abstract element playing the role of 0 if R is a prime ring) into a topological space by inducing on $\mathcal{P}(\sigma,\tau)$ the Zariski topology and taking for Open (X) the set $\{ \emptyset, \{x\} \cup V$ where $V \in$ Open $(\mathcal{P}(\sigma,\tau))\}$. In many cases, this topology will be homeomorphic with the cofinite topology on $\mathcal{P}(\sigma,\tau)$.

We will define two sheafs of rings on X. The first, $\tilde{Q}$, is the constant sheaf corresponding to Q. The second, $\tilde{R}_\rho$ (or ${}_\sigma\tilde{R}_\rho$ is we want to stress the role of σ and τ), is a sheaf of arithmetical rings defined : $\Gamma(V, \tilde{R}_\rho) = Q_V^\rho(R)$ where $\mathcal{L}^2(V) = \cap \{ \mathcal{L}^2(R-P) : P \in V \cap \mathcal{P}(\sigma,\tau)\}$. Clearly $\Gamma(V, \tilde{R}_\rho) = \cap \{Q_{R-P}^\rho(R) : P \in V \cap \mathcal{P}(\sigma,\tau)\}$ and restriction morphisms are inclusions. It is readily verified that $\tilde{R}_\rho$ is indeed a sheaf. R can be recovered from it by taking global section, for, if $s \in S$ and $s \in \Gamma(X,\tilde{R}_\rho)$, then there exists an ideal $I_p \in \mathcal{L}^2(\sigma,\tau) \cap \mathcal{L}^2(R-P)$ for any $P \in \mathcal{P}(\sigma,\tau)$ such that $I_p.s \subset R$, whence $(\Sigma I_p) s \subset R$ and $\Sigma I_p \in \mathcal{L}^2(\sigma,\tau)$ which is not contained in any $P \in \mathcal{P}(\sigma,\tau)$, hence $\Sigma I_p = R$ and therefore $s \in R$.

As was mentioned above, $Q_V^\rho(R)$ is an arithmetical (σ_ν,τ_ν)-ring. We will mainly be interested in the following two sheafs of (not necessarily Abelian) groups:

$U(\tilde{R}_\rho)$ is defined by $\Gamma(V, U(\tilde{R}_\rho)) = U(\Gamma(V, \tilde{R}_\rho))$ where U(.) denotes : taking units, restriction morphisms are inclusions.

$N(\tilde{R}_\rho)$ is defined by $\Gamma(V, N(\tilde{R}_\rho)) = N_Q^{\sigma_\nu,\tau_\nu}(Q_V^\rho(R))$ and restriction morphisms are inclusions. Let us check that $N(\tilde{R}_\rho)$ is a sheaf. If $W \subset V$, $Q_W^\rho(R)$ can be obtained from $Q_V^\rho(R)$ by localizing at a symmetric kernel functor, say ξ, such that $\mathcal{L}^2(\xi) \subset \mathcal{L}^2(\sigma_\nu,\tau_\nu)$ whence it is (σ_ν,τ_ν)- geometric yielding that for any $n \in N_Q^{\sigma_\nu,\tau_\nu}(Q_V^\rho(R))$, $Q_\xi(Q_V^\rho(R)n) = n\, Q_\xi(Q_V^\rho(R))$ whence $n \in N_Q^{\sigma_W,\tau_W}(Q_W^\rho(R))$. Now, $N(\tilde{R}_\rho)$ is clearly seperated because it is a subpresheaf of $\tilde{Q}$.

As for the second sheaf condition : let V be an open covering of V and suppose that $s \in \Gamma(V_i, N(\tilde{R}_\rho))$ for every i, then $s\, Q_V^\rho(R) =$

$= s(\cap Q^{\rho}_{V_i}(R)) \subset (\cap Q^{\rho}_{V_i}(R))\ s = Q^{\rho}_{V}(R)s$ and similarly $Q^{\rho}_{V}(R)s \subset sQ^{\rho}_{V}(R)$ whence $s \in \Gamma(V, N(\tilde{R}\rho))$. Finally, remark that $U(\tilde{R}\rho)$ is a normal subsheaf of $N(\tilde{R}\rho)$.

Definition 4.3. A Cartier divisor on X is a global section of the sheaf $N(\tilde{R}\rho)/U(\tilde{R}\rho)$. Thinking of the properties of quotient sheaves, we see that a Cartier divisor on X can be described by ginving an open cover $\{V_i\}$ of X, and for each i an element $s_i \in \Gamma(V_i, N(\tilde{R}\rho))$ such that for each i and j, $s_i s_j^{-1} = U_{ij} \in \Gamma(V_i \cap V_j, U(\tilde{R}\rho))$.
A Carter divisor is principal if it is an image of the natural map $\Gamma(X, N(\tilde{R}\rho)) \to \Gamma(X, N(\tilde{R}\rho)/U(\tilde{R}\rho))$. Two Cartier divisors are linearly equivalent if their quotient is principal. The group of Cartier divisor classes will be denoted by $\mathrm{Pic}_{\rho}(R)$ (or ${}_{\sigma}\mathrm{Pic}_{\tau}(R)$).
X is said to be locally factorial if ${}_{\sigma^{\rho}_{R-P}}\mathrm{Cl}_{\tau^{\rho}_{R-P}}(Q^{\rho}_{R-P}(R)) = 1$ for every prime $P \in P(\sigma,\tau)$.

Proposition 4.4. : If R is an arithmetical (σ,τ)-ring then :

1. ${}_{\sigma}\mathrm{Pic}_{\tau}(R) \hookrightarrow {}_{\sigma}\mathrm{Cl}_{\tau}(R)$
2. If X is locally factorial, ${}_{\sigma}\mathrm{Pic}_{\tau}(R) \cong {}_{\sigma}\mathrm{Cl}_{\tau}(R)$.

Proof

(1) : Let a Cartier divisor be given by $\{(V_i, s_i)\}$ where $\{V_i\}$ is an open covering of X, and $s_i \in \Gamma(V_i, N(\tilde{R}\rho))$.
Of course, the index set can be chosen to be finite because X is a Noetherian space. Let $P \in P(\sigma,\tau)$ and suppose that $P \in V_i$. With n_P we will denote the $Q^{\rho}_{V_i}(R)P$ power occuring in the decomposition of $Q^{\rho}_{V_i}(R)s_i$. Of course, n_p is invariant under the choice of a particular V_i because $s_i s_j^{-1} \in \Gamma(V_i \cap V_j, U(\tilde{R}\rho))$. There are only a finite number of $P \in P(\sigma,\tau)$ such that $n_p \neq o$. We define the associated fractional ideal of $\{(V_i, s_i)\}$

to be πP^{n_p}. This correspondence yields a group monomorphism f from $\Gamma(X, N(\tilde{R}\rho)/U(\tilde{R}\rho))$ into ${}_\sigma\mathbb{F}_\tau(R)$. Further, any image of the natural map $\Gamma(X, N(\tilde{R}\rho)) \to \Gamma(X, N(\tilde{R}\rho)/U(\tilde{R}\rho))$ corresponds to an element of ${}_\sigma\mathbb{P}_\tau(R)$. Thus, f induces a monomorphism $\bar{f} : {}_\sigma \mathrm{Pic}_\tau(R) \to {}_\sigma \mathrm{Cl}_\tau(R)$.

(2) Suppose that X is locally factorial and let $A = P_1^{n_1} \ldots P_k^{n_k}$ be a fractional R-ideal. For any $P \in \mathcal{P}(\sigma,\tau)$, $Q^\rho_{R-P}(R)A = Q^\rho_{R-P}(R)\, f_p$ for some $f_p \in N(\tilde{R}\rho)_p$. There exists a neighbourhood V_p of P such that f_p lives on V_p. Covering X with such open sets V_p, the element f_p gives a Cartier divisor on X. Note that if f, f' give the same fractional ideal on an open set V, then $f/f' \in \Gamma(V,U(\tilde{R}\rho))$, thus we have a well defined Cartier divisor. This construction is clearly inverse to the one of part (1) and elements of ${}_\sigma\mathbb{P}_\tau(R)$ correspond to principal Cartier divisors whence ${}_\sigma\mathrm{Pic}_\tau(R) \cong {}_\sigma\mathrm{Cl}_\tau(R)$.

Remark 4.5. : If R is an S-ring, $\tilde{R}\rho$ is the usual affine scheme of R (cfr. [21, 26]). If R is an Ω-ring, $\mathrm{Cl}_s(R)/\mathrm{Pic}_s(R)$ is a torsion group. Of course, if D is a Dedekind domain (or more generally, a bounded Noetherian Asano order), then X = Spec R is locally factorial and we recover the classical result : $\mathrm{Cl}(R) \cong \mathrm{Pic}(R)$, (in these cases $\mathrm{Pic}_s(R) \cong \mathrm{Pic}(R)$ where Pic(R) is the Fröhlich Picard group, cfr. [5]).

To end this section, we present a cohomological interpretation of ${}_\sigma\mathrm{Pic}_\tau(R)$. We have the exact sequence of non-Abelian sheaves : $1 \to U(\tilde{R}\rho) \to N(\tilde{R}\rho) \to N(\tilde{R}\rho)/U(\tilde{R}\rho) \to 1$ and ${}_\sigma\mathrm{Pic}_\tau(R) = \mathrm{Coker}(\Gamma(X,N(\tilde{R}\rho)) \to \Gamma(X, N(\tilde{R}\rho)/U(\tilde{R}\rho)))$. Whence ${}_\sigma\mathrm{Pic}_\tau(R) = \mathrm{Ker}\,(H^1(X, U(\tilde{R}\rho)) \to H^1(X, N(\tilde{R}\rho)))$ where the $H^1(.)$ are the non Abelian cohomology pointed sets of [6] . Luckily, one can also give an interpretation of ${}_\sigma\mathrm{Pic}_\tau(R)$ in terms of usual (i.e. Abelian) Čech-cohomology.

<u>Proposition 4.6.</u> : if R is an arithmetical (σ,τ)-ring we have the follwing exact sequence

$$1 \to Sh^{ab}(N(\tilde{R}\rho),\ U(\tilde{R}\rho)) \to {}_\sigma Pic_\tau(R) \to Ker(\check{H}^1(X,\ U(\tilde{R}\rho)^{ab}) \to \check{H}^1(X,\ N(\tilde{R}\rho)^{ab})) \to 1$$

where $Sh^{ab}(N(\tilde{R}\rho),\ U(\tilde{R}\rho)) = Coker\ \pi_2/\ Coker\ \pi_1$, with $\pi_1 : \Gamma(X,\ U(\tilde{R}\rho)) \to \Gamma(X,\ U(\tilde{R}\rho)^{ab})$; $\pi_2 : \Gamma(X,\ N(\tilde{R}\rho)) \to \Gamma(X,\ N(\tilde{R}\rho)^{ab})$.

<u>Proof</u>

Because fractional ideals commute in an arithmetical ring, $N(\tilde{R}\rho)/U(\tilde{R}\rho)$ is a sheaf of Abelian groups.

Therefore we have the exact diagram :

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U(\tilde{R}\rho) & \longrightarrow & N(\tilde{R}\rho) & \xrightarrow{\pi} & N(\tilde{R}\rho)/U(\tilde{R}\rho) & \longrightarrow & 1 \\
 & & & & \downarrow & & \| & & \\
1 & \longrightarrow & Ker\ \bar{\pi} & \longrightarrow & N(\tilde{R}\rho)^{ab} & \xrightarrow{\bar{\pi}} & N(\tilde{R}\rho)/U(\tilde{R}\rho) & \longrightarrow & 1
\end{array}$$

where $\tilde{M}^{ab}$ denotes the abelianized sheaf of the sheaf of groups $\tilde{M}$ (i.e the sheafification of the presheaf with sections $\Gamma(U,\ \tilde{M})/[\Gamma(U,\tilde{M}),\ \Gamma(U,\tilde{M})]$.

By local consideration, one verifies easily that $Ker\ \bar{\pi} \cong U(\tilde{R}\rho)^{ab}$, thus we get a long exact sequence using čech-cohomology

$$1 \to \Gamma(X,\ U(\tilde{R}\rho)^{ab}) \to \Gamma(X, N(\tilde{R}\rho)^{ab}) \to \Gamma(X,\ N(\tilde{R}\rho)/\ U(\tilde{R}\rho)) \to$$

$$\check{H}^1(X,\ U(\tilde{R}\rho)^{ab}) \to \check{H}^1(X,\ N(\tilde{R}\rho)^{ab})$$

whence $Coker\ (\Gamma(X,\ N(\tilde{R}\rho)^{ab}) \to\ \Gamma(X,\ N(\tilde{R}\rho)/\ U(\tilde{R}\rho))) \cong Ker(\check{H}^1(X,\ U(\tilde{R}\rho)^{ab}) \to \check{H}^1(X,\ N(\tilde{R}\rho)^{ab})$.

The result follows from applying the snake lemma (which is possible in this setting!) on the following two exact diagrams :

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & \operatorname{Ker}\beta & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \operatorname{Im}\Gamma(X, N(\tilde{R}_\rho)) & \longrightarrow & \Gamma(X, N(\tilde{R}_\rho)/U(\tilde{R}_\rho)) & \longrightarrow & {}_\sigma\mathrm{Pic}_\tau(R) & \longrightarrow & 1 \\
 & & \alpha\downarrow & & \downarrow & & \downarrow\beta & & \\
1 & \longrightarrow & \operatorname{Im}\Gamma(X, N(\tilde{R}_\rho)^{ab}) & \xrightarrow{\gamma} & \Gamma(X, N(\tilde{R}_\rho)/U(\tilde{R}_\rho)) & \longrightarrow & \operatorname{Coker}\gamma & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Coker}\alpha & & 1 & & 1 & &
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
1 & \longrightarrow & \Gamma(X, U(\tilde{R}_\rho)) & \longrightarrow & \Gamma(X, N(\tilde{R}_\rho)) & \longrightarrow & \operatorname{Im}\Gamma(X, N(\tilde{R}_\rho)) & \longrightarrow & 1 \\
 & & \downarrow\pi_1 & & \downarrow\pi_2 & & \downarrow & & \\
1 & \longrightarrow & \Gamma(X, U(\tilde{R}_\rho)^{ab}) & \longrightarrow & \Gamma(X, N(\tilde{R}_\rho)^{ab}) & \longrightarrow & \operatorname{Im}\Gamma(X, N(\tilde{R}_\rho)^{ab}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Coker}\pi_1 & & \operatorname{Coker}\pi_2 & & \operatorname{Coker}\alpha & &
\end{array}
$$

and the above description of Coker γ.

Remark 4.7. : If R is a commutative Dedekind domain, $N(\tilde{R})$ is a constant sheaf whence $\check{H}^1(X, N(\tilde{R})) = 1$ and further $Sh^{ab}(N(\tilde{R}), U(\tilde{R})) = 1$ yielding the result that Pic R $= \check{H}^1(X, U(\tilde{R}))$. In part II of this paper we will study the functorial properties of ${}_\sigma\mathrm{Pic}_\tau$ (R).

L. Le Bruyn
Dept. Mathematics UIA
Universiteitsplein 1
B-2610 Wilrijk
Belgium

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