Ω-Krull Rings, I.

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0. Introduction

In the search for a class of noncommutative rings with an arithmetical ideal theory, generalizing the classical theory of Dedekind domains, several possibilities arise; e.g. HNP-rings, Asano orders and Dedekind prime rings. An even richer gamma of possible definitions is available in case one aims to produce a noncommutative counterpart to the theory of Krull domains. A natural way to define these rings is by imposing local conditions at the minimal nonzero prime ideals. The first ramification of the theory is created by the fact that in the noncommutative case one may consider independent local conditions on the localized rings but also on the type of localization used in the construction of that local ring. Recently, some types of noncommutative Krull rings have been studied by M. Chamarie [2], R. Fossum [3], and in particular H. Marubayashi [8,9,10,11]. All these rings are supposed to be (maximal) orders in simple Artinian rings, e.g. Marubayashi-Krull rings are Goldie prime rings such that the Lambek-Michler localizations at prime ideals of height one are Asano orders.

On the other hand, F. Van Oystaeyen constructed a class of (not

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necessarily Goldie-) prime rings having properties analogeous to those of Asano-orders, namely Ω-rings, cfr [13,19]. In this note we aim to generalize Ω-rings in about the same manner Marubayashi Krull rings generalize Asano-orders, this time using symmetric localization.

In section 2 we prove that the divisor classes form an abelian group which is a direct product of infinite cyclic subgroups. One of the main motivations for studying Ω-rings is the fact that they fit nicely in the theory of primes [13,15], which is the most manageable generalization of valuation theory to the noncommutative case known to the authors. In section 3 we indicate that there is an equally sufficient valuation-like theory attached to Ω-Krull rings, which will be developed in part II of this paper.

In the last two sections, we present some examples of Ω-Krull rings. A sufficient condition is given to assure that R[T] remains Ω-Krull if R is so.

1. Ω-*Krull rings*

Throughout this note, all rings will be associative and have a unit element, modules will be unitary. Ideal will always mean twosided ideal. R-mod (resp. mod-R) stands for the category of all left (resp. right) R-modules.

An endofunctor σ in R-mod is said to be a *kernel functor* if it is a left exact subfunctor of the identity in R-mod, σ is said to be *idempotent* if $\sigma(M/\sigma(M)) = 0$ for any $M \in$ R-mod. To a kernel functor σ the filter of left ideals of R, $\mathcal{L}(\sigma) = \{L \text{ left ideal of } R : \sigma(R/L) = R/L\}$ is associated and to a filter $\mathcal{L}$ satisfying :

(K1) : If $I,J \in \mathcal{L}$, then $I \cap J \in \mathcal{L}$;

(K2) : If $I \in \mathcal{L}$ and J is a left ideal of R such that $I \subset J$, then $J \in \mathcal{L}$;

(K3) : If $I \in \mathcal{L}$ and $x \in R$, then $(I:x) = \{r \in R : rx \in I\} \in \mathcal{L}$,

one associates the kernel functor $\sigma_{\mathcal{L}}(M) = \{m \in M | \exists I \in \mathcal{L}: Im = 0\}$. Recall from [5,14,17] that this defines a one-to-one correspondence. $\sigma_{\mathcal{L}}$ will be idempotent if and only if $\mathcal{L}$ satisfies also :

(K4) : If $I \in \mathcal{L}$ and J is a left ideal of R such that $(J:x) \in \mathcal{L}$ for every $x \in I$, then $J \in \mathcal{L}$.

A kernel functor σ is called *bilateral* if its associated filter $\mathcal{L}(\sigma)$ has a cofinal set consisting of ideals, σ is said to be *symmetric* if it is both idempotent and bilateral.

It is well known that one can associate to any idempotent kernel functor σ a left exact localization functor $Q_\sigma(\cdot)$ in R-mod. $Q_\sigma(R)$ is a ring containing $R/\sigma(R)$ as a subring and there is a canonical ringhomomorphism $j_\sigma : R \to Q_\sigma(R)$. In particular, if R is a prime ring, $\sigma(R) = 0$, j_σ is the canonical embedding and R may be viewed as a subring of $Q_\sigma(R)$. An idempotent kernel functor σ is said to have *property* T if it satisfies one of the following equivalent conditions : (T1) :$Q_\sigma(\cdot)$ is right exact and commutes with direct sums ;

(T2) : For every $I \in \mathcal{L}(\sigma)$ we have : $Q_\sigma(R)j_\sigma(I) = Q_\sigma(R)$;

(T3) : For every $M \in$ R-mod : $Q_\sigma(M) \cong Q_\sigma(R) \otimes_R M$;

(T4) : $j_\sigma : R \to Q_\sigma(R)$ is a flat epimorphism of rings.

σ will be called *geometrical* if it has property T and satisfies :

(G) : For any ideal I of R, $Q_\sigma(R)j_\sigma(I)$ is an ideal of $Q_\sigma(R)$. E.g. if σ is a *central* idempotent kernel functor, i.e. $\mathcal{L}(\sigma)$ has a cofinal set consisting of centrally generated ideals, σ is geometrical.

Likewise, one can define all these concepts in mod-R. If $\mathcal{L}^2(\sigma)$ is a set of ideals of R which is multiplicatively closed and if $\mathcal{L}^\ell(\sigma)$ (resp. $\mathcal{L}^r(\sigma)$) (i.e. the filter of left (resp. right) R-ideals generated by $\mathcal{L}^2(\sigma)$) is idempotent, we will denote by $Q^\ell_\sigma(\cdot)$ (resp. $Q^r_\ell(\cdot)$) the localization functor in R-mod (resp. mod-R) associated with $\mathcal{L}^\ell(\sigma)$ (resp. $\mathcal{L}^r(\sigma)$). E.g. if P is a prime ideal of R, $\mathcal{L}^2(R-P)$ will be the multiplicatively closed set of ideals I of R not contained in P, $Q^\ell_{R-P}(\cdot)$ and $Q^r_{R-P}(\cdot)$ will be the associated localization functors.

Throughout, R will be a prime ring. The first problem encountered is to find a symmetric analogue of the Goldie theorems, i.e. to give necessary and sufficient conditions such that R may be embedded in a symmetric localization $Q_{sym}(R)$ which is a simple ring (eventually satisfying additional chain conditions) such that any localization of R at a symmetric kernel functor can be viewed as a subring of $Q_{sym}(R)$. Clearly, a sufficient condition is that σ_{R-0} is an idempotent kernel functor having property T. However, to find a necessary and sufficient intrinsic characterization in terms of elements and ideals of R might prove rather difficult.

In order to bypass this problem as well as to exclude oddities as the ones encountered in [6] and [7], arising from the fact that certain prime ideals may have trivial intersection with the center Z(R), we will limit ourselves to prime rings R satisfying Formanek's condition :

(F) : For every ideal I of R, $I \cap Z(R) \neq 0$.

lemma 1.1. : If R is a prime ring satisfying (F), then :

(1) : σ_{R-0} is an idempotent kernel functor,

(2) : $Q^{\ell}_{R-0}(R) \cong Q^{r}_{R-0}(R) \cong \{c^{-1}r = rc^{-1} | r \in R,\ 0 \neq c \in Z(R)\} = Q_{sym}(R)$ are simple rings.

proof

(1) : Suppose $I \in \mathcal{L}^2(R-0)$, J a left ideal of R such that $\sigma(I/J) = I/J$. It will be sufficient to prove that $J \in \mathcal{L}^{\ell}(R-0)$. Take $0 \neq c \in I \cap Z(R)$, then $I' = Rc \in \mathcal{L}^2(R-0)$ and $\sigma(I'/I' \cap J) = I'/I' \cap J$. There exists an ideal $I'' \in \mathcal{L}^2(R-0)$ such that $I''c = cI'' \subset I' \cap J$ whence : $I'I'' \subset I' \cap J \subset J$ and thus $J \in \mathcal{L}^{\ell}(R-0)$ because $I'I'' \in \mathcal{L}^2(R-0)$.

(2) : In view of (1) it is easy to check that σ^{ℓ}_{R-0} and σ^{r}_{R-0} have property T, using results of [19]. Therefore, $Q^{\ell}_{R-0}(R)$ and $Q^{r}_{R-0}(R)$ are simple rings. Verification of the fact that they are equal to $\{c^{-1}r = rc^{-1} | r \in R,\ 0 \neq c \in Z(R)\}$ is straightforward.

A prime ring satisfying (F) is said to be an Ω-*ring* if every ideal is a product of maximal ideals. We call a ring *quasi-local* if it has a unique maximal ideal.

Definition 1.2. : A prime ring satisfying (F) is said to be an Ω-*Krull ring* if the following conditions hold :

(1) : There exist multiplicatively closed sets of ideals $\mathcal{L}^2(\sigma_i)$ $(i \in \Lambda)$ such that :

$$R_i = Q^{\ell}_{\sigma_i}(R) = \{q \in Q_{sym}(R) | \exists I \in \mathcal{L}^{\ell}(\sigma_i) : Iq \subset R\}$$
$$= Q^{r}_{\sigma_i}(R) = \{q \in Q_{sym}(R) | \exists I \in \mathcal{L}^{r}(\sigma_i) : qI \subset R\}$$

(2) : $\forall i \in \Lambda$: R_i is a quasi-local Ω-ring;

(3) : $R = \bigcap_{i \in \Lambda} R_i$;

(4) : For every $r \in R$ there are only finitely many $i \in \Lambda$ such that $(r) = RrR \notin \mathcal{L}^2(\sigma_i)$;

(5) : $\forall i \in \Lambda;\ \forall I \in \mathcal{L}^2(\sigma_i) : R_i\, I = IR_i = R_i$.

Remark : We have used the notation $R_i = Q^{\ell}_{\sigma_i}(R) = Q^{r}_{\sigma_i}(R)$. However, there is no a priori reason why these R_i should be localizations. The following lemma proves they are :

lemma 1.3. : $\forall i \in \Lambda : \sigma_i$ is an idempotent kernel functor.

proof

(We write σ for σ_i). Suppose $I \in \mathcal{L}^2(\sigma)$, J a left R-ideal and $\sigma(I/J) = I/J$. Because $Q^{\ell}_{\sigma}(R)I = Q^{\ell}_{\sigma}(R)$ we can write :
$1 = \alpha_1\beta_1 + \ldots + \alpha_n\beta_n$ where $\alpha_i \in Q^{\ell}_{\sigma}(R)$ and $\beta_i \in I$. Take, $I' = R\beta_1 + \ldots + R\beta_n$ then $Q^{\ell}_{\sigma}(R)I' = Q^{\ell}_{\sigma}(R)$. Therefore $I' \in \mathcal{L}^{\ell}(\sigma)$ for there exists an ideal $K \in \mathcal{L}^2(\sigma)$ such that $K\alpha_i \subset R$, $1 \leqslant i \leqslant n$. Therefore $K.1 \subset R\beta_1 + \ldots + R\beta_n = I'$. Because $I' \subset I$ we have $\sigma(I'/J \cap I') = I'/J \cap I'$. So, there exists an ideal $L \in \mathcal{L}^2(\sigma)$ such that $L\beta_i \subset J \cap I'$, $1 \leqslant i \leqslant n$. Finally, we obtain : $L.I' \subset J \cap I' \subset J$ and $L.I' \in \mathcal{L}^{\ell}(\sigma)$ because $\mathcal{L}^2(\sigma)$ is closed under taking products, therefore $J \in \mathcal{L}^{\ell}(\sigma)$.

We immediately obtain that $R_i = Q^{\ell}_{\sigma_i}(R)$ is indeed the localization of R with respect to the symmetric kernel functor σ^{ℓ}_i having property (T). (5) may be strenghtened to

(5') : $\forall i \in \Lambda : \sigma^{\ell}_i$ and σ^{r}_i are geometrical, or to :

(5") : $\forall i \in \Lambda : \sigma^{\ell}_i$ and σ^{r}_i are central kernel functors.

Remark : The R_i are prime rings satisfying the condition of Formanek, so they have a symmetric ring of quotients $Q_{sym}(R_i)$. If R satisfies (5') it can be seen (using results of [16]) that $Q_{sym}(R) = Q_{sym}(R_i)$.

Each R_i is a quasi-local Ω-ring. In particular, R_i has a unique maximal ideal P_i', we will denote $P_i = R \cap P_i'$.

Proposition 1.4. :

(1) : $P_i' = P_iR_i = R_iP_i$,

(2) : P_i is a prime ideal of R.

proof

(1) : It is clear that $R_iP_i \subset P_i'$. Conversely, take $a \in P_i'$ then we can find an ideal $I \in \mathcal{L}^2(\sigma_i)$ such that $Ia \subset R \cap P_i' = P_i$. Hence $a \in R_ia = R_iIa \subset R_iP_i$. Similarly, $P_i' = P_iR_i$.

(2) : Suppose $AB \subset P_i$ where A and B are ideals of R and $B \not\subset P_i$. Then $R_iBR_i \not\subset P_i'$ whence $R_iBR_i = R_i$. Write $1 = \sum x_jb_j\, y_j$ where $x_j, y_j \in R_i$ and $b_j \in B$. There exists an ideal $I \in \mathcal{L}^2(\sigma_i)$ such that $Ix_j \subset R$ and $y_jI \subset R$ for each j. Therefore, $I^2 = I.1.I \subset RBR = B$. Finally, we obtain $A \subset AR_i = AI^2R_i \subset ABR_i \subset P_iR_i = P_i'$ and thus : $A \subset P_i' \cap R = P_i$.

Remark :

1. In case R satisfies (5'), Prop. 1.4. is nothing but a reformulation of the following result due to F. Van Oystaeyen [16] : For a geometrical kernel functor κ there is a one-to-one correspondence between Spec $Q_\kappa(R)$ and $G(\kappa)$ the set of prime ideals of R maximal with respect to not belonging to $\mathcal{L}(\kappa)$.

2. It is straightforward to check that $\mathcal{L}^2(\sigma_i) \subset \mathcal{L}^2(R-P_i)$. Conversely, if $J \in \mathcal{L}^2(R-P_i)$ then $J \not\subset P_i$. Take $x \in J-P_i$, then : $R_i\, x\, R_i = R_i$. Therefore we can find elements a_k, b_k in R_i such that $\sum a_k\, x\, b_k = 1$. There exists an ideal $I \in \mathcal{L}^2(\sigma_i)$ such that $Ia_k \subset R$ and $b_kI \subset R$ for all k. Finally, $I^2 = I.1.I \subset R\, x\, R \subset J$ whence $J \in \mathcal{L}^2(\sigma_i)$.

Now consider the following conditions :

(6) : $\forall i \neq j \in \Lambda$: $P_i \not\subset P_j$ and $P_j \not\subset P_i$;

(6') : $\forall i \neq j \in \Lambda$: $P_iR_j = P_jP_i = R_j$.

Using remark 2 above it is easy to see that these conditions are equivalent, for, $P_i \not\subset P_j$ iff $P_i \in \mathcal{L}^2(R-P_j) = \mathcal{L}^2(\sigma_j)$ iff $P_iR_j = R_jP_i = R_j$.

2. *Fractional ideals and divisor classes*

For every $i \in \Lambda$, we define a *fractional* R_i*-ideal* I_i to be a nonzero left and right R_i-submodule of $Q_{sym}(R)$ such that there exists an element $c_i \in Z(R_i)$: $c_iI_i \subset R_i$.

Proposition 2.1. : The fractional ideals of R_i form an abelian group under multiplication, for every $i \in \Lambda$.

proof

Take $0 \neq c \in P_i' \cap Z(R_i)$. Then $R_ic \cdot R_ic^{-1} = R_i = R_ic^{-1} \cdot R_ic$ (remark that c is invertible in $Q_{sym}(R)$ because $Z(R_i) \subset Z(Q_{sym}(R))$ which is a field because $Q_{sym}(R)$ is simple). R_ic is an ideal of R_i, hence, $R_ic = (P_i')^n$ for some $n \in \mathbb{N}$. Therefore, $P_i' \cdot ((P_i')^{n-1} \cdot R_ic^{-1}) = R_i = (R_ic^{-1} \cdot (P_i')^{n-1}) \cdot P_i'$ and thus $P_i'^{-1}$ exists. This implies that every ideal of R_i is invertible. That the fractional ideals of R_i form an abelian group is now easily checked.

We define a *fractional* R*-ideal* I to be a nonzero left and right R-submodule of $Q_{sym}(R)$ such that there exists an element $c \in Z(R)$: $cI \subset R$.

lemma 2.2. : When I is a fractional R-ideal, $R_iIR_i \neq R_i$ for only finitely many $i \in \Lambda$.

proof

Because $Q_{sym}(R)$ is an essential extension of R, $I \cap R \neq 0$. Take $0 \neq c \in I \cap R$, then $(c) = RcR \subset I$ and $R_i c R_i \subset R_i I R_i$. From (4) and (5) we get that $R_i \subset R_i I R_i$ for almost all $i \in \Lambda$. On the other hand, there is an element $d \in Z(R)$ such that $d(R_i I R_i) \subset R_i$ for each $i \in \Lambda$. Now, $R_i(RdR) = R_i$ for almost all $i \in \Lambda$. Hence $R_i I R_i \subset R_i$ for all but finitely many $i \in \Lambda$.

<u>lemma 2.3.</u> : Suppose that R satisfies (6) and let I_i' be a fractional R_i-ideal for each $i \in \Lambda$ such that for almost all $i \in \Lambda$: $I_i' = R_i$. Then $I = \cap I_i'$ is a fractional R-ideal and $R_i I = I R_i = I_i'$.

<u>proof</u>

Put $I_i = I_i' \cap R$. Suppose first that $I_i' \subset R_i$ for every $i \in \Lambda$. Then $I \subset R$ and $I = I_1 \cap \ldots \cap I_k$ with $I_j \subsetneq R_j$, $j \in \{1,\ldots,k\}$. We have that each $I_j' = (P_j')^{n_j} (n_j > 0)$ whence $(P_j)^{n_j} \subset I_j = (P_j')^{n_j} \cap R$. When $i \neq j$, then $R_i = (P_j)^{n_j} R_i \subset I_j R_i \subset R_i$ because of (6'). When $i = j$, $R_j I_j = I_j'$. Because R_i is a flat R-module, we get that $R_i I = R_i(I_1 \cap \ldots \cap I_k) = R_i I_1 \cap \ldots \cap R_i I_k = R_i \cap \ldots \cap I_i' \cap \ldots \cap R_i = I_i' = I R_i$. In the general case, there exist elements $c_i \in Z(R_i)$ for each $1 \leqslant i \leqslant k$ such that $c_i I_i' \subset R_i$. Furthermore, all c_i may be chosen in $Z(R)$ because $Z(R_i) \subset Z(Q_{sym}(R))$ and $Z(Q_{sym}(R))$ is the field of fractions of $Z(R)$. Put $c = c_1 \ldots c_k \in Z(R)$ then $cI = \cap cI_i'$ and hence $cIR_i = cI_i'$ whence $IR_i = I_i'$.

Let A be a fractional R-ideal. Consider the twosided R-module $A_d = \bigcap_{i \in \Lambda} R_i A R_i$. Clearly, A_d is a fractional R-ideal and we can write each $R_i A R_i = (P_i')^{n_i} (n_i \in \mathbb{Z})$, whence $A_d = \bigcap_{i \in \Lambda} (P_i')^{n_i}$ and almost all n_i are equal to 0.

<u>lemma 2.4.</u> : (1) $A \subset A_d$; (2) If $A \subset B$ then $A_d \subset B_d$; (3) $A_{dd} = A_d$.

<u>proof</u>

(1) and (2) are obvious.

(3) $A_d = \cap (P_i')^{n_i}$. Since each $(P_i')^{n_i}$ is a fractional R_i-ideal, $R_i A_d R_i = (P_i')^{n_i}$

by lemma 2.3. Hence $A_{dd} = A_d$.

Remark : It is easy to check that A_d is precisely the $\underset{i\in\Lambda}{\wedge} \sigma_i$-closure of A in $Q_{sym}(R)$, i.e. :
$A_d = \{q \in Q_{sym}(R) \mid \forall i \in \Lambda, \exists I \in \mathcal{L}^2(\sigma_i) : Iq \subset A \text{ and } qI \subset A\}$.

If $A = A_d$, then A is said to be a *divisorial ideal*. We define an equivalence relation on the set of fractional R-ideals by saying that $A \sim B$ if and only if $A_d = B_d$. Denote by $\bar{A}$ the equivalence class determined by A. The set $\mathbb{D}(R)$ of all equivalence classes forms a semigroup under $*$ defined by $\bar{A} * \bar{B} = (\overline{A_dB_d})$. The unit element of $\mathbb{D}(R)$ is of course $\bar{R}$.

Theorem 2.5. : If R is an Ω-Krull ring satisfying (6), then $\mathbb{D}(R)$ is a direct product of infinite cyclic subgroups generated by $\{\bar{P}_i\}_{i\in\Lambda}$.

proof

(1) Suppose A is a divisorial ideal, say $A = \cap P_i'^{n_i}$ where $n_i \in \mathbb{Z}$ and $n_i = 0$ for almost all $i \in \Lambda$. Put $B = \cap P_i'^{-n_i}$. From lemma 2.3 we deduce that B is a divisorial R-ideal. Then, $R_i(R_iAR_i)(R_iBR_i)R_i = (P_i')^{n_i}(P_i')^{-n_i} = R_i$. Hence, $\bar{A} * \bar{B} = \bar{R}$ and $\mathbb{D}(R)$ is a group under $*$.
(2) Suppose $A = \cap P_i'^{n_i}$ and $B = \cap P_i'^{m_i}$ are both divisorial R-ideals, then $R_i(R_iAR_i)(R_iBR_i)R_i = P_i'^{n_i+m_i} = R_i(R_iBR_i)(R_iAR_i)R_i$ whence $\mathbb{D}(R)$ is abelian.
(3) Write $A = P_1'^{n_1} \cap \ldots \cap P_k'^{n_k} \cap (\cap R_j)$. It is easy to check that $\bar{A} = (\bar{P}_1)^{n_1} * \ldots * (\bar{P}_k)^{n_k}$, whence $\mathbb{D}(R)$ is generated by $\{\bar{P}_i\}_{i\in\Lambda}$. Finally, suppose $(\bar{P}_1)^{n_1} * \ldots * (\bar{P}_n)^{n_k} = \bar{R}$, then $P_1'^{n_1} \cap \ldots \cap P_k'^{n_k} \cap (\cap R_j) = R$ whence $R_iR = R_i = (P_i')^{n_i}$ yielding that every $n_i = 0$.

Theorem 2.6. : The center of an Ω-Krull ring is a Krull domain.

proof

Because R is a prime ring, Z(R) is clearly a domain. Because the R_i are localizations of R, it is readily checked that $Z(R) = \bigcap_{i \in \Lambda} Z(R_i)$. We will prove that each $Z(R_i)$ is a discrete valuation ring. We know that $K = Z(Q_{sym}(R))$ is the field of fractions of $Z(R_i)$. Define a function v on K^* in the following way :

$v : K^* \to \mathbb{Z} : a^{-1}b \to n-m$ where $R_i a = (P_i')^m$; $R_i b = (P_i')^n$. Suppose $a^{-1}b = c^{-1}d$ where a,b,c and d are elements of $Z(R_i)$. We can write $R_i c = (P_i')^k$ and $R_i d = (P_i')^\ell$. Because the elements of $Z(R_i)$ are invertible in K, $R_i a\, R_i a^{-1} = R_i$ whence $R_i a^{-1} = (P_i')^{-m}$, similarly $R_i c^{-1} = (P_i')^{-k}$. Therefore $R_i a^{-1} R_i b = R_i c^{-1} R_i d$ whence $(P_i')^{n-m} = (P_i')^{\ell-k}$. Thus $n-m = \ell-k$ and v is well defined. It is now easy to check that v is a $\mathbb{Z}$-valued valuation. $v(a^{-1}b) \geq 0$ if and only if $R_i a^{-1}b = (P_i')^{n-m} \subseteq R_i$, whence $a^{-1}b \in R_i$. We conclude that $\{x \in K^* | v(x) \geq 0\} = Z(R_i)$ and therefore $Z(R_i)$ is a discrete valuationring.
Finally, take $c \in Z(R)$. Then $R_i(c) = R_i$ for almost all $i \in \Lambda$. But $(c) = Rc$ and $R_i(c) = R_i c = R_i$ for almost all $i \in \Lambda$. Hence c is a unit in all but finitely many i whence Z(R) is a Krull domain.

Let A be a ring, X = Spec A its prime spectrum equipped with the Zariski topology. A Zariski open set is equal to some $X(I) = \{P \in X | I \not\subset P\}$ where I is an ideal of A. A ring homomorphism $f : A \to B$ is said to be an *extension* if $B = f(A)Z_B(A)$ where $Z_B(A) = \{b \in B | \forall a \in A : bf(a) = f(a)b\}$. In that case $f^{-1}(P) \in \text{Spec } A$ for any $P \in \text{Spec } B$ and φ : Spec B $\to$ Spec A; $P \mapsto f^{-1}(P)$ is a continuous mapping.
A monomorphic extension $f : A \hookrightarrow B$ is said to be a *Zariski extension* if there exist nonempty Zariski open sets $Y(I) \subset \text{Spec } B$ and $X(J) \subset \text{Spec } A$ such that the restriction of φ yields a homeomorphism between Y(I) and X(J) with their induced topologies, such that for every ideal $H \subset \text{rad } I$, the open set $Y(H) \subset Y(I)$ corresponds to an open set $Y(H') \subset X(J)$ with $H' \subset f^{-1}(H)$. If $A \subset B$ is an extension, $\varphi(P) = P \cap A$. For nonempty open sets Y(I) and X(J), the Zariski

extension property is equivalent to $\varphi(Y(I)) = X(J)$ and $H = \text{rad } B(H \cap A)$ for every radical ideal $H \subset \text{rad } I$. A global Zariski extension is a Zariski extension such that $\varphi(I) = \text{Spec } B$.

Proposition 2.7. : Every R_i is a global Zariski extension of its center.

proof

R_i is a local Ω-ring. Hence $\text{Spec } R_i = \{0, P_i'\}$. $Z(R_i)$ is a discrete valuation ring whence $\text{Spec } Z(R_i) = \{0, p_i\}$ where p_i is the unique maximal ideal of $Z(R_i)$. Clearly, $P_i' \cap Z(R_i) = p_i$ and P_i' is the unique prime ideal lying over p_i. It is trivial to see that $\text{rad } H = \text{rad } R_i(H \cap Z(R_i))$ for every ideal H of R_i.

Remarks

1) Each minimal prime ideal of R belongs to the set $\{P_i \mid i \in \Lambda\}$. For suppose P is minimal prime. Take $0 \neq c \in P \cap Z(R)$. We have $Rc = \bigcap_i R_i c = \cap P_i'^{n_i}$. If $n_1, \ldots, n_k$ are the only integers different from zero, we easily obtain $P_1^{n_1} \ldots P_k^{n_k} \subset P_1^{n_1} \cap \ldots \cap P_k^{n_k} \subset P$ whence $P_i \subset P$ for some i. Hence $P = P_i$.

2) If each σ_i is a geometric kernel functor, we also have the converse result, namely each P_i is a minimal prime ideal. It's easy to check that $R_i A \cap R = \{x \in R \mid Ix \subset A \text{ for some } I \in \mathcal{L}^2(\sigma_i)\}$ when A is an ideal of R. Suppose A is prime and $A \subset P_i$. We claim that $R_i A \cap R = A$. If $Ix \subset A$ for some $I \in \mathcal{L}^2(\sigma_i)$, then $IR \times R \subset A$ whence $I \subset A$ or $R \times R \subset A$. The inclusion $I \subset A$ leads to a contradiction because $A \subset P_i$. Hence $x \in A$.
On the other hand, $R_i A$ is a prime ideal in R_i, for suppose $XY \subset R_i A$ with X, Y ideals of R_i. Then $(X \cap R)(Y \cap R) \subset XY \cap R \subset R_i A \cap R = A$. Because A is prime and each σ_i is a T-functor we obtain $X \subset R_i A$ or $Y \subset R_i A$. Finally, because $R_i A$ is prime, we have $R_i A = P_i'$ and therefore $A = R_i A \cap R = P_i$.

3) If the set $\{P_i : i \in \Lambda\}$ is equal to the set of minimal nonzero prime ideals, then a prime ideal is divisorial if and only if it is minimal prime. For suppose P is a prime which is not minimal. There exist elements x_i such that $x_i \in P$ but $x_i \notin P_i$ and this is true for all i. Hence $R_i x_i R_i = R_i$ and certainly $R_i P R_i = R_i$ for each i. Therefore $P \subsetneq P_d = R$. The other implication is trivial.

3. *Arithmetical pseudo valuations on Ω-Krull rings*

Throughout this section, every σ_i is supposed to be geometric. In the commutative case, valuation theory is a powerful tool in studying Krull domains. The most manageable noncommutative generalization of valuation rings known to the authors is the theory of the Van Geel-primes (cfr. [13,15]). In this section we aim to relate the so called pseudo valuation functions on the set of divisor classes to primes in $Q_{sym}(R)$. Because of its apparent importance for the definition and description of the class group, we will postpone a full account of this connexion until part II of this paper.

Let us recall some definitions. Let S be any ring. Following J. Van Geel [13,15] we will call a pair (P,S') a *prime* in S if and only if it satisfies the following properties :

(P1) : S' is a subring of S ;

(P2) : P is a prime ideal of S';

(P3) : $\forall x,y \in S$: if $xS'y \subset P$ then either $x \in P$ or $y \in P$.

If (P,S') is a prime in S, so is (P,S^P) where we denote by $S^P = \{s \in S | sP \subset P$ and $Ps \subset P\}$.

Primes are natural generalizations of commutative valuation rings, for, if $S = K$ is a field, (P,K^P) is a prime in K if and only if K^P is a valuation ring in K and P is its maximal ideal.

Extending the terminology of [13] to the Ω-Krull ring case, we define :

Definition 3.1. : An *arithmetical pseudo valuation* v on $\mathbb{D}(R)$ is a function $v : \mathbb{D}(R) \to \Gamma \cup \{\infty\}$ where Γ is a totally ordered group such that :

(V1) : $\forall I,J \in \mathbb{D}(R)$: $v(I * J) = v(I) + v(J)$;

(V2) : $\forall I,J \in \mathbb{D}(R)$: $v(\overline{I + J}) \geqslant \min(v(I),v(J))$;

(V3) : $\forall I,J \in \mathbb{D}(R)$: if $I \subset J$ then $v(I) \geqslant v(J)$;

(V4) : $v(R) = 0$ and $v(0) = \infty$.

For any $x \in Q = Q_{sym}(R)$ we will denote :

$C_x = \bigcap_{i \in \Lambda} R_i\, x\, R_i$, which is clearly a divisorial R-ideal. The next theorem is a slight adaptation of a similar result for Ω-rings (cfr [13]) :

Theorem 3.2. :

1. To any arithmetical pseudo valuation v on $\mathbb{D}(R)$ we can associate a prime in $Q_{sym}(R)$.
2. To any prime (P,Q^P) in $Q = Q_{sym}(R)$ such that $P = \bigcap_i R_i P R_i$ and $R \subset Q^P$ we can associate an arithmetical pseudo valuation on $\mathbb{D}(R)$.

proof

1. Let v be an arithmetical pseudo valuation on $\mathbb{D}(R)$. Define $P = \{x \in Q \mid v(C_x) > 0\}$. By definition of v, P is clearly a multiplicatively closed additive subgroup of Q_+ yielding that P is an ideal of Q^P. If $x,y \in Q$ such that $xQ^P y \subset P$, then $xRy \subset P$ because $R \subset Q^P$. Therefore :

$0 < v(\bigcap_{i\in\Lambda} R_i xRyR_i) = v(\bigcap_{i\in\Lambda} R_i(\bigcap_{i\in\Lambda} R_i xR)(\bigcap_{i\in\Lambda} RyR_i)R_i) = v(C_x * C_y) = v(C_x) + v(C_y)$ and thus either $v(C_x) > 0$ or $v(C_y) > 0$ yielding that (P,Q^P) is a prime in $Q = Q_{sym}(R)$.

2. If (P,Q^P) is a prime in Q such that $\bigcap_{i\in\Lambda} R_i P R_i = P$ and $R \subset Q^P$, define for any divisorial R-ideal I :

$v(I) = \{x \in Q \mid C_x * I \subset P\}$. Let Γ be the set $\{v(I) \mid I \in \mathbb{D}(R)\}$, then Γ is totally

ordered by inclusion. To show this, suppose $I,J \in \mathbb{D}(R)$ such that both $v(I) \not\subset v(J)$ and $v(J) \not\subset v(I)$. Therefore, there exist elements $x,y \in Q_{sym}(R)$ such that $C_x \divideontimes I \subset P$, $C_x \divideontimes J \not\subset P$, $C_y \divideontimes I \not\subset P$ and $C_y \divideontimes J \subset P$.
Because (P,Q^P) is a prime, we obtain :
$(C_x \divideontimes J)Q_{sym}(R)^P(C_y \divideontimes I) \not\subset P$ yielding that for some $z \in Q^P$:
$C_x \divideontimes J \divideontimes C_z \divideontimes C_y \divideontimes I \not\subset P$. But $\mathbb{D}(R)$ is an abelian group whence
$C_x \divideontimes I \divideontimes C_z \divideontimes C_y \divideontimes J \subset P$ because for any $Z \in Q^P$ we have that $C_z \divideontimes P \subset P$ and and $P \divideontimes C_z \subset P$, a contradiction. We claim that $v(I) + v(J) = v(I \divideontimes J)$ is a well defined addition on Γ which turns Γ into an ordered group with unit element $v(R)$. For if $v(I) = v(I')$ and $v(J) = v(J')$ for I,I',J and $J' \in \mathbb{D}(R)$, then for any $x \in v(I \divideontimes J)$: $C_x \divideontimes I \divideontimes J \subset P$ whence $C_x \divideontimes I \subset v(J) = v(J')$, hence $C_x \divideontimes I \divideontimes J' = C_x \divideontimes J' \divideontimes I \subset P$, finally, since $v(I) = v(I')$, $C_x \divideontimes J' \divideontimes I' = C_x \divideontimes I' \divideontimes J' \subset P$ follows, i.e. $x \in v(I' \divideontimes J')$. The fact that $v(R)$ is a unit element is obvious.
$v(I) \leqslant v(J)$ yields $v(I) + v(H) \leqslant v(J) + v(H)$ for any $H \in \mathbb{D}(R)$, for, if $x \in v(I \divideontimes H)$ then $C_x \divideontimes H \divideontimes I = C_x \divideontimes I \divideontimes H \subset P$, i.e. $C_x \divideontimes H \subset v(I) \subset v(J)$ whence $C_x \divideontimes J \divideontimes H \subset P$. The required properties (V1)-(V4) follow directly from the definition of v.

4. *Some examples*

In this section we shall give some examples of Ω-Krull rings.

A : A commutative Krull domain is an Ω-Krull ring.

B : A complete matrix ring $M_n(R)$ over an Ω-Krull ring is itself an Ω-Krull ring.

C : An Ω-ring which is a global Zariski extension of its center is an Ω-Krull ring.

D : An Azumaya algebra over a commutative Krull domain is an Ω-Krull ring.

E : If A is any simple algebra with center k and R is a commutative Krull domain containing k, then $A \otimes_k R$ is an Ω-Krull ring.

proof

A and B are straightforward.

C : Let Z(R) be the center of R and $\{P_i\}_{i\in\Lambda}$ the set of all nonzero prime ideals of R. Then all kernel functors σ_{R-P_i} are idempotent, have property T and are geometric, moreover $Q_{R-P_i}(R) = Q_{p_i}(R)$ where $p_i = P_i \cap Z(R)$ (cfr [12,19]). Therefore it will be sufficient to prove that $R = \cap Q_{p_i}(R)$. Obviously, $R \subset \cap Q_{p_i}(R)$. Conversely, suppose that $c^{-1}r \in (\cap Q_{p_i}(R)) \setminus R$ where $c \in Z(R)$ and $r \in R$. Because R is an Ω-ring, $Rc = P_1^{\ell_1}\dots P_n^{\ell_n}$, $RrR = P_1^{k_1}\dots P_n^{k_n}$ where $\ell_i, k_i \in \mathbb{N}$ for all $1 \leqslant i \leqslant n$ and $P_i \neq P_j$ for $i \neq j$. Therefore, $Rc^{-1}rR = Rc^{-1}R\, RrR = P_1^{k_1-\ell_1}\dots P_n^{k_n-\ell_n} \not\subset R$. Therefore, $k_i - \ell_i < 0$ for some i, e.g. i=1. Because $c^{-1}r \in Q_{R-P_1}(R)$ there is an ideal $I \not\subset P_1$ of R such that $Ic^{-1}r \subset R$. Therefore $I = P_2^{r_2}\dots P_s^{r_s}$ with $r_i \in \mathbb{N}$ for all $2 \leqslant i \leqslant s$. So, $Ic^{-1}rR = I.Rc^{-1}R.RrR = P_1^{k_1-\ell_1} . P_2^{k_2-\ell_2+r_2}\dots P_s^{r_s} \subset R$, a contradiction.

D : Let R be a commutative Krull domain and A an Azumaya algebra over R. Clearly $A \cong A^{**} \cong \mathrm{Hom}_R(\mathrm{Hom}_R(A,R),R)$ in R-mod. Therefore, cfr [4], if $X^1(R)$ is the set of prime ideals of R of height one, then $A = \bigcap_{p \in X^1(R)} Q_p(A)$. It is now easy to check that A is an Ω-Krull ring.

E : As in D because every ideal I of $A \otimes_k R$ is of the form $A \otimes_k J$ where J is an ideal of R.

Remark

It follows from E that any polynomial ring over a simple algebra is an Ω-Krull ring giving examples of Ω-Krull rings which are not Marubayashi-Krull.

5. *Polynomial extensions*

<u>lemma 5.1.</u> : Let R be a quasi-local Ω-ring with unique maximal ideal P such that $Z(R/P) = Z(R)/p$ where $p = P \cap Z(R)$, then $\bar{P} = Q_{p[X]}(R[X])P[X]$ is the unique maximal ideal of $\bar{R} = Q_{p[X]}(R[X])$.

<u>proof</u> :

Suppose there exists an element $x \in R[X] \setminus P[X]$ such that $\bar{P} + \bar{R}\, x\, \bar{R}$ is a proper ideal of $\bar{R}$. Let x be an element of minimal degree with this property, say $x = a_n X^n + \dots + a_o$. First, let us assume that $a_n \in P$. Because $x' = a_{n-1}X^{n-1} + \dots + a_o \in R[X]\setminus P[X]$ with $\deg x' < \deg x$ we can find elements $f_i, g_i \in \bar{R}$ and $h \in \bar{P}$ such that $1 = h + \sum_i f_i\, x' g_i$. Thus, $\sum_i f_i\, x\, g_i + h = \sum_i f_i\, a_n X^n g_i + 1$ whence $\bar{P} + \bar{R}\, x\, \bar{R} = \bar{R}$, a contradiction.

Therefore, $a_n \in R \setminus P$. Because $Ra_nR + P = R$ we can find an element $x' = X^n + \alpha_{n-1}X^{n-1} + \dots + \alpha_o$ in $\bar{P} + \bar{R}\, x\, \bar{R}$. We claim that $\alpha_i \bmod P \in Z(R/P)$ for every $0 \leqslant i \leqslant n-1$.

For, suppose there exists an element $r \in R$ and an index i, $0 \leqslant i \leqslant n-1$ such that $\alpha_i r - r\alpha_i \in R \setminus P$, then $rx'-x'r \in R[X] \setminus P[X]$ with $\deg(rx'-x'r) < \deg x$ and therefore $\bar{P} + \bar{R}(rx'-x'r)\bar{R} = \bar{R} \subset \bar{P} + \bar{R}\, x\, \bar{R}$, a contradiction.

Therefore we can find elements $c_i \in C$ and $w_i \in P$ such that $\alpha_i = c_i + w_i$, whence $x'' = X^n + c_{n-1}X^{n-1} + \dots + c_o \in \bar{P} + \bar{R}\, x\, \bar{R}$ and $x'' \in C[X] \setminus p[X]$. Therefore, $\bar{P} + \bar{R}\, x\, \bar{R}$ contains an invertible element, a contradiction. Hence $\bar{P}$ is a maximal ideal of $\bar{R}$. Next, we have to prove that $\bar{P}$ is the unique maximal ideal. Suppose Q were another maximal ideal of $\bar{R}$. Let $Q' = Q \cap R[X]$ then $0 \neq Q' \cap C[X] \in \operatorname{Spec} C[X]$ and clearly $Q' \cap C[X] \subset p[X]$. Because p is a minimal prime ideal of C, so is $p[X]$ in $C[X]$, whence $Q' \cap C[X] = p[X]$. This implies that $p = Q' \cap C$. Because R is a global Zariski extension of its center and because $Q' \cap R$ is a nonzero prime ideal of R, $Q' \cap R = P$, yielding that $\bar{P} = \bar{R}(Q' \cap R) \subset Q$. Finally, because $\bar{P}$ is a maximal ideal of $\bar{R}$, $Q = \bar{P}$ follows.

Using the same notation and assumptions we will prove :

Corollary 5.2. : $\bar{R}$ is a local Ω-ring.

proof

It will be sufficient to prove that all ideals of $\bar{R}$ are powers of $\bar{P}$. First, note that $\bar{P}$ is invertible in $Q_{sym}(R[X])$, ($Q_{sym}(R[X])$ exists because $R[X]$ satisfies Formanek's condition if R does), indeed $\bar{P}^{-1} = \bar{R} \cdot P^{-1}[X]$, where P^{-1} is the inverse of P in $Q_{sym}(R)$.

Let I be a non trivial ideal of $\bar{R}$, then $I \subset \bar{P}$ and $I\bar{P}^{-1} \subset \bar{R}$. By the ascending chain condition on twosided ideals of $\bar{R}$, either $I = \bar{P}^n$ for some $n \in \mathbb{N}$, or either $I \subset \bigcap_{n \in \mathbb{N}} \bar{P}^n$. Because $\cap \bar{P}^n = 0$ (cfr [12]) we obtain $I = \bar{P}^n$.

lemma 5.3. : Let $S = Q_{sym}(R)$, then $S[X] \cap \bar{R} = R[X]$.

proof

Obviously, $R[X] \subset \bar{R} \cap S[X]$. Conversely, let $f(X) = s_n X^n + \dots + s_o$ be any element in $\bar{R} \cap S[X]$, where $s_i \in S$ for all $0 \leqslant i \leqslant n$. Let $f(X) = h(X)^{-1} g(X)$ where $g(X) \in R[X]$ and $h(X) \in C[X] \setminus p[X]$. Because $\overline{h(X)} = h(X) \bmod P[X]$ is an element of $Z(R[X]/P[X]) \cong Z(R/P[X]) = Z(R/P)[X]$ and R/P is a simple ring, there exists an element r(X) in R[X] such that $\overline{h(X)}\,\overline{r(X)} = X^m + \bar{c}_{m-1} X^{m-1} + \dots + \bar{c}_o$. Therefore, $\overline{h(X)}\,\overline{f(X)}\,\overline{r(X)} = \overline{g(X)}\,\overline{r(X)} \in (R/P)\,[X]$ whence $\bar{s}_n \in R/P$ and thus $s_n \in R$. By induction, all $s_i \in R$ whence $f(X) \in R[X]$.

Returning to the original notation, let R be an Ω-Krull ring with defining quasi-local Ω-rings R_i. P_i' is the maximal ideal of R_i, C_i will be the center of R_i, $p_i' = P_i' \cap C_i$; $P_i = P_i' \cap R$, $p_i = p_i' \cap Z(R)$.

As above, we will denote $\bar{R}_i = Q_{P_i}(R[X])$; $\overline{P_i'} = \bar{R}_i P_i'$. Let us define a multiplicatively closed filter of ideals of R[X] : $\mathcal{L}^2(\kappa_i) = \{I[X]\alpha \mid I \in \mathcal{L}^2(\sigma_i)$; $\alpha \in C_i[X] \setminus p_i'[X]$ such that $I[X] \in R[X]\}$.

lemma 5.4. : $Q^{\ell}_{\kappa_i}(R[X]) = \bar{R}_i = Q^{r}_{\kappa_i}(R[X])$.

proof

Clearly, $\mathcal{L}^{\ell}(K_i)$ is a symmetric filter and κ_i has property T. Let $g(X)^{-1}f(X) \in \bar{R}_i$ where $f(X) \in R[X]$ and $g(X) \in C_i[X] \setminus p_i'[X]$. Therefore there exists an ideal $I \in \mathcal{L}(\sigma_i)$ such that $Ig(X)$, $If(x) \in R[X]$. Thus, $I[X]g(X) \cdot g(X)^{-1}f(X) \subset R[X]$ yielding that $\bar{R}_i \subset Q^{\ell}_{\kappa_i}(R[X])$.

Conversely, suppose $g(X)^{-1}f(X) \in Q^{\ell}_{\kappa_i}(R[X])$ with $g(X) \in C[X]$ and $f(X) \in R[X]$ and $I[X]\alpha \cdot g(X)^{-1}f(X) \subset R[X]$ for some $\alpha \in C_i[X] \setminus p_i'[X]$ and $I \in \mathcal{L}^2(\sigma_i)$.

Then, $R_iI[X]\,\alpha\, g(X)^{-1}f(X) = R_i[X]\alpha g(X)^{-1}f(X) \subset R_i[X]$ whence $g(X)^{-1}f(X) \in R_i[X]\alpha^{-1}$.

Thus, $Q^{\ell}_{\kappa_i}(R[X]) = \bar{R}_i$.

Analogously one proves that $Q^{r}_{\kappa_i}(R[X]) = \bar{R}_i$.

If R is an Ω-Krull ring and if $S = Q_{sym}(R)$, then it is easy to check that $S[X]$ is an Ω-ring (cfr [12,1]). Let $\{M_j\}_{j \in J}$ be the set of all its nonzero prime ideals, let $m_j = Z(S)[X] \cap M_j$ and define a symmetric idempotent filter as follows :

$\mathcal{L}^2(\omega_j) = \{I[X]\alpha \mid I \in \mathcal{L}^2(R-0), \alpha \in Z(S)[X] \setminus m_j$ such that $I[X]\alpha \subset R[X]\}$.

lemma 5.5. : $Q^{\ell}_{\omega_j}(R[X]) = Q_{m_j}(S[X]) = Q^{r}_{\omega_j}(R[X])$.

proof

Along the lines of lemma 5.4.

Theorem 5.6. : Let R be an Ω-Krull ring such that $Z(R_i/P_i') = C_i/p_i'$ for all $i \in \Lambda$, then $R[X]$ is an Ω-Krull ring.

proof

Let $R = \bigcap_{i \in \Lambda} R_i$. By Corollary 5.2., $\bar{R}_i$ is a quasi-local Ω-ring and $\bar{R}_i \cap S[X] = R_i[X]$ by lemma 5.3. Because $S[X]$ is an Ω-ring which is a global Zariski extension of its center, we obtain from the proof of section 4.c. that

$S[X] = \bigcap_j Q_{m_j}(S[X])$. Moreover,

$$R[X] = \bigcap_{i \in \Lambda} R_i[X] = \bigcap_{i \in \Lambda} (\bar{R}_i \cap S[X]) = (\bigcap_{i \in \Lambda} \bar{R}_i) \cap (\bigcap_{j \in J} Q_{m_j}(S[X])).$$

From lemmas 5.4. and 5.5. we know that the rings $\bar{R}_i$ and $Q_{m_j}(S[X])$ are over-rings of $R[X]$ satisfying the requirements of the definition of Ω-Krull ring. Finally, let us verify condition 4 of definition 1.2. Let $f(X) \in R[X]$. The ideal $R[X]f(X)R[X]$ contains a central element, say $g(X) = a_n X^n + \ldots + a_o$. Then $Ra_k R \in \mathcal{L}^2(\sigma_i)$ for almost all $i \in \Lambda$, yielding that $\bar{R}_i g(X) = \bar{R}_i$ for all but finitely many $i \in \Lambda$.

Also, $S[X]g(X) \in \mathcal{L}^2(Z(S)[X] - m_j)$ for almost all $j \in J$ (because $S[X]$ is an Ω-ring) implying that $Q_{m_j}(S[X])g(X) = Q_{m_j}(S[X])$ for all but finitely many j. This completes the proof.

Remark

It is not known to the authors whether the condition $Z(R_i/P'_i) = C_i/p'_i$ can be dropped in general. We conjecture that this is not the case. An other intruiging question is whether $R[T_1,\ldots,T_n]$ remains Ω-Krull if $R[T]$ is Ω-Krull.

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