# HOMOGENIZATION OF SHEAVES AND KERNEL FUNCTORS

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# 0. INTRODUCTION

In (commutative) algebraic geometry, homogenization and dehomogenization techniques are used to reduce questions about projective varieties to questions about associated affine varieties (at least if these questions can be answered by looking at the local rings, cfr. e.g. [1].

After the introduction of noncommutative affine and projective schemes (cfr [6, 4, 9]), the question arose whether a similar technique could be developped in this noncommutative setting. The aim of this note is to prove that this is indeed possible. In the first three sections we treat the most general case, i.e. homogenization and dehomogenization of (pre)-sheaves of rings. It is proved that the homogenization functor commutes with sheafification whereas the same is true for dehomogenization when the topological space is (quasi) compact. An example is given to show that this compactness hypothesis cannot be dropped.

Because the definition of noncommutative schemes benefits from the proper use of kernel functors we also investigate the (de)homogenization of kernel functors when R is a prime left. Noetherian rings we obtain relations between the rings of quotients yielding the desired noncommutative analogues of some classical commutative results, cfr. Section 7. I am indebted to Prof. F. Van Oystaeyen for several discussions which helped shape the contents of this note.

# 1. PRELIMINERIES

(1.1.): A ring R is said to be graded (of type Z) if there is a family of additive subgroup $\{R_n : n \in Z\}$ of R such that $R = \oplus R_n$ and $\forall i, j \in Z : R_i R_j \subset R_{i+j}$.

An $M \in$ R-mod is said to be a graded left R-module if there is a family $\{M_n : n \in Z\}$ of additive subgroup of M with the properties : $M = \oplus M_n$ and $R_i M_j \subset M_{i+j}$ for all $i, j \in Z$. The elements of $h(R) = \cup R_n$ and $h(M) = \cup M_n$ are called the homogeneous elements of R and M resp.

If $m \neq o$, $m \in M_i$ then m is called an homogeneous element of degree i and we write deg m = i. Any nonzero $m \in M$ may be written, in a unique way, as a finite sum $m_1 + \ldots + m_k$ with $\deg m_1 < \deg m_2 < \ldots < \deg m_k$; the elements $m_1, \ldots, m_k$ (all non-zero) are called the homogeneous components of m.

R-gr is the Grothendieck-category with objects the graded left R-modules and morphisms the gradation preserving R-module homomorphisms.

(1.2.): Let X be a fixed topological space. All sheaves and presheaves in this note will be defined over Open(X), the category obtained from X in the usual way. Let $\mathcal{R}$ be a Gr-Ring, i.e. a sheaf of graded rings over Open (X) such that for any $U, V \in$ Open(X) with $V \subset U$, the restriction morphisms $\rho_V^U : \mathcal{R}(U) \to \mathcal{R}(V)$ preserve the gradation.

We will consider the following Grothendieck-categories:

$\pi(\mathcal{R})$ (resp. $\sigma(\mathcal{R})$) will be the category of presheaves (resp. sheaves) of modules over the sheaf $\mathcal{R}$.

$g\pi(\mathcal{R})$ (resp. $g\sigma(\mathcal{R})$) will be the category of presheaves (resp. sheaves) of graded modules over the sheaf $\mathcal{R}$, i.e., $\mathcal{M} \in g\pi(\mathcal{R})$ if and only if $\underline{\mathcal{M}} \in \pi(\mathcal{R})$ where $\underline{(\ )} : g\pi(\mathcal{R}) \to \pi(\mathcal{R})$ is the functor defined by forgetting the gradation) and for all $U \in$ Open (X) : $\mathcal{M}(U) \in$ R-gr; $\forall U, V \in$ Open (X), $V \subset U$: $\mathcal{M}_V^U : \mathcal{M}(U) \to \mathcal{M}(V)$ is gradation preserving.

## 2. HOMOGENIZATION AND DEHOMOGENIZATION

(2.1.): From [4] we recollect the following definitions. Let R be a graded ring. The ring of polynomials $R[T]$ may be made into a graded ring by putting:

$$\deg T = 1;\ R[T]_n = \{ \sum_{i+j=n} r_i T^j;\ r_i \in R_i \}$$

In the same way, we build the graded module of polynomials $M[T]$ starting from an $M \in R\text{-gr}$. If we decompose $x \in M$ into homogeneous elements; $x = x_{-m} + \ldots + x_o + \ldots + x_n$ $(x_i \in M_i)$, then we may associate to it an homogeneous element $x^\star$ in $M[T]$ which is given by:
$x_\star = x_{-m} T^{m+n} + \ldots + x_o T^n + \ldots + x_n$. We say that $x^\star$ is the <u>homogenized</u> of x.
Conversely, if u is an homogeneous element of $M[T]$, say,
$u = u_{-m} T^{m+n+p} + \ldots + u_o^{n+p} + \ldots + u_n T^p$ with $u_i \in M_i$, then
$u_\star = u_{-m} + \ldots + u_o + \ldots + u_n$ is said to be the <u>dehomogenized</u> of u.
If $M \in R\text{-gr}$ and N is a (not necessarely graded) R-submodule of M, then by $N^\star$ we mean the $R[T]$-submodule of $M[T]$ generated by the $n^\star$, $n \in N$. Of course $N^\star$ is a graded submodule of $M[T]$, it is called the <u>homogenized</u> of N. Any $n \in N^\star$ is of the form $T^r n_1^\star$, $n_1 \in N$ and $r \geq o$.
Conversely, to a graded $R[T]$-submodule L of $M[T]$ we may associate $L_\star = \{u_\star;\ u \in h(L)\}$. It is clear that $L_\star$ is an R-submodule of M.

(2.2.): Now, let $\mathcal{R}$ be a Gr-Ring. We will form the Ring of polynomials $\mathcal{R}[T]$ as follows: if $U \in \text{Open}\ (X)$, we put $\mathcal{R}[T](U) = \mathcal{R}(U)[T]$ with gradation as in (2.1.), if $U, V \in \text{Open}\ (X)$ with $V \subset U$, the restriction morphism $(\rho^\star)^U_V$ is given by:

$$(\rho^\star)^U_V\,(T^{m+n+p}x_{-m} + \ldots + T^{n+p}x_o + \ldots + T^p x_n) =$$

$$T^{m+n+p}\,\rho^U_V(x_{-m}) + \ldots + T^{n+p}\,\rho^U_V(x_o) + \ldots + T^p\,\rho^U_V(x_n).$$

Because $\rho^U_V$ preserves the gradation, the same is true for $(\rho^\star)^U_V$, hence $R[T]$ is a Gr-Ring (the verification that $R[T]$ is a sheaf is proved along the lines of Prop. 2.3. below).

If $M \in g\pi(R)$, we can define the Module of polynomials $M[T]$ in a similar way.

Now, let $N \in \pi(R)$ be a subpresheaf of $M \in g\pi(R)$. For all $U \in$ Open $(X)$, define $N^\star(U) = N(U)^\star \subset M[T]$. The restriction morphisms $(N^\star)^U_V$ for $V \subset U \in$ Open$(X)$ are given by:

$$(N^\star)^U_V\,(T^{m+n+p}\,x_{-m} + \ldots + T^{n+p}\,x_o + \ldots + T^p x_n) =$$

$$T^{m+n+p}\,M^U_V\,(x_{-m}) + \ldots + T^{n+P}\,M^U_V(x_o) + \ldots + T^p\,M^U_V(x_n).$$

It follows that $N^\star$ is a graded subpresheaf of $M[T]$.

(2.3): <u>Proposition</u>: In the situation of (2.2), if $N \in \sigma(R)$, then $N^\star \in g\sigma(R[T])$.

<u>Proof.</u>

$(S_1)$: Let $U \in$ Open$(X)$ and $\{U_i;\ i \in I\}$ an open covering of U. Suppose $n \in N(U)$ and $(N^\star)^U_{U_i}(n^\star) = o$ for all $i \in I$, then:

$o = (N^\star)^U_{U_i}\,(n^\star) = (M^\star)^U_{U_i}(n^\star) = (M^U_{U_i}(n))^\star = (N^U_{U_i}(n))^\star$ and from $(x^\star)_\star = x$, it follows that $N^U_{U_i}(n) = o$ for all $i \in I$. Because $N$ is a sheaf we obtain $n = o$, thus $n^\star = o$.

$(S_2)$: Let $n_i^! \in h(M^\star(U_i))$ with compatibility conditions:

$$(N^\star)^{U_i}_{U_i \cap U_j} (n_i^!) = (M^\star)^{U_j}_{U_j \cap U_i} (n_j^!)$$

Every $n_i^!$ is of the form $n_i^! = T^{ki} (n_i^\star)$ with $n_i \in N(U_i)$.

For all i,j we have:

$$T^{ki} (N^{U_i}_{U_i \cap U_j} (n_i))^\star = (N^\star)^{U_i}_{U_i \cap U_j} (T^{ki} n_i^\star) = (M)^{\star U_j}_{U_i \cap U_j} (T^{kj} n_j^\star) =$$

$$= T^{kj} (M^{U_j}_{U_i \cap U_j} (n_j))^\star .$$

Dehomogenizing both sides yields: $N^{U_i}_{U_i \cap U_j} (n_i) = M^{U_j}_{U_i \cap U_j} (n_j)$ for all $i,j \in I$. Using the second sheaf condition for $N$ we find an $n \in N(U)$ such that $N^U_{U_i} (n) = n_i$ for all i.

From the equalities above, it follows that $k_i = k_j = k$ for all i, j. $n' = T^k n^\star$ finishes the proof.

(2.4): Let $N$ be a graded subpresheaf of $M[T]$ with $M \in g\pi(R)$. For every $U \in \text{Open}(X)$ we put: $N_\star (U) = (N(U))_\star$. For $V \subset U \in \text{Open}(X)$ the restriction morphism $(N_\star)^U_V$ is given in the following way: if $x \in M_\star (U)$, then there exists an $y \in h(M(U))$ such that $y_\star = x$, put: $(N_\star)^U_V (x) = (M^U_V(y))_\star$. One easily checks that this definition is indepent of the choice of y.

(2.5.): <u>Proposition</u>: In the situation of (2.4): if X is a compact topological space and $N \in g\sigma(R[T])$, then $N_\star \in \sigma(R)$.

<u>Proof</u>.

$(S_1)$: Let $U \in \text{Open}(X)$ and let $\{U_i;\ i \in I\}$ be an open covering of U. Suppose $x \in N_\star (U)$ such that $(N_\star)^U_{U_i}(x) = o$ for all $i \in I$. There exists an $y \in h(N(U))$ such that $y_\star = x$.

Now, $(N^{U}_{U_i}(y))_\star = (N_\star)^{U}_{U_i}(x) = o$, thus, $N^{U}_{U_i}(y) = o$ for all $i \in I$ and the fact that $N$ is a sheaf, yields $y = o$, hence $x = o$. Compactness is not necessary for this part of the proof.

$(S_2)$: The compactness hypothesis allows us to restrict to a finite covering $\{U_i;\ i = 1, \dots, n\}$. Take $x_i$ in $N_\star(U_i)$ such that

$$(N_\star)^{U_i}_{U_i \cap U_j}(x_i) = (N_\star)^{U_j}_{U_i \cap U_j}(x_j) \quad \text{for all } i, j = 1, \dots, n$$

There exist $y_i \in h(U_i))$ such that $(y_i)_\star = x_i$. Put $\deg(y_i) = d_i$ and $m = \max\ d_i;\ i = 1, \dots, n$. Replace $y_i$ by $y_i' = T^{m-d_i} y_i$, then $(y_i')_\star = x_i$ and:

$$(N^{U_i}_{U_i \cap U_j}(y'))_\star = (N_\star)^{U_i}_{U_i \cap U_j}(x_i) = (N_\star)^{U_j}_{U_j \cap U_i}(x_j) = (N^{U_j}_{U_i \cap U_j}(y_j'))_\star$$

Because $\deg(N^{U_i}_{U_i \cap U_j}(y_i')) = \deg(N^{U_j}_{U_j \cap U_i}(y_j'))$ it follows that $N^{U_i}_{U_i \cap U_j}(y_i') = N^{U_j}_{U_i \cap U_j}(y_j')$ and therefore there exists an $y \in h(N(U))$ such that $N^{U}_{U_i}(y) = y_i$ for all $i$. $y_\star$ is the required element in $N_\star(U)$.

(2.6): The compactness condition cannot be dropped:

Example

Take $\mathbb{N}$ with the discrete topology and $R$ the constant sheaf of rings $Z$ over it. Now, take $N$ to be the graded subsheaf of $Z[T]$ as follows: if $U$ is a finite open set of $\mathbb{N}$, then $N(U) = T^n Z[T]$ with $n$ the maximal element in $U$. If $U$ is infinite, $N(U) = o$. If $U = \emptyset$, $N(U) = Z[T]$. Restriction morphisms are inclusions or the zero map. It is easily checked that $N$ is a sheaf. $N_\star$ is the presheaf with $N_\star(U) = Z$ if $U$ is finite, $N_\star(U) = o$ if $U$ is infinite and inclusion or zero map for the restriction morphisms.

$(S_2)$ fails, for, take $U = \mathbb{N}$, $U_i = \{i, i+1\}$ and $x_i = 1 \in N(U_i)$. There exist no element $x$ in $N_\star(\mathbb{N})$ such that $N^{\mathbb{N}}_{U_i}(x) = 1$.

## 3. COMPATIBILITY WITH THE SHEAFIFICATION FUNCTOR

(3.1): We recall the construction of the reflector $\underline{\underline{a}}$ for the inclusion $\sigma(R) \to \pi(R)$, usually called the sheafification functor, cfr. e.g. [8]. First, define a functor $L : \pi(R) \to \pi(R)$ as follows. Let $U \in \text{Open}(X)$, we give $\text{Cov}_x(U)$, i.e. the set of all open coverings of U, the structure of a category: if $\mathcal{U} = \{U_i;\ i \in I\}$, $\mathcal{V} = \{V_j;\ j \in J\}$ are in $\text{Cov}_x(U)$, a morphism $\mathcal{U} \to \mathcal{V}$ is given by a map $\varepsilon : I \to J$ such that $U_i \subset V_{\varepsilon(i)}$ for all $i \in I$. Let $M \in \pi(\ )$ and define $M, U$ , $U \in \text{Open}(x)$, by its action on a covering $\mathcal{U} = \{U_i;\ i \in I\}$ of U:

$$[M, U](\mathcal{U}) = \text{Ker}\ \Big( \prod_{i \in I} M(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{(j,k)\ I\times I} M(U_j \cap U_k) \Big)$$

where the (j,k)-component of p is $M^{U_j}_{U_j \cap U_k}(m_j)$ and the (j,k)-component of q is $M^{U_k}_{U_j \cap U_k}(m_k)$; with $m_i : \prod_{j \in J} M(U_j) \to M(U_i)$ be the restriction morphism.

Note that $[M, U] : \text{Cov}_x(U) \to R(U)\text{-mod}$ is a contravariant functor. Hence we can define an object $LM$ of $\pi(R)$ by:

$L\,M : \text{Open}(X)^{\text{opp}} \to \text{Set}$ $\qquad U \longmapsto \varinjlim_{\mathcal{U} \in \overrightarrow{\text{Cov}}_x(U)} [M, U]\ (\mathcal{U})$

Note that $L\,M(U) = \lim\limits_{\mathcal{U} \in \overrightarrow{\text{Cov}}_x(U)} \ \varprojlim\limits_{V \in \mathcal{U}} \ (V)$

The assignment $M \to LM$ defines a left exact endofunctor of $\pi(R)$ satisfying:

1. If $M \in \varphi(R)$, i.e. the class of separated objects in $\pi(R)$ (satisfying $(S_1)$), then the canonical morphism $M \to LM$ is a monomorphism and $LM \in \sigma(R)$.
2. If $M \in \pi(R)$, then $LM \in \varphi(R)$
3. If $M \in \sigma(R)$, then $LM \cong M$ and conversely.

Finally, define $i.\underline{\underline{a}} = L \circ L$ where $i : \sigma(R) \to \sigma(R)$ is the canonical inclusion, then $\underline{\underline{a}}$ is a left adjoint of i and is called the <u>sheafification functor</u>.

Let us denote $\underline{\underline{a}}'$ for sheafification in $\pi(R[T])$, then we have:

(3.2): <u>lemma</u>: If $M \in g\pi(R[T])$, then $\underline{\underline{a}}'(M) \in g\sigma(R[T])$.

<u>Proof</u>.

For all $U \in \mathrm{Open}(X)$ : $R(U)$ - gr is closed under direct and inverse limits, hence we are done by the remarks preceding the lemma.

(3.3): <u>Theorem</u>: Let $N \in \pi(R)$ be a subpresheaf of an $M \in g\pi(R)$, then: $\underline{\underline{a}}'(N^\star) = \underline{\underline{a}}(N)^\star$, i.e. the following diagram "commutes" for suitable $N$.

$$\begin{array}{ccc} \pi(R) & \xrightarrow{(-)^\star} & g\pi(R[T]) \\ \downarrow{\scriptstyle \underline{\underline{a}}} & & \downarrow{\scriptstyle \underline{\underline{a}}'} \\ \sigma(R) & \xrightarrow{(-)^\star} & g\sigma(R[T]) \end{array}$$

<u>Proof</u>.

First step: for every $x \in X$ : $S_x(N^\star) \cong (S_x(N))^\star$ where $S_x(-)$ is the stalk at $x$.

Let $a \in h(S(N^\star))$, then we can find a neighborhood $U$ of $x$ and an element $y \in h(M^\star(U))$ representing $a$. Let $y = y_{-m} T^{m+n+p} + \dots + y_n T^p$ with $y_i \in M(U)_i$. We consider the morphism:

$$\begin{array}{rcl} f : S_x(N^\star) & \longrightarrow & (S_x(N))^\star \\ a & \longmapsto & M_x^U(y_{-m})\, T^{m+n+p} + \dots + M_x^U(y_o)\, T^{n+p} + \dots + M_x^U(y_n)\, T^p \end{array}$$

f does not depend on the choice of U and y, for, if V is another neighborhood of x and $y' = y_{-r} T^{r+q+p'} + \dots + y_o T^{q+p'} + \dots + y_q T^{p'} \in h(N^\star(V))$ representing a. Then, there is a neighborhood $N \subset U \cap V$ of x such that $(N^\star)^U(y) = (N^\star)^V_W(y')$. Hence:

$$M_x^U (y_{-m})\ T^{m+n+p} + \ldots + M_x^U (y_o)\ T^{n+p} + \ldots + M_x^U (y_n)\ T^p =$$

$$(N^\star)_x^W ((N^\star)_W^U (y)) = (N^\star)_x^W ((N^\star)_W^V (y')) =$$

$$M_x^V (y'_{-r})\ T^{r+q+p'} + \ldots + M_x^V (y'_o)\ T^{q+p'} + \ldots + M_x^V (y'_q)\ T^{p'}$$

Moreover, f is injective, for if $f(a) = o$, then $M_x^U (y_i) = o$ for all i, hence we can find a neighborhood $W \subset U$ of x such that $M_W^U (y_i) = o$ for all i. Because the definition of f does not depend on U and y, $x = o$ follows. Also, f is surjective. Indeed, if $y \in h(S_x(N)^\star)$, then y is of the form $y = T^p((y')^\star)$ with $y' \in S_x\ (N)$, letting $y' = y_{-m} + \ldots + y_n$, then for all i we can find a neighborhood $U_i$ of x and an $x_i \in h(M(U_i))$ representing $y_i$. Take $U = \cap\ U_i$ and consider:

$$b' = M_U^{U_{-m}} (y_{-m}) + \ldots + M_U^{U_o} (y_o) + \ldots + M_U^{U_n} (y_n), \text{ then:}$$

$$b\ = T^p\ (b')^\star \in h(N^\star(U)) \text{ represents } y.$$

Second step: in view of Prop. 2.3., both $(\underline{\underline{a}}(N))^\star$ and $\underline{\underline{a}}'\ (N^\star)$ are in $g\sigma(R[T])$. In order to establish the isomorphism it will be sufficient to establish isomorphisms between the stalks (cfr. e.g. [2]). Now, for all $x \in X$:

$$S_x\ (\underline{\underline{a}}'\ (N^\star)) = S_x\ (N^\star) = (S_x(N\ ))^\star = (S_x\ (\underline{\underline{a}}\ (N)))^\star = S_x\ (\underline{\underline{a}}(N)^\star)$$

(3.4): <u>Theorem</u>: Let X be a compact topological spare and $N \in g\pi(R[T])$ a graded subpresheaf of $M[T]$ with $M \in \pi(R)$, then: $(\underline{\underline{a}}'\ (N))_\star = \underline{\underline{a}}\ (N_\star)$, i.e. the following diagram "commutes" for suitable   as above:

$$\begin{array}{ccc}
g\pi(R[T]) & \xrightarrow{\ (-)_\star\ } & \pi(R) \\
\underline{\underline{a}}' \downarrow & & \downarrow \underline{\underline{a}} \\
g\sigma(R[T]) & \xrightarrow{\ (-)_\star\ } & \sigma(R)
\end{array}$$

Proof.

First step: for every $x \in X : S_x(N_\star) = (S_x(N))_\star$. Let $a \in S_x(N_\star)$, then there is a neighborhood U of x and an element $y \in N_\star(U)$ representing a. Pick $z \in h(N(U))$ such that $z_\star = y$, and define

$$f : S_x(N_\star) \longrightarrow (S_x(N))_\star$$
$$a \longmapsto (N_x^U(z))_\star$$

This definition is independent of the choices made. For, let V be another neighborhood of x and y' (with corresponding ') in $N_\star(V)$ (resp. in $h(N(V))$) representing a, then we can find an open $x \in W \subset U \cap V$ such that:

$$(N_\star)_W^U(y) = (N_\star)_W^V(y')$$

Hence, $(N_W^U(z))_\star = (N_W^V(z'))_\star$ and thus there exists a natural number k such that, $N_W^U(z) = N_W^V(z')\,T^k$.

Finally:

$$(N_x^U(z))_\star = (N_x^W(N_W^U(z)))_\star = (N_x^W(N_W^V(z')\,T^k))_\star = (N_x^V(z'))_\star$$

Now, f is injective; for if $f(a) = o$, then $N_x^U(z) = o$ and z (hence y) represents the zero morphism, thus $a = o$.

Also, f is surjective; for if $y \in (S_x(N))_\star$ then there exists an element $z \in h(S_x(N))$ with $z_\star = y$. Take an element $v \in h(N(U))$ representing z, then put $a = (N_\star)_x^U(v_\star)$ and one easily checks that $f(a) = y$.

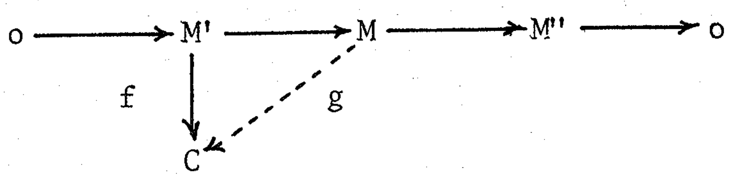
Second step: in view of Prop. 2.5 $(\underline{\underline{a}}'(N))_\star$ and $\underline{\underline{a}}(N_\star)$ are both in $\sigma(R)$. Isomorphism will follow from the stalkwise isomorphisms. For every $x \in X$:

$$S_x(\underline{\underline{a}}'(N)_\star) \approx S_x(\underline{\underline{a}}'(N))_\star \approx S_x(N)_\star \approx S_x(N_\star) \approx S_x(\underline{\underline{a}}(N_x))$$

(3.5): The example given in section 2 shows that the compactness condition cannot be dropped.

## 4. HOMOGENIZATION OF KERNEL FUNCTORS

(4.1): Let $\mathcal{C}$ be an arbitrary Grothendieck category. An *idempotent kernel functor* $\sigma$ in $\mathcal{C}$ is a left exact subfunctor of the identity such that $\sigma(C/\sigma(C)) = o$ for every $C \in Ob(\mathcal{C})$. An object C of $\mathcal{C}$ is called *$\sigma$-torsion* if $\sigma(C) = C$, *$\sigma$-torsion free* if $\sigma(C) = o$. C is said to be *$\sigma$-injective* if every diagram:




with M' $\sigma$-torsion may be completed commutatively. If g is unique as such, C is called *faithfully $\sigma$-injective* or *$\sigma$-closed.* With every object C of $\mathcal{C}$ we may associate (in an essentially unique way) a $\sigma$-closed object $Q_\sigma(C)$, containing $C/\sigma(C) = \overline{C}$ such that $Q_\sigma(C)/\overline{C}$ is $\sigma$-torsion. More details on localization may be found in [3, 5, 6, 8].

(4.2): If $\mathcal{C}$ = R-mod, there is a one-to-one correspondence between idempotent kernel functors and idempotent filters $\mathcal{L}$ of left ideals of R, i.e. satisfying:

1. $I \in \mathcal{L}$, $I \subset J$, then $J \in \mathcal{L}$
2. $I, J \in \mathcal{L}$, then $I \cap J \in \mathcal{L}$
3. $I \in \mathcal{L}$, $x \in R$, then $(I : x) \in \mathcal{L}$
4. if I is a left ideal of R, $J \in \mathcal{L}$ such that $(I : x) \in \mathcal{L}$ for all $x \in J$, then $I \in \mathcal{L}$.

With every idempotent kernel functor $\sigma$ we associate the set $\mathcal{L}(\sigma)$ of left ideals I of R such that R/I is $\sigma$-torsion. Conversely, to an idempotent filter we associate the kernel functor $\sigma$ : $\sigma(M) = \{m \in M \mid \exists I \in \mathcal{L} : Im = o\}$

(4.3): If $C$ = R-gr, we will restrict attention to <u>rigid</u> kernel functors, i.e. idempotent kernel functors such that M(n) is $\sigma$-torsion if M is $\sigma$-torsion ($(M(n))_m = M_{m+n}$). Recall from [7] that there exists a one-to-one correspondence between rigid kernel functors and filters $L$ consisting of graded left R-ideals satisfying:

1. $I \in L$, $I \subset J$, then $J \in L$
2. $I, J \in L$, then $I \cap J \in L$
3. $I \in L$, $x \in h(R)$, then $(I : x) \in L$
4. If I is a graded left ideal of R, $J \in L$ such that $(I : x) \in L$ for all $x \in h(J)$, then $I \in L$.

(4.4): In the sequel, R will be a graded ring. Let $\sigma$ be an idempotent kernel functor in R-mod with corresponding idempotent filter $L(\sigma)$. Now, define:
$L(\sigma^{\star}) = \{I \text{ graded left ideal of } R[T] : I_{\star} \in L(\sigma)\}$

(4.5): <u>Proposition</u>: $L(\sigma^{\star})$ satisfies the condition of (4.3), hence $L(\sigma^{\star})$ corresponds to a rigid kernel functor in R[T]-gr which we denote by $\sigma^{\star}$. $\sigma^{\star}$ is called the <u>homogenized</u> kernel functor of $\sigma$.

<u>Proof</u>

1. $I \in L(\sigma^{\star})$, $I \subset J$, then $I_{\star} \subset J_{\star}$ and by (4.2.1), $J_{\star} \in L(\sigma)$, whence $J \in L(\sigma^{\star})$
2. $I, J \in L(\sigma^{\star})$, it is easily checked that $I_{\star} \cap J_{\star} = (I \cap J)_{\star} \in L(\sigma)$ (by 4.2.2), whence $I \cap J \in L(\sigma^{\star})$
3. $I \in L(\sigma^{\star})$, $x \in h(R[T])$, then $(I : x)_{\star} = (I_{\star} : x_{\star}) \in L(\sigma)$ (4.3.3), thus $(I : x) \in L(\sigma^{\star})$
4. Let I be a graded left ideal of R[T] and $J \in L(\sigma^{\star})$ such that $(I : x) \in L(\sigma^{\star})$ for all $x \in h(J)$, then:
   $(I : x)_{\star} = (I_{\star} : x_{\star}) \in L(\sigma)$ for all $x_{\star} \in J_{\star}$ and therefore (4.2.4) $I_{\star} \in L(\sigma)$, whence $I \in L(\sigma^{\star})$.

(4.6) <u>lemma</u>: If M is an "admissible" graded left R[T]-module (i.e. $M \subset N[T]$ for some $N \in R\text{-mod}$), then: $(\sigma^\star(M))_\star = \sigma(M_\star)$.

<u>Proof</u>

If $x \in (\sigma^\star(M))_\star$ then we can find an element $y \in h(\sigma^\star(M))$ such that $y_\star = x$, thus there is an $I \in L(\sigma^\star)$ : $Iy = o$. Therefore $I_\star y_\star = I_\star x = o$ and by definition $I_\star \in L(\sigma)$ yielding $x \in \sigma(M_\star)$.

Conversely, if $x \in \sigma(M_\star)$ there exist an $I \in L(\sigma)$ such that $Ix = o$. Further, there is an $y \in h(M)$ : $y = x^\star T^k$ and $I^\star \in L(\sigma^\star)$. Finally, $I^\star y = (Ix)^\star T^k = o$ and thus $y \in \sigma^\star(M)$ and hence $x \in (\sigma^\star(M))_\star$.

(4.7): Recall from [4] that there exists an <u>exact</u> functor $(-)_\star : R[T]\text{-gr} \to R\text{-mod} : (M)_\star = M/(T-1)M$. When M is an admissible graded left R[T]-module (cfr. 4.6), $(M)_\star = M_\star$.

(4.8): <u>Theorem</u>: If M is an admissible graded left R-module, then $Q_\sigma(M_\star) = (Q^g_{\sigma^\star}(M))_\star$ (more details on graded modules of quotients can be found in [7]).

<u>Proof</u>

If M is an admissible graded left R-module, so is $M/\sigma^\star(M)$, for,
$M/\sigma^\star(M) \to N[T]/\sigma^\star(N\,T) \cong N[T]/\sigma(N)[T] \cong (N/\sigma(N))[T]$.

Thus, we may restrict attention to admissible $\sigma^\star$-torsion free graded left modules. It follows from lemma 4.6 that $M_\star$ is then $\sigma$-torsionfree. We have the following exact sequence in R[T]-gr (cfr. [4]):

$$o \longrightarrow M \longrightarrow Q^g_{\sigma^\star}(M) \longrightarrow \sigma^\star(E^g(M)/M) \longrightarrow o$$

where $E^g(M)$ denotes the injective envelope of M in R[T]-gr.

Exactness of $(-)_\star$ yields an exact diagram in R-mod:

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & M_\star & \longrightarrow & (Q_\sigma^{g\star}(M))_\star & \longrightarrow & (\sigma\star(E^g(M)/M))_\star & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & Q_\sigma(M^\star) & & & & & & \end{array}$$

Further, it is left to the reader to check that $(\sigma^\star(M))_\star \to \sigma((M)_\star)$ for every $M \in R[T]$-gr. Therefore, $(\sigma^\star(E^g(M)/M))_\star$ is $\sigma$-torsion, whence we get a canonical inclusion:

$$(Q_\sigma^{g\star}(M))_\star \hookrightarrow Q_\sigma(M_\star)$$

Conversely, suppose $x \in Q_\sigma(M_\star)$ then we can find a left ideal I in $L(\sigma)$ such that $Ix \in M_\star$, whence $I^\star x^\star \in (M_\star)^\star$. For every $i \in h(I^\star)$ consider the following:

$$\begin{aligned}((M : x^\star) : i) &= \{r \in R[T] : ri \in (M : x^\star)\}\\ &= \{r \in R[T] : ri\,x^\star \in M\}\end{aligned}$$

$ix^\star \in (M_\star)^\star$, thus we find a natural number k such that $T^k ix^\star \in M$, whence $(T^k) \subset ((M : x^\star) : i)$. $(T^k)_\star = R$, hence $(T^k) \in L(\sigma^\star)$ and therefore $((M : x^\star) : i) \in L(\sigma^\star)$ for all $i \in h(I^\star)$. Naturally, $I^\star \in L(\sigma^\star)$ and by (4.3.4), $(M : x^\star) \in L(\sigma^\star)$ or, equivalently, $x^\star \in Q_\sigma^{g\star}(M)$.

# 5. DEHOMOGENIZATION OF KERNEL FUNCTORS

(5.1): Let $\sigma$ be a rigid kernel functor in $R[T]$-gr with associated filter $L(\sigma)$. Now, define:

$$L(\sigma_\star) = \{I \text{ left ideal of } R : I^\star \in L(\sigma)\}$$

(5.2): Proposition: $L(\sigma_\star)$ is an idempotent filter. The corresponding idempotent kernel functor $\sigma_\star$ in R-mod will be called the dehomogenized kernel functor of $\sigma$.

Proof.

1. $I \in L(\sigma_\star)$, $I \subset J$, then $I^\star \subset J^\star$ and hence $J \in L(\sigma_\star)$
2. $I, J \in L(\sigma_\star)$, then $(I \cap J)^\star = I^\star \cap J^\star \in L(\sigma)$, thus $I \cap J \in L(\sigma_\star)$
3. $I \in L(\sigma_\star)$, $x \in R$, then $(I^\star : x^\star) \in L(\sigma)$, thus $(I : x) \in L(\sigma_\star)$
4. Let $(I : x) \in L(\sigma_\star)$ for all $x \in J \in L(\sigma_\star)$, then $(I^\star : x^\star) \in L(\sigma)$ for all $x^\star \in h(J^\star)$, whence $I^\star \in L(\sigma)$, or, $I \in L(\sigma_\star)$.

(5.3): Theorem: There is a one-to-one correspondence between idempotent kernel functors in R-mod and rigid kernel functors in $R[T]$-gr such that $(T^\ell)$ is an element of the associated filter for all $\ell \in \mathbb{N}$.

Proof.

If $\sigma$ is an idempotent kernel functor, by (Prop. 4.5) $\sigma^\star$ is a rigid kernel functor in $R[T]$-gr such that $(T^\ell) \in L(\sigma^\star)$ for all $n \in \mathbb{N}$. Clearly, $(\sigma^\star)_\star = \sigma$, for, $I \in L((\sigma^\star)_\star)$ if and only if $I^\star \in L(\sigma^\star)$ if and only if $(I^\star)_\star = I \in L(\sigma)$; whence $L(\sigma) = L((\sigma^\star)_\star)$.

Conversely, if $\sigma$ is a rigid kernel functor in $R[T]$-gr such that $(T^\ell) \in L(\sigma)$ for all $n \in \mathbb{N}$, then $(\sigma_\star)^\star = \sigma$. $I \in L((\sigma_\star)^\star)$ if and only if $I_\star \in L(\sigma_\star)$,

if and only if $(I_\star)^\star \in L(\sigma)$. To complete the proof it remains to show that $I \in L(\sigma)$ if $(I_\star)^\star \in L(\sigma)$. Take $i \in h((I_\star)^\star)$, then:
$(I : i) = \{r \in R[T] : ri \in I\}$. There exists a natural number $\ell$ such that $T^\ell i \in I$, whence $(T^\ell) \subset (I : i)$. Because $(T^\ell) \in L(\sigma)$, so is $(I : i)$ and (4.3.4) finishes the proof.

(5.4): <u>Proposition</u>: If R is a positively graded left-Noetherian ring, $M \subset N$ and $N \in R\text{-gr}$ such that N is $\sigma$-torsionfree, then;
$Q_\sigma(M) \cong Q^{g}_{\sigma^\star}(M^\star)_0$ as $R_0$-modules.

<u>Proof.</u>
By [4, 7], $Q^{g}_{\sigma^\star}(M^\star)_0 = \varinjlim \mathrm{HOM}(I, M^\star)_0$ where the direct limit is taken over all graded left ideals I contained in $L(\sigma^\star)$. Let $a \in Q^{g}_{\sigma^\star}(M^\star)_0$ be represented by a gradation preserving R[T]-module morphism $f : I \to M^\star$. Define a mapping $f_\star : I_\star \to M$ in the following way: if $x \in I_\star$ and $y \in h(I)$ such that $y_\star = x$, then $f_\star(x) = (f(y))_\star$. This mapping is well defined, for, let y' be another homogeneous element of I such that $y'_\star = x$, then we can find a natural number n such that e.g. $yT^n = y'$, whence
$(f(y'))_\star = (f(yT^n))_\star = (T^n f(y))_\star = (f(y))_\star$. Further, $f_\star$ is R-linear, for, take $y, z \in h(I)$ such that $y_\star = v$, $z_\star = w$ then there exists a natural number n such that e.g. $y + T^n z \in h(I)$ and:
$f_\star(v + w) = (f(y + T^n z))_\star = (f(y))_\star + (T^n f(z))_\star = f_\star(v) + f_\star(w)$.
For every $r \in R$ we have:
$f_\star(rv) = (f(r^\star y))_\star = (r^\star f(y))_\star = r(f(y))_\star = r f_\star(v)$.
Thus $f_\star \in \mathrm{Hom}_R(I_\star, M)$. Let b be the element of $Q_\sigma(M)$ represented by $f_\star$.
This enables us to construct a map:
$h : Q^{g}_{\sigma^\star}(M^\star)_0 \longrightarrow MQ_\sigma(M)$ by $h(a) = b$.
First, let us check that h is $R_0$-linear. Let a resp. a' be represented

by f resp. f' on I, then for every $y \in h(I)$ we have that:

$(f+f'(y))_\star = (f(y) + f'(y))_\star = (f(y))_\star + (f'(y))_\star$ and therefore

$h(a+a') = h(a) + h(a')$. For every $r \in R_o$ and every $y \in h(I)$ we have that

$((rf)(y))_\star = (r.f(y))_\star = r(f(y))_\star$, hence $h(ra) = rh(a)$. Further, h is

injective, for let a be represented by f on I and suppose that $h(a) = o$,

then we can find a left ideal $J \in L(\sigma)$ such that $f_\star \mid J = o$, whence

$f \mid J^\star \cap I = o$, implying that $a = o$.

h is also injective. Let $b \in Q_\sigma(M)$ be represented by a morphism

$f \in \mathrm{Hom}_R(I, M)$ with $I \in L(\sigma)$. Let I be generated by $i_1, \ldots, i_n$ and denote

$f(i_k) = y_k$. Now let

$r = \max\{\deg(i_1^\star), \ldots, \deg(i_n^\star), \deg(y_1^\star), \ldots, \deg(y_n^\star)\}$ and take

$i_k' = i_k^\star.T^{r-\deg(i_k^\star)}$ ; $y_k' = y_k^\star.T^{r-\deg(y_k^\star)}$.

Let I' be the graded left R[T]-ideal generated by the $i_k'$ (hence $I' \in L(\sigma^\star)$)

and left $f^\star$ be the gradation preserving mapping $f^\star : I' \to M^\star$;

$f^\star(r_1 i_1' + \ldots + r_n i_n') = r_1 y_1' + \ldots + r_n y_n'$ where $r_i \in R[T]$.

$f^\star$ is a well defined mapping, for, let $\Sigma r_i\, i_i' = \Sigma s_i\, i_i' \in h(I')$ (i.e.

$\deg r_i = \deg r_i = m$ for every i), then $\Sigma (r_i)_\star\, i_i = \Sigma (s_i)_\star\, i_i$ implying that

$\Sigma (r_i)_\star\, y_i = \Sigma (s_i)_\star\, y_i$. Now, suppose that $\Sigma r_i\, y_i' \neq \Sigma s_i\, y_i'$ then

$\Sigma (r_i)_\star\, y_i \neq \Sigma (s_i)_\star\, y_i$, a contradiction. By construction $f^\star$ is clearly R[T]-linear and $(f^\star)_\star = f$.

(5.5): <u>Proposition</u>: In the situation of (5.4) with R σ-torsion free, $Q_\sigma(R)$ and $Q_\sigma^{g\star}(R[T])_o$ are ring-isomorphic under the identification of Prop. (5.4).

<u>Proof</u>.

Recall that if $a, b \in Q_\sigma^{g\star}(R)_o$ (resp. $\in Q_\sigma(R)$) are represented by f on I and g on J where $I, J \in L(\sigma^\star)$ (resp. $\in L(\sigma)$) then a.b is the element represented by $g \circ f$ on $f(I)$. Thus, $h(a).h(b)$ is represented by $g_\star \circ f_\star$ on $f_\star^{-1}(I_\star)$, i.e. for every $x \in f_\star^{-1}(I_\star)$ and every $y \in h(f^{-1}(I))$ such that $y_\star = x$ we have: $g_\star \circ f_\star(x) = g_\star(f(y)_\star) = (g \circ f(y))_\star$, whence $h(a.b) = h(a).h(b)$.

(5.6): Proposition: In the situation of (5.5) with R $\sigma$-torsion free, $Q_\sigma(M)$ and $Q_\sigma^{g_\star}(M^\star)_0$ are isomorphic as $Q_\sigma(R)$-modules modulo the ring-isomorphism of Prop. (5.5).

Proof

Similar to the proof of Prop. 5.5.

## 6. SOME EXAMPLES

### A: LAMBEK'S KERNEL FUNCTOR

(6.1): Let R be a graded ring. Lambek's kernel functor $\sigma_L$ in R-mod is determined by the filter:

$L(\sigma_L) = \{I \text{ left ideal of } R \mid \forall a, b \in R : \text{if } (I:a) \subset \mathrm{Ann}_R\, b, \text{ then } b = o\}$

Lambek's graded kernel functor $\sigma_L^g$ on R[T]-gr (cfr. [ 4 ]) is determined by the graded filter:

$L(\sigma_L^g) = \{I \text{ graded left ideal of } R[T] \mid \forall a, b \in h(R[T]) : \text{if } (I:a) \subset \mathrm{Ann}_{R[T]}\, b,$ then $b = o\}$.

(6.2): <u>Proposition</u>: $(\sigma_L^g)_\star = \sigma_L$ and $(\sigma_L)^\star = \sigma_L^g$

<u>Proof</u>

Let $I \in L(\sigma_L)$ and suppose that there are homogeneous elements a, b of R[T] with $(I^\star : a) \subset \mathrm{Ann}_{R[T]}(b)$, then $(I : a_\star) \subset \mathrm{Ann}_R(b_\star)$ whence $b_\star = o$ and also $b = o$. Thus, $I^\star \in L(\sigma_L^g)$. Conversely let I be a left ideal in $L((\sigma_L^g)_\star)$ and suppose there are elements $a, b \in R$ such that $(I : a) \subset \mathrm{Ann}_R(b)$ then clearly $(I^\star : a^\star) \subset \mathrm{Ann}_{R[T]}(b^\star)$, thus $b^\star = o = b$. Thus $L(\sigma_L) = L((\sigma_L^g)_\star)$. The second statement is proved in a similar way.

### B: LAMBEK-MICHLER KERNEL FUNCTORS

(6.3): Let R be a graded ring and let S be a multiplicatively closed system of elements of R. With S we may associate an idempotent kernel functor $\sigma_S$ in R-mod with filter:

$L(\sigma_S) = \{I \text{ left ideal of } R \mid \forall r \in R : (I:r) \cap S \neq \emptyset\}$

If U is a multiplicatively closed system of homogeneous elements of R[T], define a kernel filter in R[T]-gr by taking

$L(\sigma_U^g)$ = {I graded left ideal of R[T] : $\forall r \in h(R[T])$, $(I:r) \cap U \neq o$}.
Let S be a multiplicatively closed subset of R and let us denote with $S^\star$ the set of all homogeneous elements X of R[T] such that $x_\star \in S$. Clearly $S^\star$ is a multiplicatively closed set of homogeneous elements of R[T].

(6.4): Proposition: $(\sigma_S)^\star = \sigma_{S^\star}^g$

Proof.
Take $I \in L((\sigma_S)^\star)$ and suppose we can find an homogeneous element r in R[T] such that $(I:r) \cap S^\star = \emptyset$. Suppose there exists an element s in $(I_\star : r_\star) \cap S$, thus $sr_\star$ in $I_\star$, then there is a natural number k such that $s^\star T^k r$ is in I and $s^\star T^k \in S^\star$, whence $(I_\star : r_\star) \cap S = \emptyset$, contradiction. Conversely, take $I \in L(\sigma_{S^\star}^g)$. For every $r \in h(R[T])$ we have $(I:r) \cap S^\star \neq \emptyset$, whence $(I_\star : r_\star) \cap S \neq \emptyset$.

(6.5): The main example: Let R be a positively graded left-Noetherian ring and let P be a prime ideal of R. The Lambek-Michler kernel functor associated with P $_p$ is determinded by the multiplicatively closed set:
$G(P) = \{x \in R \mid r \in R, rx \in P \text{ then } r \in P\}$. $P^\star$ is a graded prime ideal of R[T] (see 7.3). The graded Lambek-Michler kernel functor (cfr. [4]) associated with $P^\star$ is determided by the set:
$h(G(P^\star)) = \{x \in h(R[T]) : r \in h(R[T]), rx \in P^\star \text{ then } r \in P^\star\}$.

(6.6): Proposition: $(\sigma_p)^\star = \sigma_{p^\star}^g$

Proof.
In view of Prop. 6.4 we are left to prove that $(G(P))^\star = h(G(P^\star))$.
This is easy and it is left as an exercise to the reader.

(6.7): It is well known that the left ring of quotients $S^{-1}R$ of a ring R with respect to a multiplicatively closed set S exists if and only if S satisfies the so called Ore-conditions:

$(O_1)$: $s \in S$, $r \in R$ such that $rs = o$ then there exists an element $s' \in S$ such that $s'r = o$.

$(O_2)$: if $r \in R$ and $s \in S$ then there exist elements $r' \in R$ and $s' \in S$ such that $s'r = r's$.

If R is a graded ring and S is a multiplicatively closed set consisting of homogeneous elements, the left graded ring of quotients $S^{-1}R$ exists if and only if S satisfies the graded Ore-conditions (cfr. [4])

$(O_1^g)$: $s \in S$, $r \in h(R)$ such that $rs = o$ then there exists an element $s' \in S$ such that $s'r = o$.

$(O_2^g)$: if $r \in h(R)$ and $s \in S$ then there exist elements $r' \in h(R)$ and $s' \in S$ such that $s'r = r's$.

$S^{-1}R$ is graded in the following way:

$(S^{-1}R)_n = \{a/s : a \in h(R),\ s \in S \text{ and } \deg a - \deg s = n\}$

In [4] it is proved that S satisfies the graded Ore conditions if and only if S satisfies the Ore conditions.

(6.8): Proposition: equivalent are:

(1): S satisfies the Ore conditions in R

(2): $S^\star$ satisfies the Ore conditions in $R[T]$.

Proof.

(1) ⇒ (2): In view of (6.7) it is sufficient to check the graded Ore conditions:

$(O_1^g)$: $s \in S^\star$, $r \in h(R[T])$ such that $rs = o$, then $r_\star s_\star = o$ and by (1) we can find an element $s' \in S$ such that $s'r_\star = o$, whence $(s')^\star (r_\star)^\star = o$.

We may find a natural number k such that $(s')^{\star}\ (r_{\star})^{\star}\ T^{k} = (s')^{\star}\ r = o$.

$(O_2^g)$: Let $r \in h(R[T])$ and $s \in S^{\star}$. By (1) we find elements $r_1 \in R$, $s_1 \in S$ such that $s_1\ r_{\star} = r_1\ s_{\star}$. Finally there exist natural numbers $\ell$ and m such that: $((s_1)^{\star}\ T^{\ell})\ r = ((r_1)^{\star}\ T^{m})\ s$ finishing the proof.

(2) ⇒ (1): left as an exercise to the reader.

C: MURDOCH - VAN OYSTAEYEN KERNEL FUNCTORS

(6.4): Let R be a positively graded, left Noetherian ring and let P be a twosided prime ideal of R. $\sigma_{R-P}$ will be the symmetric kernel functor in R-mod determined by the filter:

$L(\sigma_{R-P}) = \{I \text{ left ideal of } R \mid R\, s\, R \subset I \text{ for some } s \in R - P\}$.

Let P' be a graded twosided prime ideal of R[T]. $\sigma^{g}_{R[T]-P'}$ will be the symmetric rigid kernel functor in R[T]-gr determined by the filter.

$L(\sigma^{g}_{R[T]-P'}) = \{I \text{ graded left ideal of } R[T] \mid R[T]\ s\ R[T] \subset I \text{ for some } s \in h(R[T]-P)\}$

(6.5): <u>Proposition</u>: $\sigma^{g}_{R[T]-P^{\star}} = (\sigma_{R-P})^{\star}$

<u>Proof</u>

Let I be a graded ideal in $L(\sigma^{g}_{R[T]-P^{\star}})$ then we can find an homogeneous element s in $R[T]-P^{\star}$ such that:

$R[T]\ s\ R[T] \subset I$, whence $R\ s_{\star}\ R \subset I_{\star}$. Because $s = (s_{\star})^{\star}\ T^{n}$ and $T \in P^{\star}$, $s_{\star} \in R - P$, thus $I_{\star} \in L(\sigma_{R-P})$.

Conversely, let I be a graded ideal of $L((\sigma_{R-P})^{\star})$ then there exists an element $s_{\star}$ in R - P such that $R\ s_{\star}\ R \subset I_{\star}$, whence $R[T]\ s\ R[T] \subset I$ for some element $s \in R[T] - P^{\star}$.

## 7. NONCOMMUTATIVE AFFINE AND PROJECTIVE SCHEMES

(7.1): If R is a left Noetherian prime ring, Spec R will be the set of all proper ideals of R. To any ideal I of R the set $V(I) = \{P \in \text{Spec } R : I \subset P\}$ is associated and taking the sets $X(I) = \text{Spec } R - V(I)$ for the open sets defines a topology on Spec R, called the Zariski topology.
To any ideal I of R we associate the filter:
$L(\sigma_I) = \{$left ideals L of R such that $I^n \subset L$ for some $n \in \mathbb{N}\}$.
Assigning $Q_I(R)$ to $X(I)$ for every ideal defines a sheaf of noncommutative rings on Spec R (cfr. [6]), $\tilde{R}$, called the structure sheaf. Spec R with the Zariski topology and the sheaf $\tilde{R}$ is said to be an affine scheme.

(7.2): If R is a positively graded, left Noetherian prime ring, Proj R will be the set of all proper graded prime ideals of R not containing $R_+ = \bigoplus_{n>0} R_n$. Endow Proj R with the topology induced by the Zariski topology of Spec R as follows. Put for any ideal I of R:
$V_+(I) = \{P \in \text{Proj } R : P \supset I\}$, $X_+(I) = \text{Proj } R - V_+(I)$. In these definitions we may replace I by the smallest graded ideal of R containing I, so I may be supposed to be graded.
The sets $X_+(I)$, I varying in the set of graded ideals of R, exhausts the open sets of Proj R. To an open set $X_+(I)$ the rigid kernel functor $\sigma_I^g$ is associated, i.e.
$L(\sigma_I^g) = \{$graded left ideals L of R such that $I^n \subset L$ for some $n \in \mathbb{N}\}$.
Assigning $(Q_I^g(R))_0$ to $X_+(I)$ for each graded ideal I of R defines a sheaf of noncommutative rings on Proj R, $\hat{R}$, called the structure sheaf on Proj R.

(7.3): Proposition: If R is a positively graded ring, then
Proj $R[T]$ = Proj $R \cup$ Spec $R$ and Spec $R$ is homeomorphic with $X_+((T))$.

Proof.
If $P$ is a graded prime ideal of $R[T]$ containing $(T)$, then $P = p + (T)$ where $p = P \cap R$ is a graded prime ideal of $R$ not contained in $R_+$ because $P \in$ Proj $R[T]$ and $R[T]_+ = R_+ + (T)$. Thus we obtain a one-to-one correspondence between Proj $R$ and $V_+(T)$ given by: $p \mapsto p + (T)$.
If $P \in X_+((T))$, it is easy to check that $P$ is a proper prime ideal of $R$. Further, $(P_\star)^\star = P$, for, if $\alpha \in (P_\star)^\star$ we can find a natural number $n$ such that $T^n \alpha \in P$, whence $T^n R \alpha \subset P$ and thus $\alpha \in R$ because $T^n \in P$. Therefore, $(-)^\star$ : Spec $R \to X_+((T))$ determines a one-to-one correspondence which is easily seen to be a homeomorphism since $P \in V(I)$ if and only if $P^\star \in X_+((T)) \cap V_+(I^\star)$.

(7.4): From Prop. 5.5, Prop. 6.5, the preceding remarks and lemma 14.7 of [4], it follows that:
$\widehat{R[T]} \mid X_+((T)) \cong \tilde{R}$ as ringed spaces.
In particular, if $R$ is a left Noetherian prime ring $\mathbb{P}^n(R) = \widehat{R[T_0, \ldots, T_n]}$ may be covered by $n + 1$ copies of $\mathbb{A}^n(R) = \widehat{R[S_1, \ldots, S_n]}$, yielding the desired noncommutative analogue of the classical commutative result.

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