A LOCAL-GLOBAL APPROACH TO LOCALIZATION IN GROTHENDIECK CATEGORIES

by

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0. INTRODUCTION

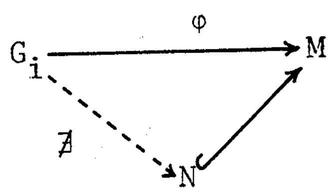
The abstract theory of localization in Grothendieck categories has reached a more or less final form, as expounded in [1, 4, 7]. However, an important gap in the theory is the lack of a manageable substitute for idempotent filters, the use of which simplifies matters a lot in the module case.
The first attempt to overcome this problem is due to A. Verschoren [8].
To an idempotent kernel funtor σ in a Grothendieck category with generator G, he associates a filter $L(G,\sigma)$ consisting of those subobjects G' of G such that G/G' is σ-torsion. However, these filters seem to be an inadequate tool in many applications such as graded modules, sheaves and presheaves, etc. In these cases, we usually have an infinite family of generators $\{G_i;\ i \in I\}$ and, although it is natural to impose conditions (Noetherianness, smallness, ...) on each of the G_i's, it seems hopeless to impose any condition on the generator $\oplus G_i$.
On the other hand, if one tries to copy results from the module case to "arbitrary" Grothendieck categories using properties of these filters (cfr. e.g. [9], [10]), the generator has to satisfy very restrictive conditions. For instance, in order to prove the claimed one-to-one correspondence between hereditary torsion theories and the filters mentioned above [8], the generator has to be a small and projective object, leaving no Grothendieck categories left but the module categories, cfr. remark 2.10 below.
The purpose of this paper is to remedy these problems by associating to each of the G_i's a "local filter". These filters behave like the classical idempotent filters, modulo the natural restriction that whenever G_i and G_j are related, the corresponding filters have to be related, too (Prop. 1.7). Again, σ-torsionness, σ-injectivity, σ-neatness etc. can be described entirely in function of these local filters (Prop. 1.8, 1.10, 3.6).

In section two, we treat the "cohesive" case (cfr. def. 2.2). In this case, we can associate to each of the local filters $L(G_i, \sigma)$ idempotent kernel functors σ_i, the "local components" of σ. σ may be recovered from them by taking the infimum, admitting a local-global lemma (2.8). A combination of these results yields a one-to-one correspondence between idempotent kernel functors and families of related local filters (Prop. 2.9). In the third section we give some examples how conditions on the local filters yield global information about the classes of σ-closed, σ-injective, σ-torsion ... object (Prop. 3.2, 3.4, 3.8). The attentive reader will verify that these proofs are mere adaptations of the classical module-proofs (cfr. e.g. [2], [3]), using a Yoneda-lemma like argument (replacing elements by morphisms) and some weak conditions on the family of generators. Using the same techniques one may derive many more local-global results which are of particular importance to localization in semi-noetherian and locally Noetherian categories. The author aims to return to some of them in a subsequent paper.

Necessary background on localization and Grothendieck categories may be found in [1, 2, 3, 4, 6].

1. LOCALLY ASSOCIATED FILTERS

(1.1): Let $\underline{C}$ be any category. Recall that a set of objects $\{G_i;\ i \in I\}$ of $\underline{C}$ is said to be a <u>family of generators</u> for $\underline{C}$ if for each object M of $\underline{C}$ and each proper subobject N of M, there exists a morphism $\varphi : G_i \to M$ for some $i \in I$ which does not factorize through the inclusion $N \to M$:



We say that G is a <u>generator</u> for $\underline{C}$ if {G} is a family of generators. Clearly, $\{G_i;\ i \in I\}$ is a family of generators if and only if $\bigoplus_{i \in I} G_i$ is a generator.

(1.2): <u>Some examples</u>

(a) The category R-mod which consists of all left modules over a ring with unit R, has a generator $G = R_s$, where R_s denotes R, viewed as a left R-module.

(b) Let R be a graded ring with unit. R is not a generator of R-gr, the category of all graded left R-modules with morphisms of degree o (cfr. [5]). We have a family of generators $\{R^{(i)},\ i \in Z\}$, where the $R^{(i)}$ are defined as follows: $(R^{(i)})_n = R_{n-i}$.

(c) Consider $\pi(R, X)$, the category of presheaves of left R-Modules over a topological space X. The presheaf of rings R is usually not a generator. For any open subset U of X define R_U by:

$$\Gamma(R_U, V) = \begin{cases} \Gamma(R, V) & \text{if } V \subset U \\ 0 & \text{if } V \not\subset U \end{cases}$$

It is easily checked that $\{R_U;\ U \in \mathrm{Open}(X)\}$ is a family of generators for $\pi(R, X)$.

(1.3): Throughout, $\underline{C}$ will be a Grothendieck category with a family of generators, hence with enough injectives.

A torsion theory for $\underline{C}$ is a pair of classes $(\mathcal{T}, \mathcal{F})$ such that:

(1.3.1): $\mathrm{Hom}_{\underline{C}}(T,F) = o$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$

(1.3.2): If $\mathrm{Hom}_{\underline{C}}(C,F) = o$ for all $F \in \mathcal{F}$, then $C \in \mathcal{T}$

(1.3.3): If $\mathrm{Hom}_{\underline{C}}(T,C) = o$ for all $T \in \mathcal{T}$, then $C \in \mathcal{F}$

$\mathcal{T}$ is called the torsion class and its objects are torsion objects, while $\mathcal{F}$ is a torsion-free class consisting of torsion-free objects. A torsion theory $(\mathcal{T}, \mathcal{F})$ will be called hereditary if and only if $\mathcal{T}$ is closed under subobjects. A class $\mathcal{T}$ is a torsion class for some hereditary torsion theory if and only if $\mathcal{T}$ is closed under quotient objects, direct sums, extensions and subobjects.

(1.4) A kernel functor in the Grothendieck category $\underline{C}$ is a left exact subfunctor σ of the identity. It is idempotent if for each object C of $\underline{C}$ we have: $\sigma(C / \sigma(C)) = o$. C is called σ-torsion if $\sigma(C) = C$, σ-torsion free if $\sigma(C) = o$.

(1.5) Recall that in a Grothendieck category $\underline{C}$ there is a one-to-one correspondence between idempotent kernel functors and hereditary torsion theories: to a kernel functor σ we associate the torsion theory $(\mathcal{T}_\sigma, \mathcal{F}_\sigma)$ where $\mathcal{T}_\sigma$ (resp. $\mathcal{F}_\sigma$) consists of the σ-torsion (resp. σ-torsion free) objects, conversely, to a torsion theory $(\mathcal{T}, \mathcal{F})$ we associate the kernel functor σ which to each $C \in \mathrm{Ob}\,(\underline{C})$ associates the largest subobject of C which lies in T.

(1.6): Let $\{G_i;\ i \in I\}$ be a family of generators for $\underline{C}$ and define for every G_i the following class of subobjects:

$$L(G_i : \sigma) = \{I < G_i : \sigma(G_i / I) = G_i / I\}$$

where . < . denotes : subobject of.

(1.7): <u>Proposition</u>: In the situation of (1.6) we have:

1. $J \in L(G_i : \sigma)$, $J \subset K$ then $K \in L(G_i : \sigma)$
2. $J, K \in L(G_i : \sigma)$ then $J \cap K \in L(G_i : \sigma)$
3. $J \in L(G_i : \sigma)$; $\varphi : G_j \to G_i$, then $\varphi^{-1}(J) \in L(G_j : \sigma)$
4. $J < G_i$ and $K \in L(G_i : \sigma)$ such that for every $G_j \leq G_i$ (i.e. $\mathrm{Hom}_{\underline{C}}(G_j, G_i) \neq 0$) and every morphism $\varphi : G_j \to K$ we have $\varphi^{-1}(J) \in L(G_j : \sigma)$ then $J \in L(G_i : \sigma)$

<u>Proof</u>

1. $0 \to G_i / K \to G_i / J$ and T_σ is closed under taking subobjects
2. $0 \to G_i / J \cap K \to G_i / J \oplus G_i / K$ and T_σ is closed under direct sums and subobjects.
3. We have the following exact diagram

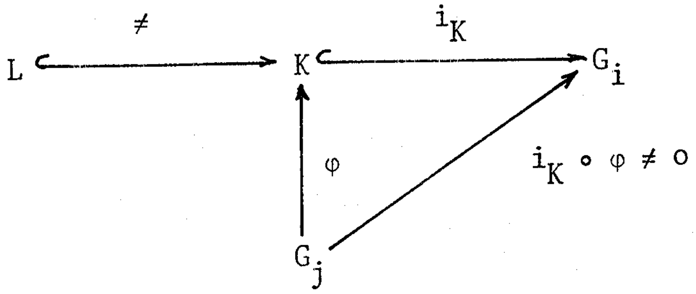
$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \varphi^{-1}(J) & \longrightarrow & G_j & \longrightarrow & G_j / \varphi^{-1}(J) & \longrightarrow & 0 \\
 & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \psi & & \\
0 & \longrightarrow & J & \longrightarrow & G_i & \longrightarrow & G_i / J & \longrightarrow & 0
\end{array}$$

where ψ is the induced mapping, which is readily seen to be injective. Hence, $G_j / \varphi^{-1}(J)$ is σ-torsion, because G_i / J is.

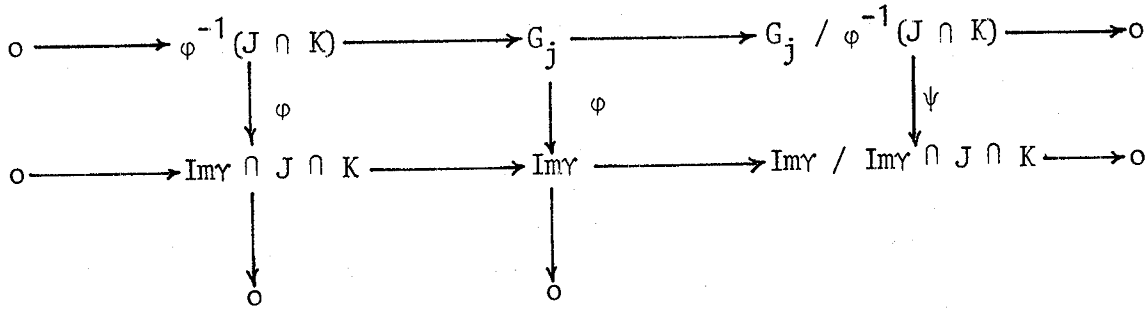
4. We have the exact sequence:

$$0 \longrightarrow K / J \cap K \longrightarrow G_i / J \longrightarrow G_i / J + K \longrightarrow 0$$

Now, clearly $G_i / J + K$ is σ-torsion as an epimorphic image of G_i / K. Suppose $\sigma(K / J \cap K) = L / I \cap K$ with $L \overset{\neq}{\hookrightarrow} K$, then we can find an index $j \in I$ and a morphism φ:



which does not factorize through L. Clearly $\varphi^{-1}(J) = \varphi^{-1}(J \cap K) \in L(G_j, \sigma)$ and we have the following exact diagram:



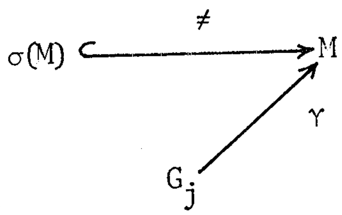
where ψ is the induced mapping, which is also surjective. Thus, $\mathrm{Im}\gamma / \mathrm{Im}\gamma \cap J \cap K \simeq \mathrm{Im}\gamma + (J \cap K) / J \cap K$ is σ-torsion as an epimorphic image of $G_j / \varphi^{-1}(I \cap K)$. Finally, $L / J \cap K \overset{\neq}{\hookrightarrow} J + \mathrm{Im}\gamma + (J \cap K) / J \cap K$, a contradiction, whence $K / J \cap K$ is σ-torsion.

Because T_σ is closed under extensions, it follows that G_i / J is σ-torsion.

(1.8): Proposition: Let σ be an idempotent kernel functor in $\underline{C}$. $M \in Ob(C)$ is σ-torsion if and only if for every $i \in I$ and every morphism $\varphi : G_i \to M$ we have that $Ker\varphi \in L(G_i : \sigma)$.

Proof

If M is σ-torsion and $\varphi : G_i \to M$, then we have an exact sequence: $o \to Ker\ \varphi \to G_i \to Im\ \varphi \simeq G_i\ /\ Ker\ \varphi \to o$. $Im\ \varphi \subset M$, hence it is σ-torsion, therefore $Ker\ \varphi \in L(G_i : \sigma)$. Conversely, suppose for every $\varphi : G_i \overset{\neq}{} M$, $Ker\ \varphi \in L(G_i, \sigma)$ and $\sigma(M) \overset{\neq}{\hookrightarrow} M$. Then we can find an index $j \in I$ and a morphism γ:

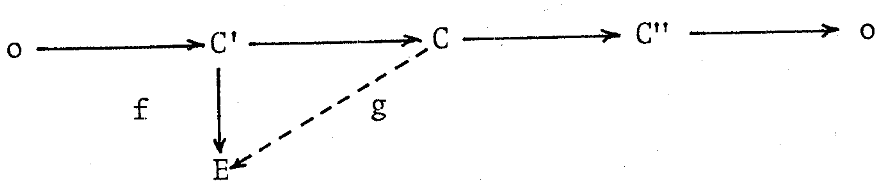


which does not factorize through $\sigma(M)$, hence $Im\gamma \not\subset \sigma(M)$.
$Ker\ \gamma \in L(G_j, \sigma)$ or, equivalently, $Im\ \gamma \simeq G_j\ /\ Ker\ \gamma$ is σ-torsion.
Finally:

$$o \longrightarrow \sigma(M) \longrightarrow \sigma(M) + Im\ \gamma \longrightarrow Im\gamma\ /\ \sigma(M) \cap Im\gamma \longrightarrow o$$

$\sigma(M)$ and $Im\gamma\ /\ \sigma(M) \cap Im\gamma$ are σ-torsion, hence so is $\sigma(M) + Im\gamma$. This forces $Im\ \gamma \subset \sigma(M)$, a contradiction. Therefore, $M = \sigma(M)$.

(1.9): An object E of $\underline{C}$ is called σ-injective if any diagram:



with exact top row and C'' σ-torsion may be completed by g to make it commutative. If g is unique as such, then E is said to be <u>σ-closed</u>. Recall from [7] that E is σ-closed if and only if C is σ-injective and σ-torsion free.

To each object C of $\underline{C}$ we may associate in an essentially unique way a σ-closed object $Q_\sigma(C)$ containing $\bar{C} = C / \sigma(C)$, such that $Q_\sigma(C) / C$ is σ-torsion. $Q_\sigma(C)$ will be called the <u>object of quotients</u> of C at σ.

If we denote by $\underline{C}(\sigma)$ the full subcategory of $\underline{C}$ consisting of all σ-closed objects in $\underline{C}$, then $Q_\sigma(-)$ is easily checked to define a left adjoint to the inclusion $i_\sigma : \underline{C}(\sigma) \to \underline{C}$. Therefore, $Q_\sigma(-)$ is a left exact endofunctor in $\underline{C}$. The following proposition is an adaptation of a similar statement due to A. Verschoren [8] for one generator.

(1.10): <u>Proposition:</u> A necessary and sufficient condition for E to be σ-injective is that each diagram

$$\begin{array}{ccccc} o & \longrightarrow & J & \longrightarrow & G_i \\ & & \downarrow & & \\ & & E & & \end{array}$$

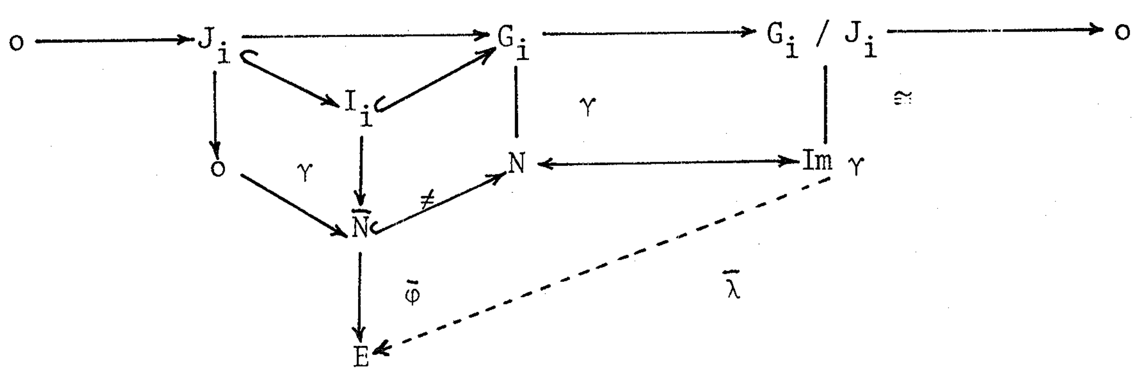
with $J \in \mathcal{L}(G_i, \sigma)$; $i \in I$, can be completed commutatively.

<u>Proof</u>

Consider a diagram

$$\begin{array}{ccccccccc} o & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & o \\ & & \downarrow{\scriptstyle \varphi} & & & & & & \\ & & E & & & & & & \end{array}$$

where N'' is σ-torsion. We have to show that φ extends to a map $\bar{\varphi} : N \to E$. Look at the set of all couples $(N^\star, \varphi^\star)$ with $N' \subset N^\star \subset N$ (hence: $\sigma(N / N^\star) = N / N^\star$) and $\varphi^\star : N^\star \to E$ a morphism extending φ. Zorn's lemma provides us with a largest couple of this kind, say $(\bar{N}, \bar{\varphi})$. Suppose that $N \overset{\neq}{\hookrightarrow} N$, then we can find an index $i \in I$ and a morphism $\gamma : G_i \to N$, which does not factorize through $\bar{N}$. We have the following situation:



where $I_i = \gamma^{-1}(\bar{N})$, $J_i = \operatorname{Ker} \gamma$. As N'' is σ-torsion, $I_i \in L(G_i, \sigma)$. The map $\varphi \circ i : I_i \to E$ extends to a map $\lambda : G_i \to E$, which factorizes through G_i / J_i, as $J = \operatorname{Ker} \gamma \subset \operatorname{Ker} \lambda$. As $G_i / J_i \simeq \operatorname{Im} \gamma$, we get a map $\bar{\lambda} : \operatorname{Im} \gamma \to E$. Because $\operatorname{Im} \gamma \not\subset \bar{N}$, $\bar{N} \subsetneq \bar{N} + \operatorname{Im} \gamma = \check{N} \subset N$. Now, define $\check{\varphi} : \check{N} \to E$ by $\check{\varphi} \mid \bar{N} = \bar{\varphi}$, $\check{\varphi} \mid \operatorname{Im} \gamma = \bar{\lambda}$. This is a well defined morphism, contradicting the maximality of $(\bar{N}, \bar{\varphi})$. Thus, $\bar{N} = N$.

2. THE COHESIVE CASE

(2.1.): An object C of $\underline{C}$ is said to be small if $Hom_{\underline{C}}$ (C, -) commutes with direct sums. A Grothendieck category $\underline{C}$ will be called locally small if it has a family of small generators.

(2.2.): Definition: A family of generators for $\underline{C}$, $\{G_i;\ i \in I\}$ will be called cohesive if every G_i is a projective, small object such that $\{G_i, \leq\}$, with $G_i \leq G_j$ if and only if $Hom_{\underline{C}}(G_i, G_j) \neq o$, is a partially ordered set.

(2.3.): Some examples:

(a): (cfr. 1.2.b), $R^{(i)} \leq R^{(j)}$ if and only if $R_{i-j} \neq o$. The family of generators $\{R^{(i)};\ i \in Z\}$ is cohesive if R satisfies the following condition: if $R_n \neq o$ and $R_m \neq o$ then $R_{n+m} \neq o$.

(b): (cfr. 1.2.c), $R_U \leq R_V$ if and only if $U \subset V$. The family of generators $\{R_U;\ U \in Open(X)\}$ is cohesive.

(2.4.): For every $i \in I$, define a class T_{σ_i} consisting of exactly those objects M of $\underline{C}$ such that for every $G_j \leq G_i$ and every morphism $\varphi : G_j \to M$ we have that $Ker\ \varphi \in L(G_j, \sigma)$.

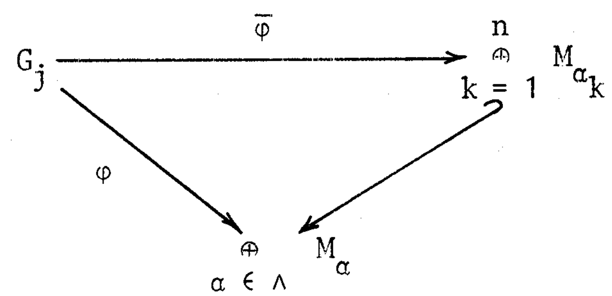
(2.5.): Proposition: If the family of generators $\{G_i : i \in I\}$ is cohesive, then T_{σ_i} is a torsion class for some hereditary torsion theory, for all $i \in I$.

Proof

(1): T_{σ_i} is closed under taking subobjects. Take $i : N \hookrightarrow M$, where $M \in T_{\sigma_i}$, then for every morphism $\gamma : G_j \to N$ with $G_j \leq G_i$ we have: $\operatorname{Ker} \gamma = \operatorname{Ker} (i \circ \gamma) \in L(G_j, \sigma)$.

(2): T_{σ_i} is closed under quotient-objects. Take $\pi : M \to N \to o$ with $M \in T_{\sigma_i}$. Let φ be any morphism from G_j to N for some $G_j \leq G_i$. Because G_j is a projective object, we can lift φ to a morphism $\bar{\varphi} : G_j \to M$. Finally, $\operatorname{Ker} \bar{\varphi} \subset \operatorname{Ker} \varphi$ and Prop. 1.7.1 finishes the proof.

(3): T_{σ_i} is closed under direct sums. Let $\{M_\alpha : \alpha \in \Lambda\}$ be a family of objects in T_{σ_i} and $\varphi : G_j \to \oplus M$ with $G_j \leq G_i$. Because G_j is a small object this morphism factorizes through a finite sum:

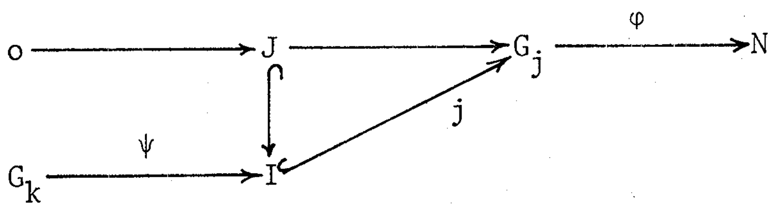


$\operatorname{Ker} \varphi = \operatorname{Ker} \bar{\varphi} \supset \bigcap_{k=1}^{n} \operatorname{Ker} (\pi_{\alpha_k} \circ \varphi)$ where π_α denotes the canonical projection. Now, the result follows immediately from 1.7.2.

(4): T_{σ_i} is closed under extensions. Assume that we have an exact sequence in $\underline{C}$.

$$o \longrightarrow N' \longrightarrow N \xrightarrow{\pi} N'' \longrightarrow o$$

with N' and N'' in T_{σ_i}. For every $G_j \leq G_i$ and $\varphi \in \operatorname{Hom}_{\underline{C}} (G_j, N)$, put $I = \operatorname{Ker} (\pi \circ \varphi)$, then $I \in L(G_j, \sigma)$, then we have to show that for each $\psi \in \operatorname{Hom}_{\underline{C}} (G_k, I)$ with $G_k \leq G_j$, we have that $\psi^{-1}(J) \in L(G_k, \sigma)$, cfr. Prop. 1.7.4.
Consider the commutative diagram with exact top row:



Clearly, $\psi^{-1}(\mathrm{Ker}\ \varphi) = \psi^{-1}(J) = \mathrm{Ker}\ (\varphi \circ j \circ \psi)$, and because $\varphi \circ j \circ \psi$ can be viewed as an element of $\mathrm{Hom}_{\underline{C}}(G_k, N')$. Since $N' \in T_{\sigma_i}$ and $\{G_i, \leq\}$ is partially ordered, we have that $\psi^{-1}(J) \in L(G_k, \sigma)$, which finishes the proof.

(2.6.): The idempotent kernel functors σ_i associated with the torsion classes T_{σ_i} will be called <u>the local components of σ</u>. We will show that σ is completely determined by its local components.
Recall that $\sigma \leq \tau$ if and only if $T_\sigma \subset T_\tau$.

(2.7.): <u>Proposition</u>:
1. If $G_j \leq G_i$, then $\sigma_i \leq \sigma_j$
2. $\sigma = \wedge\ \sigma_i$

<u>Proof</u>
1. Follows immediatly from the definitions and the fact that $\{G_i, \leq\}$ is partially ordered.
2. Is nothing but a reformulation of Prop. 1.8.

(2.8.): <u>Corollary</u> (local-global lemma)
If $M, N \in \underline{C}(\sigma)$, then a morphism $\varphi : M \to N$ is an isomorphism if and only if $Q_{\sigma_i}(\psi) : Q_{\sigma_i}(M) \to Q_{\sigma_i}(N)$ is an isomorphism for all $i \in I$.

Proof

Localizing $o \to \text{Ker } \varphi \to M \to \text{Im } \varphi \to o$ at σ_i implies Q_{σ_i} (Ker φ) = o, or equivalently, Ker φ is σ_i-torsion for all $i \in I$, hence σ-torsion. But as M is σ-torsion free, Ker φ = o.

Now, localizing $o \to M \to N \to \text{Coker } \varphi \to o$ gives for all $i \in I$:

$o \to Q_{\sigma_i}(M) \to Q_{\sigma_i}(N) \to Q_{\sigma_i}(\text{Coker } \varphi)$.

Thus, Coker φ is σ_i-torsion for all $i \in I$, hence σ-torsion. But, as the quotient of a σ-torsion free object modulo a σ-closed object has to be σ-torsion free, Coker φ = o.

(2.9.): Combining the results of section 1 and section 2 we get the following

Proposition: If $\underline{C}$ is a Grothendieck category with a cohesive family of generators $\{G_i;\ i \in I\}$ then there is a one-to-one correspondence between

1. hereditary torsion theories in $\underline{C}$
2. idempotent kernel functors in $\underline{C}$
3. families of classes $\{L_i;\ i \in I\}$ satisfying:
   (a) L_i consists of subobjects of G_i for all $i \in I$
   (b) If $J, K \in L_i$, then $J \cap K \in L_i$
   (c) If $J \in L_i$, $J < K < G_i$, then $K \in L_i$
   (d) If $J < G_i$ and $K \in L_i$ such that for every $G_j \leq G_i$ and every morphism $\varphi : G_j \to K$, we have $\varphi^{-1}(J) \in L_j$, then $J \in L_i$
   (e) If $J \in L(G_i, \sigma)$, $G_j \leq G_i$ and $\varphi \in \text{Hom}_{\underline{C}}(G_j, G_i)$, then $\varphi^{-1}(J) \in L(G_j, \sigma)$.

(2.10.): The foregoing proposition shows why it is inadequate to restrict attention to one generator. In order to get a one-to-one correspondence between idempotent kernel functors and filters one has to impose projectivity and smallness on the generator G, implying that $\underline{C}$ is a module category, for, $\mathrm{Hom}_{\underline{C}}$ (G, -) is exact and commutes with direct sums.

3. SOME LOCAL-GLOBAL RESULTS

(3.1.): Let σ be an idempotent kernel functor in $\underline{C}$, $C \in Ob(\underline{C})$. The "filter" $L(C, \sigma) = \{C' < C : \sigma(C / C') = C / C'\}$ is said to be <u>σ-Noetherian</u> if it has the following property:
if $C_1 < C_2 < \dots < C_n < \dots$ is an ascending chain of subobjects of C such that $\cup C_n \in L(C, \sigma)$, then there exists a natural number k for which $C_k \in L(C, \sigma)$.

(3.2.): <u>Proposition</u>: Let σ be an idempotent kernel functor in a Grothendieck category $\underline{C}$ with a family of generators $\{G_i;\ i \in I\}$.
Consider the following statements:
1. $L(G_i, \sigma)$ is σ-Noetherian for all $i \in I$
2. $\underline{C}(\sigma)$ is closed under taking direct sums
3. $Q_\sigma(-)$ commutes with direct sums.

If the G_i's are small objects, then the following implications hold:
(1) ⇒ (2) ⇒ (3).
If the G_i's are finitely generated, then the three statements are equivalent.

<u>Proof</u>
(1) ⇒ (2): Let $\{M_\alpha : \alpha \in \Lambda\}$ be a family of σ-closed objects. Let us denote: $M = \oplus M_\alpha$, $M' = \Pi M_\alpha$ and $\pi_\alpha : M' \to M_\alpha$ the canonical projection. Both M and M' are clearly σ-torsion free, so we have to check σ-injectivity of M.
In view of Prop. 1.10, it suffices to complete every diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & J & \xrightarrow{\ i_J\ } & G_i \\ & & \Big\downarrow f & & \\ & & M & & \end{array}$$

where $J \in L(G_i, \sigma)$. Since each M_α is σ-closed, there exist morphisms $\beta_\alpha : G_i \to M_\alpha$ extending $\pi_\alpha \circ f$. These maps define a morphism $\beta = \pi_{\beta_\alpha} : G_i \to M'$. If we can prove that $\Sigma = \{\alpha \in \Lambda : \beta_\alpha \neq o\}$ is finite then β factorizes through M and the proof is complete. Now, suppose Σ is infinite, then we can find a countable infinite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\}$ of Σ. For each positive integer j we can define a subset J_j of J:

$J_j = \mathrm{Ker}\,(\underset{k \geq j}{\pi} \beta_\alpha \circ i_J)$. Then, $J_1 < J_2 < \ldots$ is an ascending chain of subobjects of G_i with $\cup J_n = J$, for suppose $\cup J_n \overset{\neq}{\hookrightarrow} J$, then we can find an index $j \in I$ and a morphism γ

$$\begin{array}{ccccc} \cup J_n & \overset{\neq}{\hookrightarrow} & J & \overset{f}{\longrightarrow} & \oplus M_\alpha = M \\ & & \uparrow \gamma & & \\ & & G_j & & \end{array}$$

which does not factorize through $\cup J_n$. Now, $f \circ \gamma \in \mathrm{Hom}_{\underline{C}}(G_j, M) \simeq \underset{\alpha}{\oplus} \mathrm{Hom}_{\underline{C}}(G_j, M)$ because G_j is small, hence there are only finitely many α_k such that $\pi_{\alpha_k} \circ f \circ \gamma \neq o$, therefore γ factorizes through $\cup J_n$, a contradiction.

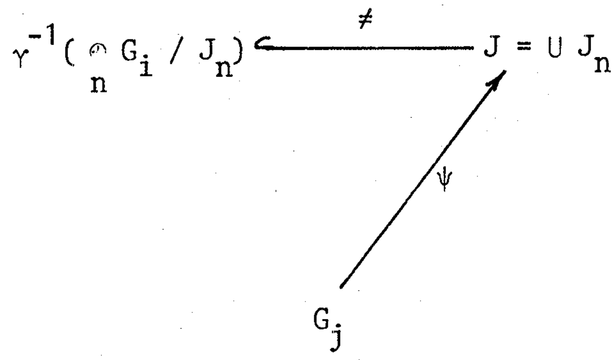
Using σ-Noetherianness of $L(G_i, \sigma)$, we can find a natural number $\ell : J_\ell \in L(G_i, \sigma)$. If $k \geq \ell$ then $\beta_{i_k}(J_\ell) = o$ and so β_{i_k} induces a mapping $G_i / J_\ell \to M'$. Now G_i / J_ℓ is σ-torsion and M' is σ-torsion free, thus this morphism must be the zero map. Therefore $\beta_{i_k} = o$ for all $k \geq \ell$, contradicting the initial assumption that Σ is infinite.

(2) $\Rightarrow$ (3): exactly as in the module case, no assumptions on the generators are necessary.

(3) $\Rightarrow$ (1): assume that every G_i is finitely generated. Let $J_1 < J_2 < \ldots$ be an ascending chain of subobjects of G_i with $J = \cup J_n$.

Define γ_n to be the composed morphism $\gamma_n : J \to J / J_n \to G_i / J_n$ and $\gamma = \Pi \gamma_n : J \to \prod_n G_i / J_n$.

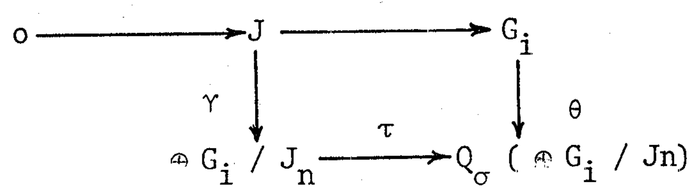
Suppose γ does not factorize through $\underset{n}{\oplus} G_i / J_n$, then we can find an index $j \in I$ and a morphism ψ:



which does not factorize through $\gamma^{-1}(\underset{n}{\oplus} G_i / J_n)$. Because G_j is finitely generated, $\psi \in \text{Hom}\,(G_j, \cup J_n) \simeq \underset{n}{\cup} \text{Hom}\,(G_j, J_n)$, hence, $\text{Im}\,\psi \subset J_m$ for some m and therefore ψ factorizes through

$\gamma^{-1}\,(\overset{m}{\underset{k=1}{\oplus}} G_i / J_k)$ a contradiction.

Since $J \in (G_i, \sigma)$, there is a morphism θ making the diagram



commute. By (3), $Q_\sigma\,(\underset{n}{\oplus} G_i\,(J_n) = \underset{n}{\oplus} Q_\sigma\,(G_i / J_n)$ and because G_i is a small object there exists an integer k such that θ factorizes through

$\overset{k}{\underset{j=1}{\oplus}} Q_\sigma\,(G_i / J_j)$.

Now, pick $h > k$. Then $\nu_k \circ \eta \circ \gamma$, where ν_h is the canonical projection

$\underset{n}{\oplus} Q_\sigma (G_i / J_n) \to Q_\sigma (G_i / J_h)$, is the zero map, whence

$\mathrm{Im}\, \gamma_h = J / J_h \subset \sigma(R / J_h)$ which implies that J / J_h is σ-torsion.

Finally, the exactness of the sequence:

$$0 \longrightarrow J / J_h \longrightarrow R / J_h \longrightarrow R / J \longrightarrow 0$$

implies $J_h \in L(G_i, \sigma)$, finishing the proof.

(3.3.): As in the module case, σ-Noetherianness of $L(G_i, \sigma)$ does not imply that $L(G_i, \sigma)$ satisfies the ascending chain condition. If we impose the ACC on every local filter we get a global result:

(3.4.): <u>Proposition</u>: If σ is an idempotent kernel functor in a Grothendieck category $\underline{C}$ with a family of f.g. generators $\{G_i; i \in I\}$, then the following conditions are equivalent:

1. $L(G_i, \sigma)$ satisfies the ascending chain condition for all $i \in I$
2. The class of σ-torsion σ-injective objects is closed under taking direct sums.

<u>Proof</u>

(1) $\Rightarrow$ (2): Let $\{M_\alpha : \alpha \in \Lambda\}$ be a family of σ-torsion, σ-injective objects and let $M = \underset{\alpha}{\oplus} M_\alpha$. Clearly, M is σ-torsion. In view of Prop. 1.10. we have to complete every diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \longrightarrow & G_i \\ & & \Big\downarrow{\scriptstyle\varphi} & & \\ & & M & & \end{array}$$

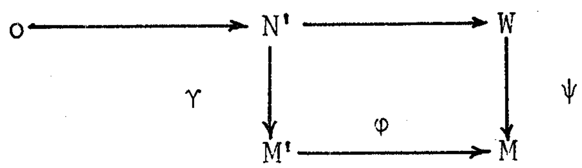
with $I \in L(G_i, \sigma)$. M is σ-torsion, hence so is $\operatorname{Im} \varphi \simeq I / \operatorname{Ker} \varphi$.
Exactness of the sequence $o \to I / \operatorname{Ker} \varphi \to R / \operatorname{Ker} \varphi \to R / I \to o$
implies that $\operatorname{Ker} \varphi \in L(G_i, \sigma)$.
For every $\alpha \in \Lambda$, $\pi_\alpha : M \to M_\alpha$ will be the canonical projection.
We claim that the set $\Sigma = \{\alpha \in \Lambda : \pi_\alpha \circ \varphi \neq o\}$ is finite.
Suppose that this is not so, then we can pick a countable infinite set
$\Sigma^\star = \{\alpha_1, \alpha_2, \dots\} \cup \Lambda$ with $\pi_{\alpha_n} \circ \varphi \neq o$ for all $n \in \mathbb{N}$.
Let $\Lambda_n = (\Lambda \setminus \Sigma^\star) \cup \{\alpha_1, \dots, \alpha_n\}$.
Define $J_n = \varphi^{-1} (\oplus \{M_\alpha; \alpha \in \Lambda_n\})$, then $J_1 < J_2 < \dots$ is a strictly increasing countable infinite chain of subobjects of G_i contained in $L(G_i, \sigma)$, since $\operatorname{Ker} \varphi \subset J_n$ for all $n \in \mathbb{N}$. Therefore φ factorizes through $\oplus \{M_\beta; \beta \in \Omega'\}$ with Ω' a finite subset of Ω. $\oplus \{M_\beta; \beta \in \Omega'\}$ is clearly σ-injective and thus φ can be extended to a morphism $\gamma : G_i \to M$.
(Remark that one does not have to impose any condition on the generators for this part of the proof).
(2) $\Rightarrow$ (1): For every object A of $\underline{C}$, define $E_\sigma(A) = \varphi^{-1}(\sigma(E(A) / A))$, where $E(A)$ is the injective hull of A and φ the canonical projection $E(A) \to E(A) / A$. Exactly as in the module case, one can show that $E_\sigma(A)$ is σ-injective and $E_\sigma(A) / A$ is σ-torsion.
Let $J \in L(G_i, \sigma)$, from the exact sequence:

$$o \longrightarrow G_i / J \longrightarrow E_\sigma(G_i / J) \longrightarrow E_\sigma(G_i / J) / (G_i / J) \longrightarrow o$$

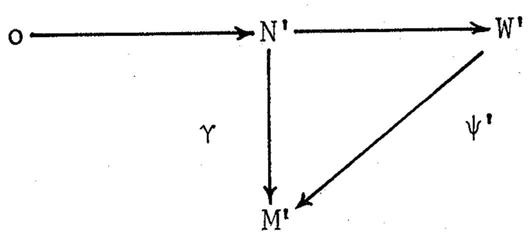
it follows that $E_\sigma(G_i / J)$ is σ-closed. Now, let $J_1 < J_2 < \dots$ be an ascending chain of subobjects of G_i contained in $L(G_i, \sigma)$ and put $J = \cup J_j$. By an argument similar to the one given in the proof of Prop. 3.2., one can factorize the natural morphism $\psi : J \to \prod_j E_\sigma(G_i / J_j)$ through $M = \oplus_j E_\sigma(G_i / J_j)$. By (2), M is σ-injective, hence ψ extends to a morphism $\gamma : G_i \to M$. Finally, using smallness of G_i, γ factors through a finite subsum, say, $\bigoplus_{j=1}^{m} E_\sigma (G_i / J_j)$, but this implies $J = J_m$ which completes the proof.

(3.5.): A morphism $\varphi : M' \to M$ is said to be <u>σ-neat</u> if and only if the following conditions are equivalent for every subobject N' of an object N such that $(N / N') = N / N'$ and for every morphism $\gamma : N' \to M'$.

1. There exists a subobject W of N properly containing N' and a morphism $\psi : W \to M$ making the following diagram commute:



2. There exists a submodule W' of N properly containing N' and a morphism $\psi' : W' \to M'$ making the following diagram commute:



(3.6.) <u>Proposition</u>: A morphism $\varphi : M' \to M$ is σ-neat if and only if the conditions above are equivalent for $N = G_i$ and $N' = J \in L(G_i, \sigma)$, for all $i \in I$.

<u>Proof</u>

Let $N' < N$ such that N / N' is σ-torsion and $\gamma \in \mathrm{Hom}_{\underline{C}}(N', M')$. Suppose there exists a subobject W of N properly containing N' and a morphism $\psi : W \to M$ making the diagram below commutative:

$$
\begin{array}{ccccc}
0 & \longrightarrow & N' & \xrightarrow{\neq} & W \\
 & & \gamma \big\downarrow & & \big\downarrow \psi \\
 & & M' & \xrightarrow{\varphi} & M
\end{array}
$$

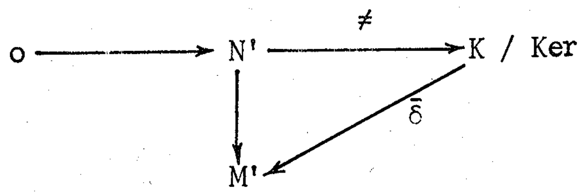
We can find an index $i \in I$ and a morphism $\beta : G_i \to W$ which does not factorize through N'. Now, let $J = \beta^{-1}(N')$, then we have the exact diagram:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \big\downarrow & & \\
0 & \longrightarrow & J & \longrightarrow & G_i & \longrightarrow & G_i / J & \longrightarrow & 0 \\
 & & \beta \big\downarrow & & \beta \big\downarrow & & \beta' \big\downarrow & & \\
0 & \longrightarrow & N' & \longrightarrow & W & \longrightarrow & W / N' & \longrightarrow & 0
\end{array}
$$

when β' is the induced morphism, which is easily seen to be injective. Since W / N' is σ-torsion, $J \in L(G_i, \sigma)$. This gives us a subobject K of G_i properly containing J and a morphism δ such that

$$
\begin{array}{ccccc}
0 & \longrightarrow & J & \longrightarrow & K \\
 & & \gamma \circ \beta \big| & \overset{\delta}{\swarrow} & \big\downarrow \psi \circ \beta \\
 & & M' & \xrightarrow{\varphi} & M
\end{array}
$$

is commutative. Because Ker $\gamma \subset$ Kerδ , δ factorizes through K / Ker $\gamma \hookrightarrow$ Im γ. Finally we have the diagram



finishing the proof.

(3.7.): In the following proposition we will characterize in a global manner those idempotent kernel functors σ for which every $L(G_i, \sigma)$ is σ-Noetherian and satisfies the ascending chain condition.

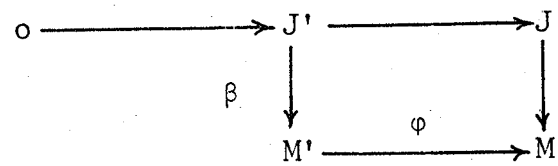
(3.8.): Proposition: Let σ be an idempotent kernel functor in a Grothendieck category with a family of generators $\{G_i; i \in I\}$. Consider the following statements:

1. $L(G_i, \sigma)$ is σ-Noetherian and satisfies the ascending chain condition for all $i \in I$.
2. Any direct sum of σ-neat morphisms is σ-neat.
3. Any direct sum of σ-injective objects is σ-injective.

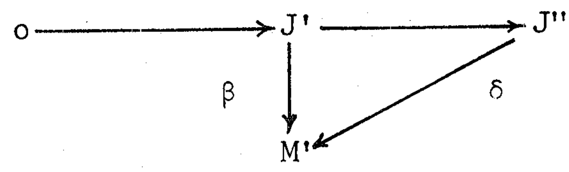
Without restrictions on the set of generators, the following implications hold: (1) $\Rightarrow$ (2) $\Rightarrow$ (3). If every G_i is finitely generated, the three statements are equivalent.

Proof

(1) $\Rightarrow$ (2): Let $\{\varphi_\alpha : M'_\alpha \to M_\alpha; \alpha \in \Lambda\}$ be a family of σ-neat morphisms; $M' = \oplus M'_\alpha$, $M = \oplus M_\alpha$ and $\varphi = \oplus \varphi_\alpha : M' \to M$. In view of Prop. 3.6. it suffices to check that every commutative diagram of the form:



with J', $J \in \mathcal{L}(G_i, \sigma)$; implies the existence of a commutative diagram:



with $J'' < J$. First, we claim that β factorizes through $\oplus \{M_\alpha;\ \alpha \in \Sigma\}$ with Σ a finite subset of Λ. Let $\pi_\alpha : M' \to M'_\alpha$ be the canonical projection and set $\Sigma = \{\alpha \in \Sigma : \pi_\alpha \circ \beta \neq 0\}$.
Suppose that Σ is infinite, then Σ countains a countably infinite subset $\Sigma' = \{\alpha_1, \alpha_2, \ldots\}$. For each $n \in \mathbb{N}$, $\Sigma_n = (\Sigma \setminus \Sigma') \cup \{\alpha_1, \ldots, \alpha_n\}$, and define $J_n = \alpha^{-1}\ (\oplus \{M_\alpha;\ \alpha \in \Sigma_n\})$.
Then, $\cup J_n = J \in \mathcal{L}(G_i, \sigma)$, and using σ-Noetherianness we have $J_k \in \mathcal{L}(G_i, \sigma)$ for some $k \in \mathbb{N}$. Finally, by the ascending chain condition of $\mathcal{L}(G_i, \sigma)$ there exists a $j \geq k$ such that $J_j = J_{j+1} = \ldots$ and this contradicts the fact that $J_1 \subset J_2 \subset \ldots$ is strictly ascending.
Thus, we are left to prove that the direct sum of a finite number of σ-neat morphisms is σ-neat. This is easy and can be proved as in the module case.
(2) $\Rightarrow$ (3): This follows simply by noting that an object E is σ-injective if and only if the zero map $E \to o$ is σ-neat.
(3) $\Rightarrow$ (1): This follows from Prop. 3.2. and Prop. 3.4. and the fact that a finitely generated object is small.

(3.9.): These propositions indicate why locally Noetherian categories behave well with respect to localization.

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