A CHARACTERIZATION OF CENTRAL Ω-KRULL RINGS

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0. INTRODUCTION

In [5] we introduced the notion of an Ω-Krull ring. It turned out that any Ω-Krull ring R is the intersection of quasi-local Ω-rings R_i which are symmetric localizations of R with respect to σ_i where $\mathcal{L}(\sigma_i) = \mathcal{L}(R \setminus P_i)$ and P_i is a prime ideal of R. If one assumes each σ_i to be a geometrical kernel functor, then the set of these prime ideals P_i equals $X^1(R)$, i.e. the set of all nonzero minimal prime ideals of R.

Throughout this note, each σ_i will be a central kernel functor. In the second section, we prove an intrinsic characterization of central Ω-Krull rings (Theorem 2.1). Two conditions in this characterization are similar to the characterizing properties in the commutative case whereas a third condition is needed in order to make the localized rings local and the localizations central. In the final section, we derive a necessary and sufficient condition under which R[T] remains central Ω-Krull if R is. Moreover, it's perhaps interesting to see that this condition may be reduced to the algebraicity of certain field extensions.

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1. PRELIMINARIES

Throughout this note, R will be a prime ring satisfying Formanek's condition, i.e. every nonzero ideal of R intersects C, the center of R, non-trivially. In this case, $Q_{sym}(R) = \{c^{-1}r = rc^{-1} | r \in R,\ 0 \neq c \in C\}$ is a simple ring. Moreover, $Q_{sym}(R) \cong Q^{\ell}_{R\backslash 0}(R) \cong Q^{r}_{R\backslash 0}(R)$, where $Q^{\ell}_{R\backslash 0}(R)$ denotes the localization of the left R-module R with respect to the symmetric filter $\mathcal{L}(R\backslash 0)$, cfr. [11] and [12].

In [5] we defined an Ω-Krull ring to be a ring R satisfying Formanek's condition such that :

(1) there exist filters of ideals of R, $\mathcal{L}^2(\sigma_i)$ $(i \in \wedge)$ such that

$$R_i = Q^{\ell}_{\sigma_i}(R) = \{q \in Q_{sym}(R) \mid \exists I \in \mathcal{L}^2(\sigma_i) : Iq \subset R\}$$
$$= Q^{r}_{\sigma_i}(R) = \{q \in Q_{sym}(R) \mid \exists I \in \mathcal{L}^2(\sigma_i) : qI \subset R\};$$

(2) for all $i \in \wedge$, R_i is a quasi-local Ω-ring (cfr. [12]), i.e. every ideal of R_i is a power of the unique maximal ideal P'_i of R_i;

(3) $R = \bigcap_{i \in \wedge} R_i$;

(4) for every $i \in \wedge$ and for all $I \in \mathcal{L}^2(\sigma_i)$: $R_iI = IR_i = R_i$;

(5) for all $r \in R$ there are only finitely many $i \in \wedge$ such that $RrR = (r) \notin \mathcal{L}^2(\sigma_i)$.

In [5] we proved that $\mathcal{L}^2(\sigma_i) = \mathcal{L}^2(R\backslash P_i)$ where $P_i = P'_i \cap R$ and $\mathcal{L}^2(R\backslash P_i) = \{I | I \text{ an ideal of } R \text{ such that } I \not\subset P_i\}$. In this paper we will assume the extra condition that every σ_i is a central kernel functor, i.e. $\mathcal{L}^2(\sigma_i)$ has a cofinal set consisting of centrally generated ideals. This is equivalent to saying that $I \in \mathcal{L}^2(\sigma_i)$ iff $R(I \cap C) \in \mathcal{L}^2(\sigma_i)$. In particular, σ_i is geometric. If R is an Ω-Krull ring such that moreover, each σ_i is a central kernel functor, then we say that R is a <u>central Ω-Krull ring</u>. In this case, $R_i = \{c^{-1}r | c \in C\backslash p_i,\ r \in R\}$

where $p_i = P_i \cap C$. Hence C_i, the center of R_i, equals $Q_{p_i}(C)$. Because C_i is a discrete valuation ring (cfr. [5]), p_i has to be a prime ideal of C of height one. Conversely, C being a Krull domain ([5]) yields that every nonzero minimal prime ideal of C has to be a p_i for some $i \in \wedge$ (cfr. [2],[4]). By the definition of R_i, it follows that P_i is the only nonzero minimal prime ideal of R lying over p_i, yielding a one-to-one correspondence between $X^1(R)$, the set of all minimal nonzero prime ideals of R, and $X^1(C)$.

A ring S is said to be related to R if $R \subset S \subset Q_{sym}(R)$ and if $cS \subset R$ for some nonzero $c \in C$. R is called a symmetric maximal order if there does not exist a ring S related to R, except for R itself.

Recall that a fractional R-ideal I is a twosided R-submodule of $Q_{sym}(R)$ such that $cI \subset R$ for some nonzero $c \in C$. If A and B are subsets of $Q_{sym}(R)$, we define $(A:_\ell B) = \{q \in Q_{sym}(R) | qB \subset A\}$ and $(A:_r B) = \{q \in Q_{sym}(R) | Bq \subset A\}$.

Lemma 1.1 : The following statements are equivalent :

(1) R is a symmetric maximal order;

(2) for any ideal I of R $(I:_\ell I) = (I:_r I) = R$;

(3) for any fractional R-ideal I $(I:_\ell I) = (I:_r I) = R$.

Proof :

(1)$\Longrightarrow$(2) : Clearly $R \subset (I:_r I) \subset Q_{sym}(R)$ and if $0 \neq c \in I \cap C$ then $c(I:_r I) \subset R$ whence $(I:_r I) = R$. Likewise, $(I:_\ell I) = R$.

(2)$\Longrightarrow$(3) : There exists a nonzero element $c \in C$ such that $cI \subset R$. Because cI is an ideal of R, $R = (cI:_r cI) = (I:_r I)$.

(3)$\Longrightarrow$(1) : Suppose S is a ring such that $R \subset S \subset Q_{sym}(R)$ and $cS \subset R$ for some nonzero $c \in C$. Then S is a fractional R-ideal and $(S:_r S) = R$. Because S is a ring we get that $S = (S:_r S) = R$.

Lemma 1.2 : If R is a symmetric maximal order in $Q_{sym}(R)$ and if I is a fractional

R-ideal, then $(R:_\ell I) = (R:_r I)$.

Proof : $x \in (R:_\ell I)$ iff $xI \subset R = (I:_r I)$ iff $IxI \subset I$ iff $Ix \subset (I:_\ell I) = R$.

Proposition 1.3 : If R is a central Ω-Krull ring, then R is a symmetric maximal order in $Q_{sym}(R)$.

Proof : Let S be a ring such that $R \subset S \subset Q_{sym}(R)$ and $cS \subset R$ for some nonzero $c \in C$. Note that cS is an ideal of R such that $R_i cS = cSR_i$ yielding that $R_i S = SR_i$ because σ_i is a central kernel functor. For each $i \in \wedge$ we have that $cSR_i \subset R_i$. Since each R_i-ideal is invertible, we have $SR_i = R_i SR_i = (cSR_i)^{-1}(cSR_i)SR_i = (cSR_i)^{-1}\, cSR_i = R_i$, whence $S \subset R_i$ for all $i \in \wedge$. Therefore $S \subset R$.

If R is a central Ω-Krull ring, we put $\mathbb{F}(R)$ to be the set of all fractional R-ideals and $\mathbb{D}(R)$ to be the set of all divisorial ideals, i.e. those fractional R-ideals I such that $I = \bigcap_i R_i I$. If $I \in \mathbb{F}(R)$ it is a straightforward computation to show that $(R:I) = \bigcap_i (R_i:R_i I)$. Because $R_i I = R_i$ for almost all $i \in \wedge$ and using lemma 2.3 of [5] we obtain that $R_i(R:I) = (R_i:R_i I)$ for all $i \in \wedge$.

Proposition 1.4 : If R is a central Ω-Krull ring, then a fractional R-ideal I is divisorial iff $I = (R:(R:I))$.

Proof : Suppose that $I = \cap R_i I = \cap (P_i')^{n_i}$. Then $(R_i:R_i I) = (R_i:P_i'^{\,n_i}) = P_i'^{\,-n_i}$. Hence $(R:I) = \cap (P_i')^{-n_i}$ and therefore $I = (R:(R:I)) = \cap (P_i')^{n_i}$. Conversely, let $I = (R:(R:I))$. Because $(R:I) = \cap (R_i:R_i I)$ and $R_i(R:I) = (R_i:R_i I)$ we obtain that $I = \cap (R_i:(R_i:R_i I)$. It follows from lemma 2.3 of [5] that $R_i I = R_i:(R_i:R_i I)$ and therefore $I = \cap R_i I$.

If S and T are rings such that $S \subset T$, we say that T has the intersection property (i.p.) with respect to S if any nonzero ideal of T has a non-trivial intersection with S.

Lemma 1.5 : If R is a central Ω-Krull ring, then the center of R/P, Z(R/P), has the i.p. with respect to $C/P \cap C$ for all $P \in X^1(R)$.

Proof : Choose $P \in X^1(R)$. First we show that R/P satisfies Formanek's condition. Let $0 \neq (I/P)$ be an ideal of R/P, then $I \not\subset P$ and therefore $(I \cap C) \not\subset (P \cap C)$. In particular $(I/P) \cap (C/P \cap C) \neq 0$ whence $(I/P) \cap Z(R/P) \neq 0$. Let A be an ideal of Z(R/P) and let a be a nonzero element of A. $J = Z(R/P)a$ is an ideal of Z(R/P) and $(R/P)J = (R/P)a$. Furthermore, $(R/P)J \cap Z(R/P) = J$ because a is invertible in $Q_{sym}(R/P)$ (which exists since R/P is prime and satisfies Formanek's condition). By the first part of the proof $0 \neq (R/P)J \cap (C/P \cap C) = J \cap (C/P \cap C)$ whence $0 \neq J \cap (C/P \cap C) \subset A \cap (C/P \cap C)$.

Finally, we recall the definition of an arithmetical pseudovaluation on $\mathbb{D}(R)$. A function $v : \mathbb{D}(R) \to \Gamma \cup \{\infty\}$ where Γ is a totally ordered group is said to be an arithmetical pseudovaluation if it satisfies :

(1) $\forall\, I,J \in \mathbb{D}(R)$: $v(I * J) = v(I) + v(J)$;

(2) $\forall\, I,J \in \mathbb{D}(R)$: $v(\overline{I + J}) \geq \min(v(I),v(J))$ where $\overline{I + J} = \cap R_i(I + J)$;

(3) $\forall\, I,J \in \mathbb{D}(R)$: if $I \subset J$ then $v(I) \geq v(J)$;

(4) $v(R) = 0$ and $v(0) = \infty$.

For more information on pseudovaluations, the reader is referred to [9] and [10].

2. CHARACTERIZATION OF Ω-KRULL RINGS

In this section we aim to prove the following result :

Theorem 2.1 : R is a central Ω-Krull ring if and only if :

(1) R is a symmetric maximal order in $Q_{sym}(R)$;

(2) R satisfies the ascending chain condition on divisorial ideals contained in R;

(3) for each $P \in X^1(R)$ and for any ideal I of R we have $I \subset P$ if and only if $(I \cap C) \subset (P \cap C)$.

If R is commutative, assumption (1) is equivalent to saying that R is completely integrally closed in its field of fractions (cfr. [4] and lemma 1.1). Therefore, condition (1) and (2) state that R is a Krull domain, in the commutative case. Condition (3) is necessary in order to force a closer relationship between the ring and its center.

If follows from proposition 1.3, theorem 2.5 of [5] and the fact that the σ_i are central, that these conditions are necessary. We will prove the converse implication by a series of lemmas.

First, we aim to establish that the set of divisorial ideals, $\mathbb{D}(R)$, is a commutative group. To this end we use the group of Artin. The construction we give runs along the lines of G. Maury and J. Raynaud [6]. Denote by $\mathbb{F}(R)$ the set of all nonzero fractional R-ideals. Clearly, if A and B are in $\mathbb{F}(R)$, so is A.B. $\mathbb{F}(R)$ with this multiplication and ordered by inclusion satisfies the following properties :

(1) $\mathbb{F}(R)$ is an associative semigroup with identity element R;

(2) $\mathbb{F}(R)$ is a lattice; $A + B = \sup(A,B)$ and $A \cap B = \inf(A,B)$;

(3) if $A,B,C \in \mathbb{F}(R)$, then $A \leqslant B$ implies $AC \leqslant BC$ and $A(B + C) = AB + AC$, $(B + C)A = BA + CA$;

(4) if $\{A_\alpha : \alpha \in I\}$ is a nonzero family of elements of $\mathbb{F}(R)$ such that $\sum A_\alpha \in \mathbb{F}(R)$, then for any $C \in \mathbb{F}(R)$, $\sup\{CA_\alpha : \alpha \in I\}$ and $\sup\{A_\alpha C : \alpha \in I\}$ exist in $\mathbb{F}(R)$ and $\sup\{CA_\alpha : \alpha \in I\} = \sum CA_\alpha = C(\sum A_\alpha) = C \sup\{A_\alpha : \alpha \in I\}$. Also, $\sup\{A_\alpha C : \alpha \in I\} = \sup\{A_\alpha : \alpha \in I\}C$;

(5) for every $A \in \mathbb{F}(R)$, $(R:A) \in \mathbb{F}(R)$ (note that $(R:_\ell A) = (R:_r A) = (R:A)$ by lemma 1.2);

(a) $A(R:A)A \leqslant A$; (b) $\forall X \in \mathbb{F}(R) : AXA \leqslant A$ implies $X \leqslant (R:A)$;

(6) $R^2 \leqslant R$; moreover if $R \leqslant S$, $S^2 \leqslant S$ and $S \in \mathbb{F}(R)$, then $S = R$ because R is a symmetric maximal order.

Now, we define an equivalence relation on $\mathbb{F}(R)$ by saying that $A \sim B$ iff $(R:(R:A)) = (R:(R:B))$. The set of equivalence classes is isomorphic to the set of divisorial ideals $\mathbb{D}(R)$, i.e. those fractional ideals I such that $I = (R:(R:I))$, which becomes an associative semigroup by defining $A * B = (R:(R:AB))$ if $A,B \in \mathbb{D}(R)$. Proposition 1.4 of [6] yields that $\mathbb{D}(R)$ is a commutative group.

Now, we reverse the ordering on $\mathbb{D}(R)$, i.e. $A \leqslant B$ iff $A \supset B$. It is readily verified that each finite nonempty subset of $\mathbb{D}(R)$ has a supremum (resp. infimum), $A_1 \cap \ldots \cap A_n$ (resp. $(R:(R:(A_1 + \ldots + A_n))))$.

Proposition 2.2 : $\mathbb{D}(R) \cong \mathbb{Z}^{(\wedge)}$ for a certain index set $\wedge$ and this isomorphism is order preserving.

Proof : We already know that $\mathbb{D}(R)$ is a commutative, ordered group such that any two elements have a supremum and an infimum. Moreover, condition (2) of Theorem 2.1 states that any nonempty subset of positive elements of $\mathbb{D}(R)$ (i.e. divisorial ideals contained in R) has a minimal element. A well-known theorem on commutative ordered groups satisfying these properties (cfr. [1]) yields that $\mathbb{D}(R) \cong \mathbb{Z}^{(\wedge)}$ for some index set $\wedge$ and the isomorphism is order preserving.

Of course, the order relation on $\mathbb{Z}^{(\wedge)}$ is defined by $(\alpha_\lambda)_{\lambda \in \wedge} \leqslant (\beta_\lambda)_{\lambda \in \wedge}$ iff $\alpha_\lambda \leqslant \beta_\lambda$ for all $\lambda \in \wedge$. Let $\psi : \mathbb{D}(R) \to \mathbb{Z}^{(\wedge)}$ be an order preserving isomorphism. Put $e_i = (\delta_{i\lambda})_{\lambda \in \wedge}$ and let $P_i = \psi^{-1}(e_i)$. Thus, any element A of $\mathbb{D}(R)$ can be written as $A = P_1^{n_1} * \ldots * P_k^{n_k}$ $(n_i \in \mathbb{Z})$.

Lemma 2.3 : P_i is a prime ideal of R.

Proof : Let $x,y \in R$ such that $xRy \subset P_i$. It is straightforward to check that $\overline{RxR} \ast \overline{RyR} = \overline{RxRyR} \subset P_i$ where $\bar{A} = (R:(R:A))$. Further, $\psi(\overline{RxR}) = \sum n_j e_j$ and $\psi(\overline{RyR}) = \sum m_j e_j$ where $n_j, m_j \geqslant 0$. In particular, $\psi(\overline{RxR}) + \psi(\overline{RyR}) = \sum (n_j + m_j)e_j \geqslant \psi(P_i) = e_i$. Therefore, $n_i \geqslant 1$ or $m_i \geqslant 1$ yielding that either $x \in \overline{RxR} \subset P_i$ or $y \in \overline{RyR} \subset P_i$.

Lemma 2.4 : Let P be a nonzero prime ideal of R, then P contains P_i for some $i \in \wedge$.

Proof : $Rc \subset P$ for some nonzero $c \in C$. Since $Rc \in \mathbb{D}(R)$ we may write $P \supset Rc = P_1^{n_1} \ast \ldots \ast P_k^{n_k} \supset P_1^{n_1} \ldots P_k^{n_k}$ and all $n_i \geqslant 0$. Therefore $P \supset P_i$ for some i.

Proposition 2.5 : $\mathbb{D}(R)$ is generated by the prime ideals of height one of R.

Proof : Suppose P is a height one prime. By the previous lemma, $P = P_i$ for some i. Conversely, let P be a prime generator of $\mathbb{D}(R)$. If P is not a height one prime, $0 \neq Q \subsetneq P$ for some prime ideal Q of R. Again by the previous lemma, $P_i \subsetneq P$ for some $i \in \wedge$. Therefore $\psi(P) \lneq \psi(P_i) = e_i$, a contradiction because $\psi(P) > 0$.

Let us define for all i

$$v_i : \mathbb{D}(R) \to \mathbb{Z} : A = P_1^{n_1} \ast \ldots \ast P_k^{n_k} \mapsto n_i$$

Proposition 2.6 : v_i is an arithmetical pseudovaluation on $\mathbb{D}(R)$.

Proof : It is easy to check that $A = P_1^{n_1} \ast \ldots \ast P_k^{n_k} = (R:(R:P_1^{n_1} \ldots P_k^{n_k}))$. Because $\mathbb{D}(R)$ is a commutative group, it's trivial to see that for all $I,J \in \mathbb{D}(R)$ $v_i(I \ast J) = v_i(I) + v_i(J)$. Now, let $I \subset J$ and let $\psi(J) = \sum m_j e_j$, $\psi(I) = \sum n_j e_j$. Since $\psi(J) \leqslant \psi(I)$ we have that $v_i(J) \leqslant v_i(I)$.

Next, we have to establish that $v_i(\overline{I+J}) \geqslant \min(v_i(I), v_i(J))$. Suppose first that both I and J are contained in R. Then $I = (R:(R:P_1^{n_1} \dots P_k^{n_k}))$ $\subset (R:(R:P_i^{n_i})$ because all $n_i \geq 0$. Similarly, $J = (R:(R:P_1^{m_1} \dots P_k^{m_k}))$ $\subset (R:(R:P_i^{m_i}))$ whence $(I + J) \subset (R:(R:P_i^{n_i})) + (R:(R:P_i^{m_i})) = (R:(R:P_i^{k_i}))$ where $k_i = \min(n_i, m_i)$, yielding that $v_i(\overline{I + J}) \geq \min(v_i(I), v_i(J))$. If $I \not\subset R$ or $J \not\subset R$, there exists an element $c \in C$ such that $cI \subset R$ and $cJ \subset R$. Hence $v_i(\overline{(cR)I + (cR)J}) \geq \min(v_i((cR)I), v_i((cR)J)) = \min(v_i(cR) + v_i(I), v_i(cR) + v_i(J))$. Therefore $v_i(cR) + v_i(\overline{I + J}) \geq v_i(cR) + \min(v_i(I), v_i(J))$ completing the proof.

Lemma 2.7 : Let v be an arithmetical pseudovaluation on $\mathbb{D}(R)$ and let $\{I_j\}$ be an arbitrary set of divisorial ideals such that $\sum I_j \in \mathbb{F}(R)$, then $v(\overline{\sum I_j}) = \inf\{v(I_j)\}$.

Proof : One inequality is obvious since $v(\overline{\sum I_j}) \leq v(I_j)$ for every j. The converse implication is proved in a way similar to the proof of Proposition 2.6.

Corollary 2.8 : If v is an arithmetical pseudovaluation on $\mathbb{D}(R)$ and if $I \in \mathbb{D}(R)$, then $v(I) = \inf\{v(\overline{RxR}) \mid x \in I\}$.

Now, let us consider again the arithmetical pseudovaluation $v_i : \mathbb{D}(R) \to \mathbb{Z} : A = P_1^{n_1} * \dots * P_k^{n_k} \mapsto n_i$ and suppose $i = 1$ for the sake of simplicity. Denote $Q_1 = \{x \in Q_{sym}(R) \mid v_1(\overline{RxR}) > 0\}$ and $R_1 = \{x \in Q_{sym}(R) \mid xQ_1 \subset Q_1$ and $Q_1x \subset Q_1\}$. It is straightforward to check that $Q_1 \cap R = P_1$.

Proposition 2.9 : $R_1 = \{x \in Q_{sym}(R) \mid v_1(\overline{RxR}) \geq 0\} = \{x \in Q_{sym}(R) \mid xI \subset R$ and $Ix \subset R$ for some ideal I of R not contained in $P_1\}$.

Proof : (1) Suppose $x \in Q_{sym}(R), v_1(\overline{RxR}) \geq 0$ and $y \in Q_1$, i.e. $v_1(\overline{RyR}) > 0$. We immediately have that $RxyR \subset RxRRyR \subset \overline{RxR} * \overline{RyR}$, hence $\overline{RxyR} \subset \overline{RxR} * \overline{RyR}$. This yields $v_1(\overline{RxyR}) \geq v_1(\overline{RxR}) + v_1(\overline{RyR}) > 0$. Therefore $xQ_1 \subset Q_1$. Similarly $Q_1x \subset Q_1$. Conversely, suppose $x \in R_1$. In particular $xP_1 \subset Q_1$. Corollary

2.8 yields that there is an element $y \in P_1$ such that $v_1(\overline{RyR}) = 1$. Hence $RxRyR \subset Q_1$. We claim that $v_1(\overline{RxRyR}) > 0$. If $RxRyR \subset R$ we may write $\overline{RxRyR} = \overline{Rxr_1yR} + \ldots + \overline{Rxr_nyR}$ (because the divisorial ideals satisfy the ascending chain condition). If $RxRyR \not\subset R$, then it may be multiplied by a central element such that the image is in R; then the argument used before may be repeated. Since $xr_jy \in Q_1$ for all j, Lemma 2.7 yields that $v_1(\overline{RxRyR}) > 0$. Therefore $v_1(\overline{RxRyR}) = v_1(\overline{RxR}) + v_1(\overline{RyR}) \geqslant 1$ and $v_1(\overline{RyR}) = 1$. Hence $v_1(\overline{RxR}) \geqslant 0$.

(2) Let $x \in R_1$, i.e. $v_1(\overline{RxR}) \geqslant 0$. Write $\overline{RxR} = P_1^{n_1} * \ldots * P_k^{n_k}$ and $n_1 \geqslant 0$. Multiply this equality by those $P_i^{n_i}$ with $n_i < 0$. Then $I * \overline{RxR} = P_1^{n_1} * \ldots$ where $I \in \mathbb{D}(R)$ and on the right hand side of this equality all n_i are positive. Hence $I * \overline{RxR} \subset R$. Now I is the product (in $\mathbb{D}(R)$) of positive powers of P_i and $i \neq 1$. Hence $I \not\subset P_1$. Conversely, suppose $Ix \subset R$ and $I \not\subset P_1$. Take $y \in I \backslash P_1$. Then $yRx \subset R \subset R_1$. Similarly as in the first part of the proof we have $v_1(\overline{RyRxR}) = v_1(\overline{RyR}) + v_1(\overline{RxR}) \geqslant 0$. But $v_1(\overline{RyR}) = 0$ since $y \in R\backslash P_1$. Therefore $v_1(\overline{RxR}) \geqslant 0$ and $x \in R_1$.

<u>Proposition 2.10</u> : R_1 is a quasi-local Ω-ring with unique maximal ideal Q_1.

<u>Proof</u> : Using condition (3) of theorem 2.1 we may write $R_1 = \{x \in Q_{sym}(R) \mid xI \subset R$ and $Ix \subset R$ for an ideal I of R such that $(I \cap C) \not\subset (P \cap C)\} = \{c^{-1}r = rc^{-1} \mid c \in C\backslash p_1, r \in R\}$ where $p_1 = P_1 \cap C$. Hence it's clear that $Q_1 = R_1P_1 = P_1R_1$ and that R_1 is a quasi-local ring.

Suppose I is an ideal of R_1 and $I \neq R_1$. Then $I \subset Q_1$. Put $Q_1' = \{x \in Q_{sym}(R) \mid v_1(\overline{RxR}) \geqslant -1\}$. Because $v_1(P_1) = 1$ we have $v_1(R:P_1) = -1$. Hence $(R:P_1) \subset Q_1'$. Using Corollary 2.8 we obtain that Q_1' contains an element of valuation -1. We clearly have $Q_1Q_1' \subset R_1$ and $Q_1'Q_1 \subset R_1$. Since $I \subset Q_1$ we have $IQ_1' \subset Q_1Q_1' \subset R_1$. Therefore $IQ_1' = R_1$ or $IQ_1' \subset Q_1$. In the second case, $IQ_1'^2 \subset R_1$. Suppose $IQ_1'^n \subset R_1$ for all $n \in \mathbb{N}$. Take $x \in I$, then $v_1(\overline{RxR}) = m \in \mathbb{N}$.

Choose $n > m$. Arguing in a way similar to the argument used in Proposition 2.9, we find an element in $IQ_1'^n$ which has negative valuation m-n, a contradiction. Therefore $IQ_1'^n = R_1$ for some $n \in \mathbb{N}$. Similarly $Q_1'^m I = R_1$ for some $m \in \mathbb{N}$. Hence $Q_1'^m = Q_1'^n$. Since Q_1' contains an element of valuation -1, the latter only happens if m = n. It is easily verified that $I = Q_1^n$ (since here $Q_1 Q_1' = Q_1' Q_1 = R_1$). So finally R_1 is a quasi-local Ω-ring.

We are now able to finish the proof of Theorem 2.1. For all $i \in \wedge$ $R_i = Q^{\ell}_{\sigma_i}(R) = Q^{r}_{\sigma_i}(R)$ where $\mathcal{L}^2(\sigma_i) = \mathcal{L}^2(R\backslash P_i)$ (Proposition 2.9) and R_i is a quasi-local Ω-ring (Proposition 2.10). Moreover $R = \bigcap_i R_i$, for suppose $x \in \cap R_i$, then $v_i(\overline{RxR}) \geq 0$ for all i. Hence $x \in \overline{RxR} \subset R$ because $\mathbb{D}(R) \cong \mathbb{Z}^{(\wedge)}$. If $I \in \mathcal{L}^2(R\backslash P_i)$ we already have established that $R_i I = IR_i = R_i$ (Proposition 2.10). Finally, take $r \in R$; $\overline{RrR} = P_1^{n_1} * \ldots * P_k^{n_k}$ and all $n_i \geq 0$. An easy computation learns that $n_i > 0$ iff $r \in \overline{RrR} \subset P_i$. Because there are only finitely many $n_i > 0$, it follows that $(r) \in \mathcal{L}^2(\sigma_i)$ for almost all i. This proves that R is Ω-Krull and condition (3) yields that R is central Ω-Krull.

Remark

It is still an open question whether or not condition (3) of theorem 2.1 can be weakened or even dropped.

The following generalizations of commutative properties may be derived as applications of Theorem 2.1.

Proposition 2.11 : Let R be a ring satisfying the following conditions :

(1) $R = \bigcap_i R_i$ where all R_i are quasi-local Ω-rings contained in $Q_{sym}(R)$ such that $IR_i = R_i I$ for all ideals I of R;

(2) for any ideal I of R and any $P \in X^1(R)$ $I \subset P$ iff $(I \cap C) \subset (P \cap C)$;

(3) for all $r \in R$ $R_i(r) = R_i$ for almost all i.

Then R is a central Ω-Krull ring.

Proof : First, it's easy to check that $C = \cap C_i$ where $C = Z(R)$ and $C_i = Z(R_i)$. To prove that R is a central Ω-Krull ring we check conditions (1) to (3) of Theorem 2.1. The fact that R is a symmetric maximal order in $Q_{sym}(R)$ is proved as in Proposition 1.3 (use the fact that ideals of R_i are invertible for all i, cfr. Proposition 2.1 of [5]). It remains to verify whether the ascending chain condition on divisorial ideals holds. This is proved in the same way as in the commutative case (cfr. [4], Chapter 1, Theorem 3.6).

Proposition 2.12 : Under the conditions of the preceding proposition we have that for all $P \in X^1(R)$ there exists an index i such that $Q_{R\setminus P}(R) = Q_{C\setminus p}(R) = R_i$ where $p = P \cap C$.

Proof : Because $R = \cap R_i$, we have that $C = \cap C_i$. Since R is Ω-Krull, C is a Krull domain [5]. For all $p \in X^1(C)$, there is an index i such that $Q_{C\setminus p}(C) = C_i$ ([2],[8]). There is exactly one $P \in X^1(R)$ lying over p, because R is central Ω-Krull. Therefore $\bar{R} = Q_{R\setminus P}(R) = Q_{C\setminus p}(R) = RC_i \subset R_i$. $\bar{R}$ is a quasi-local Ω-ring since R is an Ω-Krull ring. Let $\bar{P}$ (resp. P_i') be the unique maximal ideal of $\bar{R}$ (resp. R_i). First we prove that $P_i' \subset \bar{P}$. Suppose $x \in R_i$ and $x \notin \bar{P}$. Then $\bar{R}x\bar{R} \not\subset \bar{P}$, hence $\bar{R} \subset \bar{R}x\bar{R}$ and this entails $R_i = R_i\bar{R}R_i \subset R_i\bar{R}x\bar{R}R_i = R_ixR_i \subset R_i$ and $x \notin P_i'$. Therefore $P_i' \subset \bar{P}$. We claim that $P_i' \cap R$ is a prime ideal of R. Indeed, take $x,y \in R$ and suppose that $xRy \subset P_i' \cap R$. Then $R_ixRy = RxR_iy \subset P_i'$ (note that $IR_i = R_iI$ if I is an ideal of R). So, $x \in P_i'$ or $y \in P_i'$. Hence $x \in P_i' \cap R$ or $y \in P_i' \cap R$. Since $(P_i' \cap R) \subset (\bar{P} \cap R)$ and prime ideals of R contained in $\bar{P} \cap R = P$ extend to prime ideals of $\bar{R}$, we may derive from $0 \neq \bar{R}\,(P_i' \cap R) \subset \bar{P} = \bar{R}(\bar{P} \cap R)$ that $\bar{P} = \bar{R}(P_i' \cap R)$. Therefore $\bar{P} \subset \bar{R}P_i' \subset R_iP_i' = P_i'$ and we conclude that $\bar{P} = P_i'$. Since $\bar{R}$ and R_i are quasi-local Ω-rings, they both are symmetric maximal orders in $Q_{sym}(R)$. Finally, using lemma 1.1 we obtain $\bar{R} = (\bar{P} : \bar{P}) = (P_i' : P_i') = R_i$.

3. POLYNOMIAL EXTENSIONS

In [5] we gave a sufficient condition on an Ω-Krull ring R for R[T] to be Ω-Krull, namely for all $P \in X^1(R)$ we have $Z(R/P) = C/(P \cap C)$. This condition is not necessary as the following example due to Professor D.S. Passman and Professor P.F. Smith shows.

Example : Let $\mathbb{R}$, resp. $\mathbb{C}$, be the field of real, resp. complex, numbers. And let $\bar{\ }$ be the extension of the complex conjugation to the formal power series $\mathbb{C}[[t]]$. Consider the following subset of the two by two matrices over $\mathbb{C}[[t]]$:

$$A = \left\{ \begin{pmatrix} \alpha & \beta \\ t\bar{\beta} & \bar{\alpha} \end{pmatrix} \middle|\ \alpha, \beta \in \mathbb{C}[[t]] \right\}$$

Clearly A is a ring for the usual addition and multiplication, and A is closed under taking the adjoint. Now consider the following ring homomorphism $\psi : A \longrightarrow \mathbb{C}$ which maps $\begin{pmatrix} \alpha & \beta \\ t\bar{\beta} & \bar{\alpha} \end{pmatrix}$ to $\alpha(0)$, where $\alpha(0)$ is the constant term of α. So kernel $\psi = P = \left\{ \begin{pmatrix} t\alpha & \beta \\ t\bar{\beta} & t\bar{\alpha} \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}[[t]] \right\}$ is a maximal ideal of A and every element outside P is invertible. Indeed, $a = \begin{pmatrix} \alpha & \beta \\ t\bar{\beta} & \bar{\alpha} \end{pmatrix} \in A \setminus P$ iff $\alpha(0) \neq 0$; so if $a \in A\setminus P$, then $\det a \in \mathbb{R}[[t]] \setminus t\,\mathbb{R}[[t]]$. Therefore $a^{-1} = (\det a)^{-1} . \operatorname{adj} a \in A$.

Because $P^2 \subset tA$ one immediately obtains $\bigcap_n P^n = 0$. To justify the following we embed A in the ring of two by two matrices over the quotient field of $\mathbb{C}[[t]]$. Let $W = t^{-1}P$. We claim that $W = P^{-1}$, in the sense that $PW = WP = A$. Indeed, since $\begin{pmatrix} \alpha & \beta t^{-1} \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \bar{\alpha} t & \bar{\beta} \end{pmatrix}$ and since $\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} \in P$ we obtain $A \subset WP = PW = t^{-1}P^2 \subset A$; so the claim follows. Therefore one proves as in the proof of Corollary 5.2 in [5] that A is a quasi-local Ω-ring.

Note that A satisfies Formanek's condition because $\begin{pmatrix} t^n & 0 \\ 0 & t^n \end{pmatrix} \in P$ for all $n \in \mathbb{N}_0$.

Now one easily checks that Z(A) is the set of diagonal matrices with entries in $\mathbb{R}[[t]]$. Moreover $A/P \cong \mathbb{C}$ and $Z(A)/Z(A) \cap P \cong \mathbb{R}$. So we can consider Z(A/P) as a finite Galois extension of $Z(A)/Z(A) \cap P$. By making a slight adaption of the proof of Lemma 5.1 in [5], one proofs as in [5] that A[T] is an Ω-Krull ring.

In this section we aim to give necessary and sufficient conditions on a central Ω-Krull ring R in order to have that a polynomial extension R[T] is a central Ω-Krull ring.

Lemma 3.1 : If R is a symmetric maximal order in $Q_{sym}(R)$, $A \in \mathbb{D}(R)$ and $I \in \mathbb{F}(R)$, then $(A:_\ell I) = (R:I) \ast A = A \ast (R:I) = (A:_r I)$.

Proof : By symmetry it suffices to prove that $(A:_\ell I) = A \ast (R:I) = R:(R:A(R:I))$. Since $(A:_\ell I) = (R:A'):_\ell I = (R:_\ell IA') = (R:_r IA')$ where $A' = (R:A)$, we obtain that $(A:_\ell I)$ is divisorial. Now suppose $x \in (R:I)$ and $y \in A$. Then $yxI \subset yR \subset A$. So $A(R:I) \subset (A:_\ell I)$. Since $(A:_\ell I)$ is divisorial we have $A \ast (R:I) \subset (A:_\ell I)$. Conversely, let $x \in (A:_\ell I)$. Then $xI \subset A$ and $xI(R:A) \subset A(R:A) \subset R$. So $x \in (R:I(R:A))$. Let $\alpha = (R:(R:I))$ and $\beta = (R:A)$, then $(A:_\ell I) \subset (R:I(R:A)) = (R:\alpha\beta) = R:(R:(R:\alpha\beta)) = (R:(\alpha \ast \beta)) = (\alpha \ast \beta)^{-1} = \beta^{-1} \ast \alpha^{-1} = A \ast (R:I)$. The result follows.

Proposition 3.2 : If R is a symmetric maximal order, then R[T] is a symmetric maximal order.

Proof : In view of Lemma 1.1 it is sufficient to prove that $(I:_\ell I) = (I:_r I) = R[T]$ for any nonzero ideal I of R[T]. Suppose that $q \in Q_{sym}(R[T])$ is such that

$qI \subset I$. Clearly, $IQ_{sym}(R)$ is a twosided ideal of $Q_{sym}(R)$ [T], therefore it is principal and generated by a central element (cfr. [3], Prop. 5.1.3), say $IQ_{sym}(R) = Q_{sym}(R)[T]g$. So g is invertible in $Q_{sym}(R[T])$. It follows from $qI \subset I$ that $qg \in Q_{sym}(R)[T]g$ and $q \in Q_{sym}(R)[T]$. Put $q = q_pT^p + \dots + q_0$ with $q_i \in Q_{sym}(R)$. Let C(I) denote the set of all leading coefficients of elements of I together with the zero element. It is straightforward to check that C(I) is an ideal of R such that $q_p \in (C(I):_{\ell}C(I)) = R$. Hence $q' = q_{p-1}T^{p-1} + \dots + q_0 \in (I:_{\ell}I)$ and repeating the above argument yields $q_{p-1} \in R$, etc. Thus $q \in R[T]$.

Proposition 3.3 : If R is a symmetric maximal order satisfying the ascending chain condition on divisorial ideals contained in R, then R[T] satisfies the ascending chain condition on divisorial ideals contained in R[T].

Proof : Let $\{I_n : n \in \mathbb{N}\}$ be an ascending chain of divisorial ideals contained in R[T]. Then $\{Q_{sym}(R)I_n : n \in \mathbb{N}\}$ is an ascending chain of ideals of $Q_{sym}(R)[T]$. Since $Q_{sym}(R)[T]$ has ACC on twosided ideals, there is a natural number $n' \in \mathbb{N}$ such that $Q_{sym}(R)I_m = Q_{sym}(R)I_{n'}$ for all $m \geq n'$. Denote $I_d = (R:(R:I))$ for any ideal I of R. The chain $C(I_n)_d$, $n \in \mathbb{N}$ where $C(I_n)$ is the set of all leading coefficients of elements of I_n together with the zero element, is an ascending chain of divisorial ideals in R. Therefore, there is a natural number n" such that $C(I_m)_d = C(I_{n''})_d$ for all $m \geq n''$. Let $n = \sup(n',n'')$ and let $k \geq n$. Let $f \in Q_{sym}(R[T])$ be such that $fI_n \subset I_k$. Then $fI_nQ_{sym}(R) \subset I_kQ_{sym}(R) = I_nQ_{sym}(R)$. Since any ideal of $Q_{sym}(R)[T]$ is principal $I_nQ_{sym}(R) = Q_{sym}(R)[T]g$. Thus $fg \in Q_{sym}(R)[T]g$. Now g is a unit in $Q_{sym}(R[T])$, thus $f \in Q_{sym}(R)[T]$, say $f = f_pT^p + f_{p-1}T^{p-1} + \dots + f_0$. Let a be any element of $C(I_n)$, i.e. there exists an element $h \in I_n$ such that $h = aX^m + a_{m-1}X^{m-1} + \dots + a_0$. Then $fh = f_paX^{m+p} + h'$ where h' is an element of degree less than m + p. But $fI_n \subset I_k$ implies that $fC(I_n) \subset C(I_k)$ and thus also $fC(I_n)_d \subset C(I_k)_d = C(I_n)_d$, whence it follows that $f_p \in R$ by Lemma 1.1. Repeating the argument, we obtain that $f \in R[T]$. Thus $(I_k:I_n) \subset R[T]$. Consequently, by lemma 3.1 $(R[T]:I_n) \ast I_k \subset R[T]$. The latter entails that $I_k \subset I_n$, and this finishes the proof.

Professor D.S. Passman pointed out the following lemma.

Lemma 3.4 : If R is a prime ring satisfying Formanek's condition, I an ideal of R[T] and $\alpha \in I$, then there exists a nonzero element $c \in Z(R)$ such that $c\alpha \in R(I \cap Z(R)[T])$.

Proof : If $\alpha = \sum a_i T^i \in R[T]$, we put supp $\alpha = \{T^i | a_i \neq 0\}$.

(1) Let α be an element of I with the property that there is no proper subset S' of S = supp α such that S' = supp β with $\beta \in I$. Let $\alpha = \sum_{i=0}^{n} a_i T^i$ and $a_n \neq 0$. Then $Ra_nR \cap Z(R) \neq 0$. Hence there is an element $\alpha' = \sum_{i=0}^{n} a_i' T^i \in I$ with $0 \neq a_n' \in Z(R)$ and supp $\alpha' \subset$ supp α. We have $b\alpha' - \alpha' b \in I$ for all $b \in R$. But supp$(b\alpha' - \alpha' b) \subsetneq$ supp $\alpha' \subset$ supp α. Therefore $\alpha' b = b\alpha'$ for all $b \in R$ and hence $\alpha' \in Z(R)[T]$. Finally $a_n'\alpha - a_n\alpha' \in I$ and supp$(a_n'\alpha - a_n\alpha') \subsetneq$ supp α. This yields that $a_n'\alpha = a_n\alpha' \in R(I \cap Z(R)[T])$.

(2) Now let $\alpha = \sum_{i=0}^{n} a_i X^i$, $a_n \neq 0$, be an arbitrary nonzero element of I. Suppose that for all proper subsets S' of S = supp α and S' = supp β for some $\beta \in I$, there exists an element $c \in Z(R)$ such that $c\beta \in R(I \cap Z(R)[T])$. Pick a subset S' of S such that S' = supp β for some $\beta \in I$ and S' is minimal with this property. If S' = S then (1) yields the result. If S' $\neq$ S, then S' = supp β, $\beta = \sum_{i=0}^{m} b_i T^i \in I$, $b_m \neq 0$. Similarly as in (1) we may assume $\beta \in Z(R)[T]$. Then $b_m\alpha - a_m\beta \in I$ and supp$(b_m\alpha - a_m\beta) \subsetneq$ supp α. Hence $c(b_m\alpha - a_m\beta) \in R(I \cap Z(R)[T])$ for some $c \in Z(R)$. Finally, $(cb_m)\alpha \in R(I \cap Z(R)[T])$ and $cb_m \in Z(R)$.

Suppose R satisfies the assumptions of the foregoing lemma, then we have :

Corollary 3.5 : If $P \in$ Spec R[T] such that $P \cap R = 0$ and if I is an ideal of

$R[T]$, then $I \cap (Z(R)[T]) \subset P \cap (Z(R)[T])$ implies that $I \subset P$.

Proof : Let $\alpha \in I$. By the foregoing lemma $c\alpha \in R(I \cap Z(R[T])) \subset R(P \cap Z(R[T])) \subset P$ for some nonzero $c \in Z(R)$. Therefore $cR[T]\alpha \subset P$ and since $c \notin P$ we obtain $\alpha \in P$.

The following observation is clear :

Lemma 3.6 : Let R be a central Ω-Krull ring and $P \in X^1(R)$. The following statements are equivalent :

(1) $Z(R/P)[T]$ has the intersection property (i.p.) with respect to $(C/P \cap C)[T]$,

(2) $Q(Z(R/P))[T]$ has the i.p. with respect to $Q(C/P \cap C)[T]$ (where $Q(Z(R/P))$ denotes the field of fractions of $Z(R/P)$),

(3) $Q(Z(R/P))$ is an algebraic field extension of $Q(C/P \cap C)$.

We are now ready to state :

Theorem 3.7 : Let R be a central Ω-Krull ring. Then $R[T]$ is a central Ω-Krull ring if and only if for all $P \in X^1(R)$ $Q(Z(R/P))$ is an algebraic field extension of $Q(C/P \cap C)$.

Proof : (1) If $R[T]$ is a central Ω-Krull ring, we only have to prove that for any $P \in X^1(R)$ $P[T]$ is a height one prime ideal of $R[T]$ (because of Lemma 1.5 and Lemma 3.6). Suppose $0 \subsetneqq Q \subset P[T]$ where Q is a prime ideal of $R[T]$. We have $0 \subsetneqq Q \cap C[T] \subset p[T]$ where $p = P \cap C \in X^1(C)$. Therefore $p[T] \in X^1(C[T])$ because C is a Krull domain. Hence $Q \cap C[T] = p[T]$. Therefore $Q \cap R \neq 0$ and from $P \in X^1(R)$ we derive that $Q \cap R = P$. Thus $P[T] = Q$.

(2) Conversely, in view of Theorem 2.1, Propositions 3.2 and 3.3 and Corollary 3.5 it remains to prove that for any ideal I of $R[T]$ and any $P \in X^1(R)$ such that $(I \cap C[T]) \subset (P \cap C)[T]$, we have $I \subset P[T]$. Because $(I \cap C[T]) \subset (P \cap C)[T]$, $I\overline{R[T]}$ (where $\overline{R[T]} = R[T]\ (C[T] \setminus (P \cap C)[T])^{-1}$) is a proper ideal of $\overline{R[T]}$ and is contained in some maximal ideal $\bar{M}$. Then $\bar{M} \cap R[T] = M$ is a prime ideal

of $R[T]$ such that $(M \cap C[T]) \subset (P \cap C)[T]$. Now, $(P \cap C)[T] \in X^1(C[T])$ whence $M \cap C[T] = (P \cap C)[T]$. This implies that $M \cap C = P \cap C$; thus $(M \cap R) \subset P$ because R is central Ω-Krull. From $M \cap R \in \mathrm{Spec}(R)$ and $P \in X^1(R)$ it follows that $M \cap R = P$ and $P[T] \subset M$. Now we claim that $M \subset P[T]$. Suppose not, then $M \not\subset P[T]$ and $\varphi(M) \neq 0$ where φ denotes the canonical map $\varphi : R[T] \rightarrow R[T]/P[T] \cong (R/P)[T]$. Because R is a central Ω-Krull ring, R/P satisfies Formanek's condition and so does $(R/P)[T]$. Therefore $\varphi(M) \cap Z(R/P)[T] \neq 0$. The assumption and Lemma 3.6 yield that $\varphi(M) \cap (C/P \cap C)[T] \neq 0$. So $M \cap C[T] = (M + P[T]) \cap C[T] \not\subset (P \cap C)[T]$, a contradiction. Thus $M = P[T]$ and $I \subset P[T]$.

Note that we have proved the following fact : if R is a central Ω-Krull ring and $P \in X^1(R)$ then the following statements are equivalent :

(1) $(I \cap C[T]) \subset (P \cap C)[T]$ implies $I \subset P[T]$ where I is an ideal of $R[T]$.

(2) $Q(Z(R/P))$ is algebraic over $Q(C/P \cap C)$.

The next proposition answers an open question posed in [5] in the case that R is a central Ω-Krull ring.

Proposition 3.8 : Let R be a central Ω-Krull ring and suppose that for all $P \in X^1(R)$ $Q(Z(R/P))$ is algebraic over $Q(C/P \cap C)$. Then $R[X_1,\dots,X_n]$ is a central Ω-Krull ring for all $n \in \mathbb{N}$.

Proof : We proceed by induction on n. The case $n = 1$ follows from Theorem 3.7. By induction, we may assume that $R[X_1,\dots,X_n]$ is a central Ω-Krull ring. We need to prove that for all $P \in X^1(R[X_1,\dots,X_n])$ and for all ideals I of $R[X_1,\dots,X_n]$ $I \subset P$ iff $I \cap C[X_1,\dots,X_n]) \subset P \cap C[X_1,\dots,X_n]$. If $P \cap R[X_1,\dots,X_{n-1}] = 0$, then $I \subset P$ by Corollary 3.5. The same result holds if $P \cap R[X_1,\dots,\hat{X}_i,\dots,X_n] = 0$ where $\hat{X}_i$ indicates that X_i does not occur. Hence we may assume that $P \cap R[X_1,\dots,X_{n-1}] \neq 0,\dots, P \cap R[X_2,\dots,X_n] \neq 0$. Therefore

$P = (P \cap R)[X_1,\ldots,X_n] = ((P \cap R)[X_1,\ldots,X_{n-1}])[X_n]$ and $P \cap R \in X^1(R)$. Note that $(P \cap R)[X_1,\ldots,X_{n-1}] \in X^1(R[X_1,\ldots,X_{n-1}])$ because $R[X_1,\ldots,X_{n-1}]$ is a central Ω-Krull ring. We still need to prove that $Z(R[X_1,\ldots,X_{n-1}]/(P \cap R)[X_1,\ldots,X_{n-1}])[X_n] \cong Z(R/P \cap R)[X_1,\ldots,X_n]$ has the i.p. with respect to $(C[X_1,\ldots,X_{n-1}]/(P \cap C)[X_1,\ldots,X_{n-1}])[X_n] \cong (C/P \cap C)[X_1,\ldots,X_n]$. But this is clearly satisfied since $Q(Z(R/P \cap R))$ is algebraic over $Q(C/P \cap C)$.

Remark

The authors do not know if the condition of Theorem 3.7 is always fulfilled or not. We already noted that it's still an open question whether or not condition (3) of Theorem 2.1 can be weakened or even dropped. If the last case is true, then $R[T]$ is always a central Ω-Krull ring if R is.

Lemma 3.9 : Let R be a prime ring satisfying Formanek's condition. If $A \in \mathbb{F}(R)$, then $(R[T]:_{\ell}A[T]) = (R:_{\ell}A)[T]$.

Proof : It is clear that $(R:_{\ell}A)[T] \subset (R[T]:_{\ell}A[T])$. Conversely, let $\alpha \in R[T]:_{\ell}A[T]$. We may write $\alpha = g(T)^{-1}f(T) \in Q_{sym}(R[T])$ with $f(T) \in R[T]$ and $g(T) \in C[T]$. $A \cap C \neq 0$ since $A \in \mathbb{F}(R)$. Choose $0 \neq c \in A[T] \cap C$. Then $g(T)^{-1}f(T)c = h(T) \in R[T]$ yielding that $g(T)^{-1}f(T) = c^{-1}h(T) = \sum_j c^{-1}d_jT^j$ with $h(T) = \sum_j d_jT^j$. Since $(\sum_j c^{-1}d_jT^j)A \subset R[T]$, we have $c^{-1}d_j \in (R:_{\ell}A)$ for all j and hence $\alpha \in (R:_{\ell}A)[T]$.

We end this note with the following nice observation.

Proposition 3.10 : If $R[T]$ is a central Ω-Krull ring, then R is also a central Ω-Krull ring.

Proof : We check the conditions of Theorem 2.1. First, let S be a ring such that $R \subset S \subset Q_{sym}(R)$ and $cS \subset R$ for some $0 \neq c \in C$. Then $R[T] \subset S[T] \subset Q_{sym}(R[T])$

and $cS[T] \subset R[T]$. Hence $R = S$ since $R[T]$ is a symmetric maximal order. Suppose $A_1 \subset \ldots \subset A_n \subset \ldots \subset R$ is an ascending chain of divisorial ideals. By the preceding lemma $A_1[T] \subset \ldots \subset A_n[T] \subset \ldots \subset R[T]$ is a chain of divisorial ideals of $R[T]$. Since $R[T]$ has ACC on divisorial ideals, $A_n = A_{n+1} = \ldots$ for some n. Finally, let $P \in X^1(R)$, I an ideal of R and $(I \cap C) \subset (P \cap C) = p$. Then $(I \cap C)[T] \subset P[T] \cap C[T]$. If we can prove that $P[T] \in X^1(R[T])$, then $I \subset P$ since $R[T]$ is central Ω-Krull. Now $P[T] \in \mathrm{Spec}(R[T])$ and $P[T] \in \mathbb{D}(R[T])$ by Lemma 3.9. Hence $P[T] \in X^1(R[T])$.

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