A Note on Noncommutative Krull Domains

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ABSTRACT

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We define a kind of non-commutative Krull domains using serial over-rings, which turn out to be localizations at prime ideals of height one. We globalize some properties of valuation rings to these so-called S-Krull domains in particular, every element is normalizing. We study divisorial ideals and introduce the class group and establish that the class group (as well as the notion of S-Krull domain) behaves well under localization at T-functors. We point out some special cases and the relation to existing theories of R. Fossum, M. Chamarie and H. Marubayashi.

0. Introduction

In the search for a class of noncommutative rings with an arithmetical ideal theory, generalizing the theory of commutative Dedekind rings, several possibilities arise : hereditary Noetherian prime (HNP) rings Asano orders, maximal orders, and Dedekind prime rings. An even richer gamma of possible definitions is available in case one aims to produce a noncommutative counterpart to the theory of Krull domains. It is natural to define these rings by local conditions at prime ideals of height one. The first ramification of the theory is created by the fact that in the noncommutative case one may consider independent local conditions on the localized rings but also on the type of localization used in the construction of that local ring. Certain types of noncommutative Krull domains have been studied by M Chamarie, [4] , R. Fossum, (tame orders) [7] , and H. Marubayashi, [11] . The rings we are studying in this note are S-Krull rings (S for : serial) which are described "locally" (at prime ideals of height one) by serial rings. One of our main aims is the introduction of the class group of an S-Krull domain. To our knowledge there have been no attempts to introduce and study the class group in the noncommutative case, in the absence of a polynomial identity, not even for Dedekind prime rings. The main structural property of S-Krull rings is that they turn out to be intersections of principal valuation rings in skewfields in O. Schilling's sense, cf. [16] . Some of the properties of principal valuation rings may be globalized to S-Krull rings, in particular : left ideals are ideals. After some structure theory in Section 1 we study divisorial ideals and introduce the class group in section 2. Some of the methods used in this note indicate that it is rewarding to work with less restrictive definitions e.g. prime Goldie rings which are "locally" HNP rings allow, under mild conditions on the localizations at prime ideals, a structure theory generalizing in many senses the theory of tame orders as described in [7], but this will not be included in this note.

1. Left S-Krull Domains

Throughout this note R will be a prime Goldie ring with simple Artinian (left and right) ring of fractions Q. Let us point out that, the seemingly weaker hypothesis that R is a right order in a simple ring, will lead to the same theory for S-Krull domains. Recall that a ring S is said to be left serial if the set of its left ideals is totally ordered. A subring Λ of a skewfield Δ is said to be a total subring if for $x \in U(\Delta)$ we have either $x \in \Lambda$ or $x^{-1} \in \Lambda$.

1.1. Proposition. If S is a left serial overring of R in Q then R is a domain, Q is a skewfield and S is a total subring of the skewfield Q.

Proof. Put $I_l(S) = \{s \in S, \mathrm{Ann}_l(s) \neq 0\}$, then $I_l(S)$ is an ideal of S. Indeed, take $a, b \in I_l(S)$ and $\mathrm{Ann}_l a \subset \mathrm{Ann}_l b$ say, then any nonzero $c \in \mathrm{Ann}_l a$ annihilates $a + b$.

Consequently $I_l(S)$ is a right ideal of S. Now consider $a \in I_l(S)$, $s \in S$; then either $S s \subset \mathrm{Ann}_l(a)$ or $\mathrm{Ann}_l(a) \subset Ss$. In the first case $sa=0 \in I_l(S)$; in the second case every nonzero $c \in \mathrm{Ann}_l(a)$ is of the form ts for some $t \in S$, thus : $(ts)a = t(sa) = 0$ i.e. $s a \in I_l(S)$.

Now suppose that R is not a domain, then $I_l(S) \cap R$ is a nonzero (twosided) ideal of R. Since R is a prime Goldie ring, $I_l(S) \cap R$ is essential and contains a regular element, x say. But x cannot be a right zero divisor in S and an invertible element in Q, so we reach a contradiction. It follows that R is a domain and Q is a skewfield. An arbitrary $q \in Q$ may be written as $q = rs^{-1}$ where r and s are regular in R. Now $Sr \subset Ss$ would entail $q \in S$, while on the other hand $Ss \subset Sr$ entails $q^{-1} = sr^{-1} \in S$ is a total subring of Q. $\square$

1.2. Definition R is said to be a left (resp. right) quasi Krull ring if there is a family of overrings of R in Q, $\{O_v, v \in V\}$ say, such that : QK_1 = For all $v \in V$, O_v is left (resp. right) serial; QK_2 : $R = \bigcap_{v \in V} O_v$.

1.3. Corollary. In view of Prosposition 1.1., left (resp. right, quasi Krull rings are domains. Moreover, since total subrings of a skewfield are both left and right serial, it follows that a left quasi-Krull ring is a right quasi Krull ring and vice versa.

Recall, cf. [16], that a total subring of a skewfield Q is a valuation ring of Q if and only if it is invariant under inner automorphisms of Q and moreover, valuation rings of Q correspond bijectively to valuation functions $v : U(Q) \to \Gamma$ for some ordered group Γ. If $\Gamma = Z$ then v is called a principal valuation and the corresponding valuation ring O_v of Q is a principal valuation ring.

1.4. Remark (P.M. Cohn [5]) If Q satisfies a polynomial identity then every total subring of Q is a valuation ring. In general however total subrings may be far from being valuation rings.

A ring S is said to be strongly left serial if the set of its left ideals is well ordered. Note that this is just another way of introducing a left Noetherian hypothesis. However, strongly left serial domains need not be right Noetherian, cf. [2], hence they certainly need not be valuation rings in general. Nevertheless in the situation we are considering we have :

1.5. Proposition An overring S of R in Q is a principal valuation ring of Q if and only if it is stronly left serial.

Proof From the fact that S is left Noetherian and left serial it follows that S is a left principal ideal domain. Thus, S has a unique maximal left ideal M which must then be the Jacobson radical of S, hence an ideal. Write $M = Sm$; we will proceed to show that $M = Sm = mS$. Since $mS \subset Sm = M$ let us suppose there is an $s \in S$ such that $sm \not\in mS$. Since Q is a skewfield we can find $u, v \in R$ such that $sm = muv^{-1}$. If $Su \subset Sv$ then $sm = suv^{-1}$ is in mS, contradiction, hence $Sv \subset Sv$ then $sm = muv^{-1}$ is in mS, contradiction, hence $Sv \subset Su$ or $uv^{-1} = \alpha^{-1}$ for some $\alpha \in S$ and $\alpha^{-1} \not\in S$. The proper left ideal $S\alpha$ is contained in $Sm = M$, tehrefore $\alpha = \beta m$ for some $\beta \in S$. Consequently, $sm = m(\beta m)^{-1} = \beta^{-1}$ with $\beta \in S$, a contradiction. In the terminology of L. Lesieur, [10], M is a good ideal of S and therefore $\bigcap_{n \in \mathbb{N}} (Sm)^n = 0$.

For $x \in S$ put $v(x) = n$ where n is the minimal positive integer such that $x \not\in Rm^{n+1}$; if $x \not\in S$ put $v(x) = -v(x^{-1})$. It is straightforward (and well-known) to check that v is a principal valuation of Q with corresponding valuation ring S. □

1.6. <u>Definition</u>. R is a <u>left (resp. right) S-Krull domain</u> if there is a family of overrings of R in Q, $\{O_v, v \in V\}$ say, such that the following conditions are met :

K1. : For all $v \in V$, O_v is strongly left (resp. right) serial.

K2. : $R = \bigcap_{v \in V} O_v$ (we may assume $O_v \neq Q$ for all v)

K3. : For all $r \in R$, the set $E(r) = \{v \in V, r$ is not a unit of $O_v\}$ is finite.

1.7. <u>Corollary</u> . In view of Proposition 1.5., R is a left S-Krull domain if and only if it is a right S-Krull domain and, as notation suggested, the O_v are principal valuation rings of Q.

Let K be the center of Q, C the center of R. Let V' be the set of valuations in V that are non-trivial on K.

In case Q is algebraic over K (e.g. if R is integral over C or if R satisfies a polynomial identity) then $V = V'$, but this may be false in general (for example for some rings of twisted formal power series we do have $V \neq V'$!).

1.8. Lemma Let R be an S-Krull domain. Then C is a Krull domain (but it may be a field!).

Proof It is clear that $C=Z(R) \subset \bigcap_{v\in V} Z(0_v)$. On the other hand, $\bigcap_{v\in V} Z(0_v) \subset R$ hence $C = \bigcap_{v\in V} Z(0_v)$ follows.

Now for $v\in V$, $Z(0_v)$ is either equal to K or a principal valuation ring of K. It is clear that the family $\{Z(0_v), v\in V'\}$ satisfies the approximation property in K (see K_3), hence C is a Krull domain.

1.9. Remark From $k = k[x^{-1}] \cap k[x]$ in $k(x) = K$ it is clear that C may be a field and still different from K!

2. Divisorial Ideals

Although many of the results of this section remain valid for rings satisfying K_1, K_2 only we suppose throughout that R is an S-Krull domain defined by the set $\{0_v, v\in V\}$ of strongly left serial overrings. If $A, B \subset Q$, write $(A \underset{l}{:} B) = \{x\in Q, xA \subset B\}$, $(A \underset{r}{:} B) = \{x \in Q, Ax \subset B$

A (nonzero) left (resp. right) fractional R-ideal is a nonzero left (resp. right) R-submodule A of Q such that there is a $q \in Q$ such that $Aq \subset R$ (resp. $qA \subset R$).

A is said to be a left (resp. right) divisorial R-ideal if it is left (resp. right) fractional and $(A \underset{r}{:} R) \underset{l}{:} R = A$ (resp. $(A \underset{l}{:} R) : R = A$.

If A is a left fractional R-ideal then it is readily verified that $(A \underset{r}{:} R) \simeq \mathrm{Hom}_R(A,R)$ is a right fractional R-ideal.

2.1. Proposition 1. For any subset A of Q, $(A \underset{r}{:} R) = (A \underset{l}{:} R)$

2. A left divisorial R-ideal is a right divisorial ideal.

Proof : Take $s \in (A \underset{r}{:} R)$. Then $As \subset R$ yields $0_v As \subset 0_v$ for every $v \in V$, i.e. $s \in \bigcap_{v\in V} (0_v A \underset{r}{:} 0_v)$. Conversely of $s \in \bigcap_{v\in V} (0_v A \underset{r}{:} 0_v)$ then we obtain $As \subset \bigcap_v 0_v A s \subset \bigcap_v 0_v = R$. Similarly, $(A \underset{l}{:} R) = \bigcap_{v\in V} (0_v A \underset{l}{:} 0_v)$.

We claim : $(O_v A \underset{l}{:} O_v) = (O_v A \underset{r}{:} O_v)$. Indeed, if $sO_v A \subset O_v$ then we have :

$O_v A s = s^{-1}(s\, O_v A)s \subset s^{-1} O_v s \subset O_v$, therefore $s \in (O_v A \underset{r}{:} O_v)$. The other inclusion will follow from a left-right symmetrical argumentation.

2. May be easily deduced from 1; let us establish a little more here i.e. for every left divisorial R-ideal B we have that $B = \bigcap_{v \in V} O_v B$. Since $O_v B$ is a left fractional O_v-ideal and since O_v is a (left) principal ideal domain, it follows that $O_v B$ is a finitely generated projective O_v-module.

Hence $O_v B = (O_v B \underset{r}{:} O_v) \underset{l}{:} O_v$. Pick $s \in \bigcap_v ((O_v B \underset{r}{:} O_v) \underset{l}{:} O_v)$ then we have :

$$s(B \underset{r}{:} R) \subset \bigcap_{v \in V} s(O_v B \underset{r}{:} O_v) \subset \bigcap_{v \in V} O_v = R,$$

whence $s \in ((B \underset{r}{:} R) \underset{l}{:} R) = B$. Thus $B = \bigcap_{v \in V} O_v B$.

2.2. <u>Corollaries</u> 1. One-sided R-submodules of Q are two-sided.

2. Every prime ideal P of R is completely prime and R satisfies the left and right Ore conditions for R-P.

3. R is fully left and right bounded.

<u>Proof</u> : 1. If $a \in Q$ them Ra is a left divisional R-ideal and therefore it is a two-sided R-module.

2. That a prime ideal is completely prime is immediate from 1. Furthermore, for given $e \in R$, $s \in R - P$ it is clear that there exist $r' \in R$, $s' \in R-P$ and $r'' \in R$, $s'' \in R-P$ such that $s'r = r's$ and $rs'' = sr''$.

3. For every prime ideal P, one-sided ideals of R/P are ideals, hence R/P is a bounded ring, consequently R is fully left bounded.

By corollary 2.2.2. and 2.2.3. it follows that (even in the absense of a Noetherian condition) the J. Lambek - G Michler kernel functor K_P associated to a prime ideal P of R coincides with the symmetric K_{R-P}. Therefore, the left and right localizations at P may, without ambiguity, be described as follows :

$Q_P^l(R) = \{a'b,\ a \in R-P,\ b \in R\}$

$Q_P^r(R) = \{a\, b^{-1},\ \ a \in R,\ b \in R - P\}$

2.3. Proposition If P is a prime ideal of the S-Krull domain R then $Q_P^l(R) \simeq Q_P^r(R)$.

2. The prime ideals of height one (denoted by $X^1(R)$) are precisely the maximal divisorial ideals.

Proof. 1. Take $x \in Q_P^l(R)$. Then $a\,x \in R$ for some $a \in R-P$.
Now $a\,O_v = O_v a$ yields $a\,O_v x \subset O_v$. Hence for all $v \in V$ we obtain :
$a^{-1}(a\,O_v\,x)\,a \subset a^{-1}\,O_v a \subset O_v$. This implies $x\,a \in \bigcap_{v \in V} O_v = R$ or $x \in Q_P^r(R)$.
A symmetrical argument finishes the proof of 1.

2. May be checked along the lines of the corresponding commutative statement, see for example Theorem 3.12 of [6].

Let D(R) be the set of divisorial R-ideals in Q. Define an operation $D(R) \times D(R) \to D(R)$ by $(A,B) \to ((AB : R) : R)$. It is easily verified that this is an associative operation admitting R as an identity element.

2.4. Proposition 1. As S-Krull domain satisfies the ascending chain condition on divisorial ideals.

2. For any (left) fractional R-ideal A we have $(A : A) = R$

3. D(R) is a group.

Proof : 1 Straightforward (or mimic Theorem I.3.6. of [6]).

2. If $\lambda \neq 0$ in Q is such that $A\,\lambda \subset R$ then we have : $(A\lambda \underset{l}{:} A\lambda) = (A \underset{l}{:} A)$, hence we may restrict ourselves to consider the case where A is a left ideal of R. Now if $x \in (A \underset{l}{:} A)$ then $x\,O_v A = O_v\,x\,A \subset O_v A$. Now $O_v A = O_v a$ for some $a \in O_v$, whence $x\,O_v\,a \subset O_v a$ i.e. $x\,O_v \subset O_v$ and thus $x \in O_v$.

Therefore $(A \underset{l}{:} A) \subset R$ and $(A \underset{l}{:} A) = R$ follows.

3. Let A be a divisioral R-ideal. We have :

$(((A:R)A) : R) = ((A:R) : (A : R)) = R$, and also :

$((A(A:R)) : R) = R$. Thus $(A:R)$ is an inverse for A.

2.5. Proposition. 1 If $P \in X^1(R)$ then $Q_P(R)$ is a principal valuation ring.

2. $R = \bigcap_{P \in X^1(R)} Q_P(R)$.

Proof. 1. If $Q_P(R)$ is not a total subring of Q then there exists a $q \in Q$ such that $Rq \cap R \subset P$ and $R \cap Rq^{-1} \subset P$.

Thus $(P:R) \subset ((Rq \cap R) : R) \cap ((Rq^{-1} \cap R) : R)$.

Since $(Rq^{-1} \cap R)q = Rq \cap R$ it follows that :

$((Rq \cap R) : R) \cap ((Rq^{-1} \cap R) : R) = (Rq \cap R) : (Rq \cap R) = R$, which contradicts the fact that P is a divisorial R-ideal. Now let $x \in (P:P)-R$, then $x \notin (P : P)$ and $x\, PQ_P(R) \subset Q_P(R)$. Suppose the latter inclusion is proper, then $x\, PQ_P(R) \subset PQ_P(R)$ yields $x\, P \subset x\, PQ_P(R) \cap R = P$, a contradiction. Therefore $x\, PQ_P(R) = Q_P(R)$ and also $PQ_P(R) = x^{-1}\, Q_P(R)$. In the same way one proves that $Q_P(R)P = PQ_P(R) = Q_P(R)x^{-1}$. Since P is divisioral, $P = \bigcap_{v \in V} O_v P$ i.e. $O_{v_o} P \neq O_{v_o}$ for some $v_o \in V$. In that case

$\bigcap_{n=o}^{\infty} (O_{v_o} P)^n = 0$ entails $\bigcap_{n=o}^{\infty} P^n = 0$ and then one easily constructs a principal valuation function as in Proposition 1.5..

2. If $y \in \bigcap_{P \in X'(R)} Q_P(R)$ then $Ry^{-1} \cap R$ is a divisorial ideal not in a maximal divisorial ideal, hence $Ry^{-1} \cap R = R$ and thus $y \in R$.

2.6. Corollary For each $P \in X^1(R)$ there exists a $v \in V$ such that $Q_P(R) = O_v$ (supposing $X^1(R) \neq \emptyset$ i.e. R not a field).

Proof Principal valuation rings of skewfields are maximal subrings. Obviously $\{Q_P(R),\, P \in X^1(R)\}$ satisfies K_1, K_2, K_3, hence we may assume that R is given by the family of its localizations at prime ideals of height one.

It follows that an ideal J is divisorial if and only if $J = \bigcap_{P \in X^1(R)} Q_P(R)J =$

$= \bigcap_{P \in X^1(R)} Q_P(J)$.

2.7. Lemma . 1. $D(R)$ is an abelian group.

2. $D(R)$ is the free abelian group on $X^1(R)$.

Proof Consider divisorial ideals A and B of R.

Since $((AB : R) : R)$ may be identified with $\mathrm{Hom}_R(\mathrm{Hom}_R(AB,R), R)$, it follows from the exactness of the functor Q_P, $P \in X^1(R)$, and the fact that $R \to Q_P(R)$ is a flat ring epimorphism, that $Q_P(R)\,((AB : R) : R) =$ $= Q_P(((AB:R):R)) = (Q_P(A\,B) : Q_P(R)) : Q_P(R)$ and also $Q_P(AB) = Q_P(R)AB =$ $= Q_P(R)AQ_P(R)B = Q_P(A)Q_P(B)$.

In $Q_P(R)$, products of ideals are commutative because $Q_P(R)$ is a principal valuation ring and thus all ideals are powers of the radical. Therefore we obtain

$$\begin{aligned} ((AB : R) : R) &= \bigcap_{P \in X^1(R)} ((Q_P(A)Q_P(B) : R) : R) \\ &= \bigcap_{P \in X^1(R)} ((Q_P(B)Q_P(A) : R) : R) = \bigcap_{P \in X^1(R)} Q_P(R)(((BA):R) : R) \\ &= ((BA : R) : R) \end{aligned}$$

And this proves that $D(R)$ is abelian.

2. This is now easy, the proof is similar to the proof given in the commutative case cf. [6].

It is now clear how to define the class group of a S-Krull domain :

2.8. Definition The class group of an S-Krull domain R is defined to be the abelian group $D(R)/\mathrm{Prin}(R)$ where $\mathrm{Prin}(R)$ is the subgroup of $D(R)$ consisting of left principal ideals of R (these are two-sided!). This group is denoted by $\mathrm{Cl}(R)$. If $\mathrm{Inv}(R)$ is the subgroup of $D(R)$ consisting of invertible divisorial ideals then $\mathrm{Pic}(R) = \mathrm{Inv}(R) / \mathrm{Prin}(R)$ may be identified with a subgroup of $\mathrm{Cl}(R)$.

2.9. Proposition Let R be an S-Krull domain, then :

1. The following sequence of groups is exact :

   $1 \to U(R) \to U(Q) \xrightarrow{d} Prin(R) \to 1$, where d is defined by $d(u) = Ru$, $u \neq 0$ in Q.

2. The noncommutative approximation theorem : given a finite set $\{n_1,\ldots,n_p\} \subset Z$, there exists an $x \in U(Q)$ such that $v_i(x) = n_i$, $i = 1,\ldots, p$ and $v(x) \geq 0$ for all $v \in V - \{v_1,\ldots,v_p\}$ where v_i corresponds to $Q_{P_i}(R)$.

Proof 1. Obvious. Note that U(R) is normal in U(Q).

2. Follow the lines of proof of Theorem 5.8 in [6], bearing in mind that, if $P \subset \bigcup_{i=1}^{n} P_i$ for prime ideals P, P_i of R then $P \subset P_i$ for some i, also in the noncommutative case; moreover the products of prime ideals appearing in the proof for the commutative case reappear in the non-commutative version in such a way that the order of the factors does not matter at all.

2.10 Proposition : Let R be an S-Krull domain and let K be a kernel functor on R-mod which satisfies property T (i.e. Q_K is a perfect localization).

Then the following properties hold :

1. $Q_K(R)$ is an S-Krull domain
2. The canonical ring morphism $R \to Q_K(R)$ defines a morphism $Cl(R) \to Cl(Q_K(R))$.

Proof 1. By property T for K, the canonical $j_K : R \to Q_K(R)$ is a right flat ring epimorphism. From Corollary 2.2.1. it follows that K is symmetric. Consider a maximal ideal M of $Q_K(R)$ and put $P = M \cap R$.
Since Prin(R) is a commutative group, left principal ideals commute and thus a left principal ideal Ra will commute to any ideal but then all ideals of R commute. If $RaRb \subset P$ consider $aqb \in Q_K(R)$ for some

arbitrary $q \in Q_K(R)$. Then $Iq \subset P$ for some $I \in \mathcal{L}(K)$ yields $Iaqb = aIqb \subset P$ because $IRa = RaKI = aRI$. By property T, $Q_K(R)Iaqb = Q_K(P)ayb \subset Q_K(P) = M$ follows. So $aqb \in M$ for all $q \in Q$, hence a or b is in M i.e. a or b is in P and P in a prime ideal. Since any $I \in \mathcal{L}(K)$ cannot be in P it follows that we have a canonical monomorphism $Q_K(R) \to Q_P(R)$. Now if J is an ideal of $Q_K(R)$ which is not contained in H then $J \cap R \not\subset P$ hence $Q_P(R)J = Q_P(R)(J \cap R) = Q_P(R)$. It is (well known and) straight forward to show that the fact that ideals commute in R entails that ideals of R extend to ideals of the localization at a kernel functor having property T. Now by the Ore conditions with respect to $R - P = G(P)$ this property holds for K as well as K_P. Now from $Q_P(R)(J \cap R) = Q_P(R)$ it follows that $J \cap R$ contains a finitely generated (left)ideal J' not in P and such that $Q_K(R)J'$ is finitely generated on the left but also an ideal of $Q_K(R)$ not in M and contained in J. It is therefore easy to see that the filter $\mathcal{L}(K_{Q_K(R)-M})$ based upon the set of ideals of $Q_K(R)$ not contained in M has finite type i.e. each left ideal in it contains an ideal in it which is finitely generated as a left $Q_K(R)$-module, hence it is an idempotent filter, cf. [17], and $\gamma = K_{Q_K(R)-M}$ is a symmetric kernel functor. Obviously $Q_\gamma(Q_K(R)) \subset Q_P(R)$. Conversely of $x \in Q_P(R)$ then $J'x \subset R$ for some J' in R not contained in P, say $J' = Rs$ for some $s \in R - P$. Then $Q_K(R)s$ is an ideal of $Q_K(R)$ not contained in M such that $Q_K(R)s\,x \subset Q_K(R)$. If we establish that $Q_\gamma(Q_K(R))s = Q_\gamma(Q_K(R))$ then $x \in Q_\gamma(Q_K(R))$ and $Q_P(R) = Q_P(R) = Q_\gamma(Q_K(R))$ follows. Note that $Q_K(R) - M$ is multiplicatively closed because a left ideal L of $Q_K(R)$ equals $Q_K(R)(L \cap R)$ which is an ideal of $Q_K(R)$ satisfies the Ore conditions with respect to $Q_K(R)-M$. Finally this implies that γ has property T and therefore $Q_K(R)\,s \in \mathcal{L}(\gamma)$ entails $Q_\gamma(Q_K(R))s = Q_\gamma(Q_K(R))$. Well known properties of symmetric localization entail that $Q_K(R) = \bigcap_{M \in \Omega} Q_{Q_K(P)-M}(Q_K(R))$, where Ω is the set of maximal ideals of $Q_K(R)$. In order to establish 1. it will therefore be sufficient to prove that $Q_\gamma(Q_K(R))$ is an S-Krull domain. But $Q_\gamma(Q_K(R)) = Q_{R-P}(R) = $ $= RS^{-1}$ where $S = R - P$, and we claim that $RS^{-1} = \bigcap_{v \in W} O_v$ where W is the set of valuations in V such that $\cup (O_v)$ S. That $RS^{-1} \subset \bigcap_{v \in W} O_v$ is obvious Conversely, take $0 \neq x \in \bigcap_{v \in W} O_v$. The set F of valuations v such that $v(x) < 0$ is finite. For every $v \in F$ (note $F \subset V-W$) there exist, $s_v \in S$, $n_v \in \mathbb{N}$ such that $v(s_v^{n_v} x) \geq 0$, so if s is the product of the $s_v^{n_v}$, $v \in F$ then $v(sx) \geq 0$ holds for all $v \in V$ i.e. $s\,x \in R$ or $x \in RS^{-1}$. Finally we established that RS^{-1} hence $Q_K(R)$ is a subintersection for R, hence a Krull domain.

2. Property T for R entail exactness of Q_K and $Q_K(M) \simeq Q_K(R) \otimes_R M$ for any $M \in R$-mod. If M is a divisioral ideal of R one easily deduces from the foregoing remarks that $Q_K(M)$ is a divisorial ideal of $Q_K(R)$. Thus $j_K : R \to Q_K(R)$ induces an epimorphism $Cl(j_K) = Cl(R) \to Cl(Q_K(R))$.

2.11. <u>Corollaries</u> 1. Examples of K as in Proposition 2.10 are : $K = K_S$ the kernel functor associated to an arbitrary multiplicatively closed set S in R, $K = K_P$ where S is taken to be $R-P$, $K = \bigvee_i K_i$ where $\{K_i\}$ is a set of kernel functors of type K_S of K_P (note that the sup of geometric T functors is a geometric T functor, cf. [17]).

2. A kernel functor K is said to be a <u>divisorial kernel functor</u> if $L \in \mathcal{L}(K)$ if and only if $L^{**} \in \mathcal{L}(K)$. For example K_P with $P \in X^1(R)$ is divisorial. First let us point out that a divisorial kernel functor over an S-Krull domain is Noetherian. Indeed, consider a sequence $L_1 \subset L_2 \subset \dots L_i \subset$ of left ideals of R such that $\bigcup_i L_i \in \mathcal{L}(K)$, then because R satisfies the ascending chain condition on divisioral ideals (check as in the commutative case) cf. N. Bourbaki [1], it follows

that $L_1^{**} \subset L_2^{**} \subset \ldots L_n^{**} \subset \ldots$ is stationary i.e. $L_n^{**} = L_{n+1}^{**} = \ldots$. So $L_n^{**} \in \mathcal{L}(K)$, but K is divisorial hence $L_n \in \mathcal{L}(K)$. Consequently, for a divisorial kernel functor K to have property T it is necessary and sufficient that every $L \in \mathcal{L}(K)$ contains a K-projective L' also in $\mathcal{L}(K)$. Obviously if Cl(R) is a torsion group then every divisorial kernel functor with a filter basis of divisorial ideals has property T. Actually every kernel functor γ such that $\gamma \geq \bigcap_{P \in X^1(R)} K_P$ is divisorial. Therefore if $\bigwedge_{P \in X^1(R)} K_P = \xi$ the trivial kernel functor (e.g. in case R is a Dedekind ring) then all kernel functors are divisorial. Note also that, just as in the commutative case, the divisorial ideals of R are intersections of principal (left) ideals hence K is a divisorial exactly then when $L \in \mathcal{L}(K)$ if and only if $Rx \in \mathcal{L}(K)$ for all $Rx \supset L$. It seems reasonable to expect that divisorial kernel functors of finite type have property T, however we have no proof of this as yet.

2.12 Remark Let $j : R \to S$ be a faithfully flat extension of S-Krull domains then Ker(Cl(j) equals Ker(Pic(j)).

Proof If $I \in \mathcal{D}(R)$ is such that $S\,I \in$ Prin(S)then faithfully flatness of j entails that I is a finitely generated projective ideal of R i.e. $I \in$ Inv(R).

2.13. The P.I. Case Suppose that R is an S-Krull domain satisfying a polynomial identity.

Since the center of R is a Krull domain (possibly a field) it follows from a theorem of A. Braun that R is a finitely generated Z(R)-module Now the left and right Ore conditions with respect to each prime ideal of R yield that R is a Zariski central ring. If $U \subset$ Spec R = X $V \subset$ Spec Z(R) = Y are the open sets of birationality given by the central kernel then there is a one-to-one correspondence $V \cap Y^1(Z(R)) \leftrightarrow$

$\to U \cap X^1(R)$ given by $p \to \text{rad } Rp = P$, $P \to P \cap Z(R) = p$. The valuation ring $Q_P(R) = Q_p(R)$ corresponding to $P \in U \cap X^1(R)$ is an Azumaya algebra hence its associated valuation is unramified over the corresponding central valuation i.e. $U \cap X^1(R)$ describes the subset of V of centrally unramified valuations. Note that in the P.I. case theory of S-Krull domain reduces to a very special case of R. Fossum's theory of orders in particular tame orders over domains. One may check that our class group then coincides with the one introduces there.

2.14 The Zariski central case Although more general than the P.I. case one can easily convince oneself that the theory of Zariski Central S-Krull domains is a particular case of the theory esthablished by M. Chamaric [4], as well as of the theory expanded by H. Marubayashi [11],[12], [13]. If we exclude the case $R = Q$ then $V = V'$ (see after Corollary 1.7.) and all valuations valuations $v \in V$ in some sense determined by a central valuation.

Without further finiteness conditions, the class of Zariski central S-Krull domains is not a trivial one, there do remain several intringing problems, e.g. the determination of the class groups $Cl(R)$, $Cl(Z(R))$ and their interrelation, which we hope to attack in a subsequent paper.

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