DEP. WISKUNDE
ARCHIEF

UNIVERSAL BIALGEBRAS ASSOCIATED WITH ORDERS.

L. LE BRUYN(1)

University of Antwerp, U.I.A. Belgium.

February 1981. 81-03

0. Introduction.

In [9], M.E. Sweedler associated to every algebra A over a field K a universal measuring bialgebra $M_K(A,A)$ and its maximal cocommutative pointed subbialgebra $H_K(A,A)$. These objects may be used in several domains, e.g. to obtain a beautiful Galois theory, cfr. [8].
Over arbitrary commutative rings, these constructions cannot be generalized and one has to restrict attention to Galois objects, as introduced in [4], in order to get a more or less satisfactory Galois theory. However, the condition of being a Galois object, puts severe restrictions on the ringextension. A lot of "nice" extensions, e.g. the integral

(1) The author is supported by an NFWO/FNRS-grant (Belgium).

closure of a Dedekind ring in a finite Galois extension of its field of fractions, do not necessarely fit into this Galois-object framework, as an example due to Janutz [6] shows.
Therefore, it would be interesting to extend Sweedler's construction to a nice class of rings, e.g. Dedekind domains. And with this note we put the first steps in this direction.
In the first two sections we associate bialgebras to orders, in the sence of I. Reiner [7]. These constructions are similar to the ones in [9], modulo some technical difficulties, mainly stemming from projectivity conditions.
In section 3, these bialgebras are applied to yield a Galois theory for Dedekind rings which is, as one would expect, closely related to the Galois theory of the corresponding fields of fractions.
In the final section, we have a brief look at the bialgebras associated to orders in a central simple algebra Σ. It is proved that for almost all prime ideals these bialgebras are orders in the Hopf-algebra $H_K(\Sigma,\Sigma)$.
However, these results are far from being complete. For instance, one is tempted to conjecture that these bialgebras contain some information about the corresponding noncommutative curves, in the sense of [1], [10].
Further, it might be possible to replace these bialgebras by automorphism schemes, as in L. Béqueri [2].
At this time, however, these claims are rather speculative and the author aims to return to them in a subsequent paper.

1. Construction of $M_D(A,B)$.

In this section we will associate to a pair of finitely generated projective D-algebras A and B, a universal measuring D-coalgebra $M_D(A,B)$. Our construction runs along the lines of M.E. Sweedler [9] modulo some technical difficulties.

The main problem in generalizing Sweedler's construction to the ring case is to find a suitable substitute for $A°$, (cfr. definition below). However, when D is a Dedekind-ring, this problem may be succesfully solved.

Definition 1.1 : Let A be any D-algebra,

$A° = \{g \in A^\star : \text{Ker } g \text{ contains an ideal } I;\ A/I \text{ is f.g. and torsion free}\}$

Remark 1.2 : $A°$ is a D-submodule of $A^\star = \text{Hom}_D(A,D)$. Clearly $A°$ is closed under scalar-multiplication. The sum of any two elements of $A°$ is again in $A°$ since $A/I \cap J \hookrightarrow A/I \oplus A/J$ and therefore $A/I\cap J$ is again f.g. and torsion free.

Proposition 1.3 : Let A, B be D-algebras and $f \in \text{Alg}_D(A,B)$.

(a) The dual map of f, $f^\star : B^\star \to A^\star$ sends $B°$ in $A°$

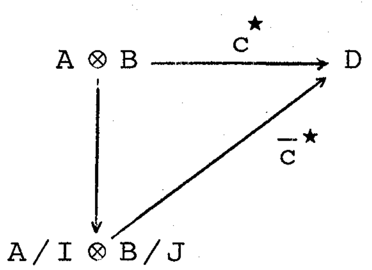
(b) The map $A^\star \otimes B^\star \hookrightarrow (A \otimes B)^\star$ restricts to $A° \otimes B° \cong (A \otimes B)°$

(c) If $M : A \otimes A \to A$ is the multiplication, then $M^\star(A°) \subset A° \otimes A°$

proof.

(a) It is easy to show that if $b^\star \in B°$ has $J \subset \text{Ker } b^\star$, then $\text{Ker } f^\star(b) \supset f^{-1}(J)$. Further, $A/f^{-1}(J) \hookrightarrow B/J$ and therefore it is f.g. and torsion free.

(b) Let K be any ideal in $A \otimes B$ with $A \otimes B/K$ is f.g. and torsion free. Let $I = \{a \in A : a \otimes 1 \in K\}$ then A/I is f.g. and torsion free, because (see part a) it is the inverse image of K under the algebra map $a \mapsto a \otimes 1$ of A to $A \otimes B$. Similarly, if $J = \{b \in B : 1 \otimes b \in K\}$ then B/J is f.g. and torsion free. Note that $A \otimes J + I \otimes B \subset K$ and by [3, A. II. 59, Prop.6] $A \otimes B / A \otimes J + I \otimes B \cong A/I \otimes B/J$ so $A \otimes B / I \otimes B + A \otimes J$ is again f.g. and torsion free (if A', B' are f.g, torsion free over a Dedekind ring D, $A' \cong I_1 \oplus \dots \oplus I_n$; $B' \cong J_1 \oplus \dots \oplus J_m$ with I_i, J_j fractional ideals, so $A' \otimes B' \cong \underset{i,j}{\oplus} (I_i \otimes J_j) \cong \oplus I_i J_j$ and is thus f.g. and torsion free).
Now suppose $c^\star \in (A \otimes B)^\circ$ with $K \subset \text{Ker } c^\star$, I and J as above. Then $c^\star$ factors throught $A/I \otimes B/J$. That is, there exists a unique $\bar{c}^\star$ such that the diagram below is commutative :



(1)

Thus, $\bar{c}^\star \in (A/I \otimes B/J)^\star \cong (A/I)^\star \otimes (B/J)^\star$ (D is Dedekind ring and [3, A. II.80, Coroll.1].
Via this isomorphism, write $\bar{c}^\star = \Sigma\, \bar{d}_i^\star \otimes \bar{e}_i^\star$ with $\bar{d}_i^\star \in (A/I)^\star$, $\bar{e}_i^\star \in (B/J)^\star$.
In particular, for $a \in A/I$, $b \in B/J$ we have :
$< \bar{c}^\star, a \otimes b > = \sum_i < \bar{d}_i^\star, a > < \bar{e}_i^\star, b >$.
Now, if π_1, π_2 are the natural projections $A \to A/I$ and $B \to B/J$ the commutativity of (1) comes down to :
$< c^\star, a \otimes b > = < \bar{c}^\star, \pi_1(a) \otimes \pi_2(b) > = \sum_i < \bar{d}_i^\star, \pi_1(a) > < \bar{e}_i^\star, \pi_2(b) >$. (2)
Let $d_i^\star = \bar{d}_i^\star \circ \pi_1$, then $d_i^\star \in A^\circ$ because $I = \text{Ker } \pi_1 \subset \text{Ker } d_i^\star$.

Similarly, $e_i^\star = \bar{e}_i^\star \circ \pi_2 \in B°$. (2) then becomes : $c^\star = \Sigma\, d_i^\star \otimes e_i^\star$, thus $(A \otimes B)° \subset A° \otimes B°$.

Conversely, if $d^\star \in A°$ (resp. $e^\star \in B°$) with $I \subset \operatorname{Ker} d^\star$: A/I is f.g. and torsion free (resp. $J \subset \operatorname{Ker} e^\star$: B/J is f.g. and torsion free) then $A \otimes J + I \otimes B \subset \operatorname{Ker}(d^\star \otimes e^\star)$ and $A \otimes B / A \otimes J + I \otimes B$ is f.g. and torsion free. So $A° \otimes B° \subset (A \otimes B)°$.

(c) For $a^\star \in A^\star$; $a, b \in A$: $< M^\star(a^\star), a \otimes b > = < a^\star, ab >$. If $I \subset \operatorname{Ker} a^\star$ with A/I f.g. and torsion free, then $A \otimes I + I \otimes A \subset \operatorname{Ker} M^\star(a^\star)$ and $A \otimes A / A \otimes I + I \otimes A$ is f.g. torsion free. Thus, $M^\star(A°) \subset (A \otimes A)° = A° \otimes A°$.

Now, define $\Delta = M^\star|A° : A° \to A° \otimes A°$ and $\varepsilon : A° \to D$ by $\varepsilon(a^\star) = < a^\star, 1 >$.

Proposition 1.4 : $(A°, \Delta, \varepsilon)$ is a D-coalgebra.

proof.

Similar to, M.E. Sweedler [9].

If A, B are D-algebras and $f \in \mathrm{Alg}_D(A,B)$. Proposition 1.3.a. states that $f^\star|B°$ is a map from $B°$ to $A°$. Denote $f° = f^\star|B°$. A diagram chase shows that $f°$ is a coalgebra map. For any D-algebra A, $A^\star$ is a left A-module with scalar multiplication defined by $< b \rightharpoonup a^\star, a > = < a^\star, ab >$ for $a^\star \in A$, $a, b \in A$. The right action is defined by $< a^\star \leftharpoonup b, a > = < a^\star, ba >$. This makes $A^\star$ into an A-A-bimodule.

Proposition 1.5 : Let A be a D-algebra. For any $f \in A^\star$ the following are equivalent :

(a) $f \in A^\circ$

(b) $M^\star(f) \in A^\circ \otimes A^\circ$

(c) $M^\star(f) \in A^\star \otimes A^\star$

(d) $A \rightharpoonup f$ is f.g. and torsion free

(e) $f \leftharpoonup A$ is f.g. and torsion free

<u>proof</u>.

(a) $\Rightarrow$ (b) : since $M(A^\circ) \subset A^\circ \otimes A^\circ$ (Prop. 1.3.c)

(b) $\Rightarrow$ (c) : trivially

(c) $\Rightarrow$ (d) : Let $M^\star(f) = \sum_{i=1}^{n} a_i^\star \otimes b_i^\star$, where $a_i^\star, b_i^\star \in A^\star$. By the definition of $M^\star$ we have : $< f, ab > = \sum_{i=1}^{n} < a_i^\star, a > < b_i^\star, b >$.
Hence $b \rightharpoonup f = \sum_{i=1}^{n} a_i^\star < b_i^\star, b >$, thus $(A \rightharpoonup f) \subset Da_i^\star + \ldots + Da_n^\star$ and so it is finitely generated. Now suppose that $A \rightharpoonup f$ is not torsion free, hence for some $b \in A$, $d \in D$: $d\,(b \rightharpoonup f) \equiv 0$, thus for all a in A : $d < f, ab > = 0$. But this implies $< f, ab > = 0$ for all a, or, $b \rightharpoonup f = 0$.

(d) $\Rightarrow$ (a) : $M = (A \rightharpoonup f)$ is f.g. and torsion free. Then $I = \{a \in A : a \rightharpoonup M = 0\}$ is an ideal of A with A/I is f.g. and torsion free (because I is the kernel of the map $\pi : A \to End_D(M)$ given by $\pi(a)\,[m] = a \rightharpoonup m$. Hence, $A/I \hookrightarrow End_D(M)$ and thus A/I f.g. and torsion free).
But for any $a \in I : < f, a > = < a \rightharpoonup f, 1 > = < 0, 1 > = 0$.
So, $I \subset Ker\ f$, whence $f \in A^\circ$. This proves the equivalence (a) - (d).
Obviously, (e) $\Leftrightarrow$ (a) follows by left-right symmetry from (a) $\Leftrightarrow$ (d).

<u>Proposition 1.6</u> : If C is a D-coalgebra such that C is a projective D-module. Let $C^\star$ be the dual algebra. The natural map $C \to C^{\star\star}$ maps C to $C^{\star\circ}$.

proof.

Let $c^{\star} \in C^{\star}$, $c \in C$ and c' the image of c in $C^{\star\star}$. The definitions of $\rightharpoonup$ and of the multiplication in $C^{\star}$ imply :

$c^{\star} \rightharpoonup c' = \sum_{(i)} c'_{(1)} < c^{\star}, c'_{(2)} >$. Thus $C \rightharpoonup c'$ is f.g. and also torsion free, since $C^{\star\star}$ is torsion free.

The injection $A^{\circ} \hookrightarrow A^{\star}$ induces a map $A^{\star\star} \to A^{\circ\star}$. Define : $\pi : A \to A^{\circ\star}$ to be the composition map $A \to A^{\star\star} \to A^{\circ\star}$. Note that π is an algebra map.

Proposition 1.7 : Let A, C be projective D-modules, A a D-algebra and C a D-coalgebra, then there is a one to one correspondence between $Alg_D(A, C^{\star})$ and $Coalg_D(C, A^{\circ})$.

proof.

Given $f \in Alg_D(A,C^{\star})$, let $\psi(f) \in Coalg_D(C,A^{\circ})$ be the composite :

$\psi(f) : C \to C^{\star\circ} \overset{f^{\circ}}{\to} A^{\circ}$.

If $g \in Coalg_D(C,A^{\circ})$, let $\Phi(g) \in Alg_D(A,C^{\star})$ be the composite :

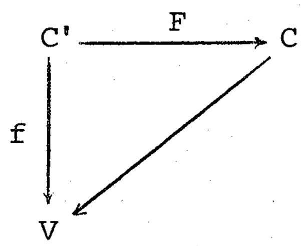
$\Phi(g) : A \overset{\pi}{\to} A^{\circ\star} \overset{g^{\star}}{\to} C^{\star}$.

It is easily verified that $\Phi(\psi(f)) = f$ and $\Phi(\psi(g)) = g$, since

$\psi(f)(c) : A \to D \quad a \mapsto < f(a), c >$; and $\Phi(g)(a) : C \to D \quad c \mapsto < g(c), a >$.

In the above proposition we showed that $(-)^{\circ}$ has the required properties to complete Sweedler's construction, this time for finitely generated projective D-algebras.

Definition 1.8 : If V is a D-module, a pair (C, π) where C is a D-coalgebra and $\pi : C \to V$ a D-module morphism, is called a cofree coalgebra on V if for any projective D-coalgebra C' and D-module morphism $f : C' \to V$ there is a unique coalgebra map F making the following diagram commutative :



If it exists, C is clearly unique up to D-coalgebra isomorphism.

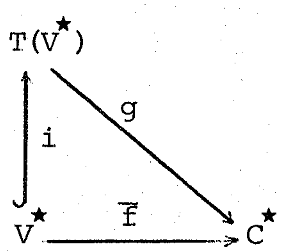
<u>Theorem 1.9</u> : If V is a f.g. projective D-module, then the cofree coalgebra on V exists.

<u>proof</u>.

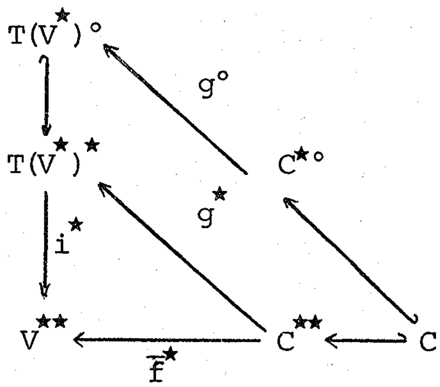
Let $T(V^*)$ be the tensor algebra on V^*, which is a projective D-module since V^* is f.g. and projective. Let $i : V^* \to T(V^*)$ be the natural injection. Let π be the composite

$$\pi : T(V^*)^\circ \to T(V^*)^* \overset{i^*}{\to} V^{**}$$

We claim that $(T(V^*)^\circ, \pi)$ is the cofree coalgebra on $V^{**} \cong V$. Let C be a projective D-coalgebra and $f : C \to V^{**}$ a D-module morphism. Denote by $\overline{f}$ the composite $V^* \to V^{***} \overset{f^*}{\to} C^*$. Because of the universal mapping property for $T(V^*)$, there is a unique algebra map g such that the following diagram is commutative :



Dualizing this diagram we obtain :



The vertical composite is nothing but $\pi : T(V^*)^\circ \to V^{**}$, the top diagonal composite is the unique coalgebra map $F : C \to T(V^*)^\circ$ corresponding to $g : T(V^*) \to C^*$ (see Prop. 1.7).

The bottom horizontal composite is $f : C \to V^{**}$ since there is a one to one correspondence between $\mathrm{Hom}_D(C,V^{**})$ and $\mathrm{Hom}_D(V^*,C^*)$ given by

$$\psi : \mathrm{Hom}_D(C,V^{**}) \rightleftarrows \mathrm{Hom}_D(V^*,C^*) : \Phi$$

$$f \longmapsto (V^* \to V^{***} \overset{f^*}{\to} C^*)$$

$$(C \to C^{**} \overset{g^*}{} V^{**}) \longmapsfrom g$$

and the horizontal composite is $\Phi(\psi(f)) = f$. Thus, $(T(V^*)^\circ, \pi)$ is the cofree coalgebra on $V^{**} \cong V$.

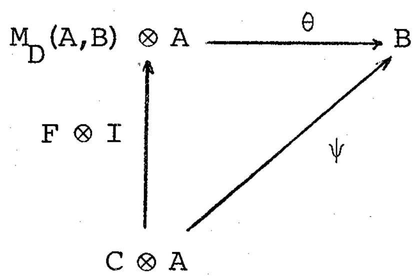
Let us recall the definition of "measuring". Let A, B be D-algebras, M a D-coalgebra and $\psi : M \otimes A \to B$ a D-module morphism. M is said to measure A to B if ψ satisfies :

(1) $\psi\ (m \otimes a a') = \sum_{(m)} \psi\ (m_{(1)} \otimes a)\quad (m_{(2)} \otimes a')$

(2) $\psi\ (m \otimes 1) = \varepsilon(m)\ 1_B$

For all $a, a' \in A$; $m \in M$.

<u>Theorem 1.10</u> : Let A, B be finitely generated projective D-algebras. There is a D-coalgebra $M = M_D(A,B)$ and a D-module morphism $\theta : M \otimes A \to B$ measuring A to B and with the following universal property : If C is a projective D-coalgebra and (C,ψ) measures A to B then there is a unique coalgebra map $F : C \to M$ such that the following diagram is commutative :



<u>proof</u>.

As in M.E. Sweedler [9], using the foregoing results.

<u>Remark 1.11</u> : If A is a f.g. projective D-algebra, then $M_D(A,A)$ has a unique algebra structure such that it is a bialgebra and $\theta : M_D(A,A) \otimes A \to A$ makes A into an $M_D(A,A)$-module.

2. Bialgebras associated with D-orders.

From now on we will restrict attention to D-orders in the sense of I. Reiner [7], i.e. D is a Dedekind ring, K its field of fractions, A a K-algebra and Λ a subring of A such that Λ is a f.g. D-module and $K.\Lambda = A$.
Remark that Λ is a f.g. projective D-module since it is finitely generated and torsion free.
First, we want to investigate the connection between $M_D(\Lambda,\Lambda)$ (as defined in section 1) and $M_K(A,A)$ (as defined by Sweedler in [9]).

Proposition 2.1 : $M_D(\Lambda,\Lambda) \otimes_D K$ is a subbialgebra of $M_K(A,A)$.

proof.
$(M_D(\Lambda,\Lambda), \Delta, \varepsilon, \mu, m)$ is the D-bialgebra constructed in section 1. We will give $M_D(\Lambda,\Lambda) \otimes_D K$ a K-bialgebra structure in the following way :

$$\overline{\Delta} : M_D(\Lambda,\Lambda) \otimes K \to M_D(\Lambda,\Lambda) \otimes K \otimes_K M_D(\Lambda,\Lambda) \otimes K \cong M_D(\Lambda,\Lambda) \otimes M_D(\Lambda,\Lambda) \otimes K$$
$$m \otimes k \mapsto \Delta m \otimes k$$

$$\overline{\varepsilon} : M_D(\Lambda,\Lambda) \otimes K \to K$$
$$m \otimes k \mapsto \varepsilon(m)\, k$$

$\overline{\mu}$, $\overline{m}$ as usual. It is easily verified that these maps are well defined and that $(M_D(\Lambda,\Lambda) \otimes_D K, \overline{\Delta}, \overline{\varepsilon}, \overline{\mu}, \overline{m})$ is a K-bialgebra.
Further $\psi : M_D(\Lambda,\Lambda) \otimes \Lambda \to \Lambda$ is a D-measuring. Now, define
$\overline{\psi} : (M_D(\Lambda,\Lambda) \otimes_D K) \otimes_K A \to A$ by $\overline{\psi}\,(m \otimes k \otimes k'\lambda) = kk'\,(\psi\,(m \otimes \lambda))$.
$\overline{\psi}$ is well defined and a K-measuring. Applying the universal mapping property of $M_K(A,A)$ yields a unique K-coalgebra map F such that the following diagram is commutative :

$$\begin{array}{ccc} M_K(A,A) \otimes A & \xrightarrow{\varphi} & A \\ {\scriptstyle F \otimes I}\uparrow & \nearrow{\scriptstyle \overline{\psi}} & \\ (M_D(\Lambda,\Lambda) \otimes_D K) \otimes A & & \end{array}$$

It is easy to check that $F(M_D(\Lambda,\Lambda) \otimes_D K)$ is a subbialgebra of $M_K(A,A)$. □

From now on, we will identify $M_D(\Lambda,\Lambda)$ with its image in $M_K(A,A)$.

<u>Definition 2.2</u> : C is a torsion free D-coalgebra, then

C is called <u>irreducible</u> if any two non-zero subcoalgebras have non-zero intersection.

C is <u>simple</u> if it has no non-zero subcoalgebras.

C is <u>pointed</u> if all simple subcoalgebras of C are free D-modules of rank one.

<u>Lemma 2.3</u> : If H is a torsion free D-coalgebra and G(H) is the set of its group-like elements, then :

(1) D G(H) is a free D-module

(2) G(H) corresponds bijectively to the free subcoalgebras of rank one.

<u>proof</u>.

(1) Suppose D G(H) is not free, hence there are $g_1,\ldots,g_n \in G(H)$: $Dg_1+\ldots+Dg_n$ is not free. By induction on n we may suppose however that $Dg_1+\ldots+Dg_{n-1}$ is free. Thus $d_n\, g_n = \Sigma\, d_i\, g_i$ with $d_n \neq 0$, then : $\Delta\, d_n\, g_n = \Sigma\, d_i\, \Delta\, g_i = \Sigma\, d_i\, g_i \otimes g_i$, on the other hand, $\Delta\, d_n\, g_n = d_n\, g_n \otimes g_n$, hence $\Sigma\, d_n\, d_i\, g_i \otimes g_i = \Sigma\, d_i\, d_j\, g_i \otimes g_j$.

Thus $d_n\, d_i = d_i^2$, hence $d_n = d_i$ or $d_i = 0$ and $d_i\, d_j = 0$ of $i \neq j$ so there is just one $i : d_i \neq 0$. Thus $g_n = g_i$, done.

(2) Let H' be a free subcoalgebra of rank one, $H' = Dh$ and $\Delta h = d(h \otimes h)$. Take $h' = dh$, then $\Delta h' = h' \otimes h'$, hence $\varepsilon(h') = 1$ and this implies that d is invertible in D. Finally $Dh' = Dh = H'$.

Recall from [9] that $H_K(A,A)$ is the maximal cocommutative pointed subcoalgebra of $M_K(A,A)$.

Definition 2.4 : $H_D(\Lambda,\Lambda) = \{m \in M_D(\Lambda,\Lambda) : m \otimes 1 \in H_K(A,A)\}$.

Proposition 2.5 : If L is a cocommutative pointed D-subcoalgebra of $M_D(\Lambda,\Lambda)$, then $L \subset H_D(\Lambda,\Lambda)$.

proof.

Let I be a simple K-subcoalgebra of $L \otimes_D K$, and let $I' = \{i \in L : i \otimes 1 \in I\}$. Then, $0 \neq I'$ and I' is a D-subcoalgebra of L, hence, there is a simple D-subcoalgebra $J = Db \subset I'$.

Thus $J \otimes K = Kb \subset I$ and since I is simple, $Kb = I$, therefore every simple subcoalgebra of $L \otimes_D K$ is 1-dimensional, so $L \otimes K$ is a pointed cocommutative K-subcoalgebra of $M_K(A,A)$, hence $L \otimes_D K \subset H_K(A,A)$.

Therefore $L \subset H_D(\Lambda,\Lambda)$.

Proposition 2.6 : $H_D(\Lambda,\Lambda)$ is pointed.

proof.

Let L be a simple D-subcoalgebra of $H_D(\Lambda,\Lambda)$, $L \otimes_D K \subset H_K(A,A)$ a K-subcoalgebra. Since $H_K(A,A)$ is pointed, there is a

$g \in G\ (H_K(A,A))\ :\ Kg \subset L \otimes_D K$.

Let $L' = \{1 \in L\ :\ 1 \otimes 1 \in K\ g\}$, then L' is a nonzero D-subcoalgebra, hence $L = L'$.

If we are able to prove that $g \in L \otimes D_P$ for all P prime ideals of D, then $g \in \cap\ L \otimes\ D_P = L$ and then $Dg \subset L$ is a D-subcoalgebra, so $D g\ = L$, done.

Now, L is a f.g. projective D-module with basis, say $\alpha_1,\ldots,\alpha_n$.

$\alpha_i\ = k\,.\,g$ for some $k \in K$, thus, $\Delta\,\alpha_i = k\ g \otimes g = k^{-1}\ \alpha_i \otimes \alpha_i$. Since $L \otimes D_P$ is a D_P-coalgebra $\Delta\,\alpha_i\ \in \bar{L} = (L \otimes D_P) \otimes (L \otimes D_P)$ and $\bar{L}$ has D_P-basis $\alpha_i \otimes \alpha_j$. Thus, finally, $k^{-1} \in D_P$, so $g \in D_P\ \alpha_i \subset L \otimes D_P$.

Remark 2.8 :

(1) $H_D\ (\Lambda,\Lambda) \hookrightarrow End_D(\Lambda)$ so $H_D(\Lambda,\Lambda)$ is finitely generated and torsion free, hence it is a f.g. projective D-module.

(2) For all m in $M_K(A,A)$, there exists an element d in D; $d\ m\ :\ \Lambda \to \Lambda$, for, $\Lambda = D\,\lambda_1 + \ldots + D\,\lambda_n$ and $m(\lambda_i)\ = \Sigma\ k_{ij}\ \lambda_j$ with $k_{ij} \in K$, so for all i we can find $d_i \in D$ such that $d_i\,m\,(\lambda_i) \in \Lambda$. Finally put $d = \pi\ d_i$, then $d\ m\ :\ \Lambda \to \Lambda$.

Theorem 2.9 : If $m \in H^1_K(A,A)$ (i.e. the pointed irreducible component of $H_K(A,A)$ with group like element 1, cfr. [9]), then there exist a $d \in D$, and a D-coalgebra $C \subset H_K(A,A)$ which is a f.g. D-module with $d\ m \in C$ and C measures Λ to Λ.

proof.

First, let $m \in C^+_n(H^1_K(A,A))$ (for notation and properties the reader is referred to [9]).

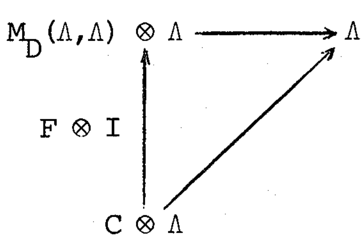
$n = 1$: Then $\Delta m\ = m \otimes 1 + 1 \otimes m$. We can find a d in D with $dm\ :\ \Lambda \to \Lambda$.

Take $C = D\,.\,1 + D\,.\,dm$. Then C measures Λ to Λ, $dm \in\ C$ and $(C,\ \Delta|_C,\ \varepsilon|_C)$ is a finitely generated D-coalgebra.

$n > 1$: Then $\Delta m = m \otimes 1 + 1 \otimes m + \sum_i n_i \otimes m_i$ with $n_i, m_i \in C^+_{n-1}(H^1_K(A,A))$.
By the induction hypothesis, we can find $d_i, d'_i \in D$ and C_i, C'_i f.g. D-subcoalgebras measuring Λ to Λ such that $d_i\, n_i \in C_i$; $d'_i\, m_i \in C'_i$.
Take $C' = \Sigma\, C_i + \Sigma\, C'_i$, then C is a f.g. D-subcoalgebra of $H_K(A,A)$ measuring Λ to Λ.
Further, there exists an element $d' \in D$ such that $d'm : \Lambda \to \Lambda$. Now, take $d = d' \pi d_i \pi d'_i$ and $C = C' + D\, d\, m$, then C satisfies the requirements of the theorem.
Let $m \in C_n(H^1_K(A,A)$, then $m - \varepsilon(m)\, 1_A \in C^+_n(H^1_K(A,A))$, so there is a $d \in D$ and C with $d(m - \varepsilon(m)\, 1_A)$ in C. Let $d'\ \varepsilon(m) \in D$, $d'' = dd'$, then $d''\, m \in C$.
Finally, $H^1_K(A,A) = \bigcup_n C_n(H^1_K(A,A))$ finishes the proof.

<u>Theorem 2.10</u> : In the situation of (2.9) we have :
$H^1_K(A,A) \subset H_D(\Lambda,\Lambda) \otimes_D K$.

<u>proof</u>.
Let $m \in H^1_K(A,A)$, then by the previous theorem there is an element d of D and a f.g. D-coalgebra $C \subset H_K(A,A)$ measuring Λ to Λ and $dm \in C$. By the universal mapping property of $M_D(\Lambda,\Lambda)$ there is a D-coalgebra map F such that the diagram below is commutative :



Hence, we can view dm as an element of $M_D(\Lambda,\Lambda)$ and since $dm \otimes 1 = dm \in H_K(A,A)$ we get that $dm \in H_D(\Lambda,\Lambda)$. Finally $dm \otimes \frac{1}{d} \in H_D(\Lambda,\Lambda) \otimes_D K$.

3. Some Galois Theory for Dedekind Rings.

In this section we will apply the foregoing in order to get a satisfactory Galois theory for Dedekind rings. Throughout we will consider the following situation. D is a Dedekind ring having K for its field of fractions, E another Dedekind ring with field of fractions L such that E is a f.g. D-module (hence E is the integral closure of D in L).
If (H, Δ, ε) is a D-coalgebra and $\varphi : H \otimes E \to E$ a D-measuring, then $(H \otimes K, \overline{\Delta}, \overline{\varepsilon})$ is a K-coalgebra and $\overline{\varphi} : (H \otimes K) \otimes_K L \to L$ a K-measuring, with :

$$\overline{\Delta} : H \otimes K \to H \otimes H \otimes K \quad : \quad h \otimes k \mapsto \Delta h \otimes k$$

$$\overline{\varepsilon} : H \otimes K \to K \quad : \quad h \otimes k \mapsto k\ \varepsilon(h)$$

$$\overline{\varphi} : (H \otimes K) \otimes_K L \to L \quad : \quad h \otimes k \otimes k'e \mapsto kk' . \varphi(h \otimes e)$$

It is easy to check that all these mappings are well defined.

<u>Definition 3.1</u> : Define the fixed elements of an algebra A under a coalgebra C which measures A to A to be the set
$A^C = \{a \in A \mid c . a = \varphi(c \otimes a) = \varepsilon(c)\ a ; \forall c \in C\}$.

<u>Proposition 3.2</u> : In the above situation, E^H is the integral closure of D in $L^{H \otimes_D K}$.

<u>proof</u>.
Let L' be the field of fractions of E^H. Then $L' \subset L^{H \otimes_D K}$, because, if $l' = d/d' \in L'$, then :
$\psi((h \otimes k) \otimes d/d') = k/d'\ \psi(h \otimes d) = k/d'\ \varepsilon(h)\ d = \varepsilon(h \otimes k)\ d/d'$.

Now, suppose that $L' \subsetneq L^{H \otimes K}$, and let D' be the integral closure of D in $L^{M \otimes K}$, we have $D \subset E$ and for every $d' \in D'$, $h \in H$: $\psi(h \otimes d') = \overline{\psi}(h \otimes 1 \otimes d') = \overline{\varepsilon}(h \otimes 1)\ d' = \varepsilon(h)\ d'$, thus $D' \subset E^H$, but this contradicts $L' \subsetneq L^{H \otimes K}$ and therefore $L' = L^{H \otimes K}$. Conversely, if $x \in L^{H \otimes K}$ and x is integral over D then $x \in E$ and for every $h \in H : \psi(h \otimes x) = \varepsilon(h)\ x$, so $x \in E^H$.

Proposition 3.3 : $H_D(E,E)$ is a D-order in $H_K(L,L)$.

proof.

In the foregoing section, we established $H_K^1(L,L) \subset H_D(E,E) \otimes K$.

Further, by a theorem of Konstant we have :

$H_K(L,L) = KG \# H_K^1(L,L)$, where G is the set of group-like elements of $H_K(L,L)$ and .#. denotes the smashed product, cfr. [9]. So, it remains to prove that $G \subset H_D(E,E)$.

If $g \in G$, then g is a K-automorphism of L. If $e \in E$, then there exist $d_0,\ldots,d_n \in D$ such that :

$d_n\ e^n + \ldots + d_1\ e + d_0 = 0$, hence, $d_n\ g(e)^n + \ldots + d_1\ g(e) + d_0 = 0$.

Since E is integrally closed in L, we have $g(e) \in E$. Thus, DG is a cocommutative pointed measuring bialgebra and by the universal property of $H_D(E,E)$; $DG \hookrightarrow H_D(E,E)$. Finally, because DG is pointed, $DG \hookrightarrow H_D(G,G)$.

Now, let $H \subset H_D(E,E)$ then by definition : $H \otimes K \subset H_K(L,L)$ and by Konstants theorem :

$$H \otimes_D K \cong KG\,(H \otimes_D K)\ \#\ H_K^1(H \otimes_D K).$$

Now, we are able, as in the foregoing section, to proof that $G(H) = G(H \otimes K)$.

Put : $H^1 = \{h \in H : h \otimes 1 \in H^1(H \otimes K)\}$.

Clearly, H^1 is a D-subcoalgebra of H and $H^1 \otimes_D K = H^1(H \otimes_D K)$.

Definition 3.4 :

Let E be a ring extension of D such that E is a finitely generated D-module, E is called a Galois extension with Galois group G if there is a representation of G by D-automorphisms of E leaving D elementswise fixed.

E is called a purely inseparable extension if for every $x \in E$ there is a natural number p^e with $x^{p^e} \in D$, p the characteristie of D.

Remark : Our definition of a Galois extension is not the same as the one given in De Meyer-Ingraham [5]. The extension $\mathbb{Z}[\sqrt{2}]$ of $\mathbb{Z}$, e.g., is Galois in the sense of (3.4) but not in the sence of [5].

Theorem 3.5 : (Galois theorem for Dedekind rings)

Let D be a Dedekind ring of characteristic p, H a cocommutative bialgebra measuring a Dedekind extension E of D, $H \subset H_D(E,E)$, $G = G(H)$ and H^1 as above, then :

(a) E^{H^1} is Galois over E^H

(b) E^{DG} is purely inseparable over E^H

(c) E and $E^{H^1} \otimes_{E^H} E^{DG}$ have the same field of fractions L.

proof.

(a) By Prop. 3.2, E^{H^1} is the integral closure of D in $L^{H^1_K(H \otimes K)}$. Now, by [9, Prop. 10.2.3], $L^{H^1_K(H \otimes K)}$ is Galois over $L^{H \otimes K}$. Hence there are $L^{H \otimes K}$-automorphisms $\varphi_1, \ldots, \varphi_n$ of $L^{H^1_K(H \otimes K)}$ leaving exactly $L^{H \otimes K}$ fixed. For all i; $\varphi_i(E^{H1}) \subset E^{H1}$. Thus, the elements of E^{H^1} fixed under all φ_i build E^H.

(b) E^{DG} is the integral closure of D in L^{KG}. By [9, 10.2.3] L^{KG} is purely inseparable over $L^{H \otimes K}$. If $x \in E^{DG}$, there exist $d_{n-1}, \ldots, d_0$ D : $x^n + \ldots + d_1 x + d_0 = 0$, hence there is natural number p^e : $x^{p^e} \in L^{H \otimes K}$ and further : $(x^n + \ldots + d_0)^{p^e} = 0$ hence, $(X^{p^e})^n + \ldots + d_0^{p^e} = 0$ so x^{p^e} is in the integral closure of D in $L^{H \otimes K} = E^H$

(c) Because $L^{H^1(H \otimes K)}$ and L^{KG} are lineary independend over $L^{H \otimes K}$ there is an isomorphism $E^{H^1} \otimes E^{DG} \cong E^{H^1} E^{DG}$.

The field of fractions of $E^{H^1} E^{DG}$ equals $L^{H^1(H \otimes K)} L^{KG} = L$.

4. Orders in central simple algebra.

First we prove two theorems which are of some independent interest :

Theorem 4.1 : Let H be a pointed irreducible K-coalgebra with unique group like element 1 and H measures B to B. For all natural numbers n and for all $m \in C_n^+(H)$, we can find a natural k and an injection $\psi : B \hookrightarrow M_k(B)$, with $M_k(B)$ the k x k matrices with coefficients in B, ψ is an upper triangular matrix for every $a \in B$ with constant diagonal element a, $\psi(a)_{1,k} = m \cdot a$ and $\psi(a)_{ij} = p \cdot a$ with $p \in C_l^+(H)$, $l < n$, for $i > j$.

proof. (by induction on n)

n = 1 : Recall from Sweedler [9] that $C_1^+(H) = P(H)$, the primitive elements of H. $m \in P(H)$ implies that m is a derivation on B, therefore we have an algebra morphism :

$$\psi_m : B \hookrightarrow H_2(B) \qquad a \longmapsto \begin{pmatrix} a & m \cdot a \\ 0 & a \end{pmatrix}$$

satisfying the requirements of the theorem.

n > 1 : If m $C_n^+(H)$, we have : $c = c \otimes 1 + 1 \otimes c + \sum_{i=1}^{h} p_i \otimes q_i$; with $p_i, q_i \in C_{n-1}^+(H)$. By the induction hypothesis we can find algebra monomorphisms ψ_{p_i}, ψ_{q_j} satisfying the requirements of the theorem.

$$\psi_{p_i} : B \to M_{k_i}(B)$$
$$\psi_{q_j} : B \to M_{l_j}(B)$$

Now, construct a mapping $\psi_m : B \to M_K(B)$ with $k = \sum_{i=1}^{n} (k_i + l_i) - 2h + 1$ in the following way :

let $k_0 = l_0 = 0$ and $v_\alpha = \sum_{i=0}^{\alpha} k_i + \sum_{j=0}^{\alpha-1} l_j - 2\alpha + 1$,

$w_\alpha = \sum_{i=0}^{\alpha} (k_i + l_i) - 2\alpha$. Now, define :

$\psi_m(a)_{v_\alpha+i,\ v_\alpha+j} = \psi_{q_\alpha}(a)_{i,j}$ $\quad 1 \leq i \leq l_\alpha - 1$

$\quad 1 \leq j \leq l_\alpha$

$\psi_m(a)_{w_\alpha+i,\ w_\alpha+j} = \psi_{p_\alpha+1}(a)_{i,j}$ $\quad 1 \leq i \leq k_{\alpha+1}$

$\quad 2 \leq j \leq k_{\alpha+1}$

$\psi_m(a)_{i,j} = a$, for all i

$\psi_m(a)_{1,\ w_\alpha+i} = \psi_{p_\alpha+1}(a)_{1,i}$ $\quad 2 \leq i \leq k_{\alpha+1}$

$\psi_m(a)_{k,\ v_\alpha+i} = \psi_{q_\alpha}(a)_{l_\alpha,i}$ $\quad 1 \leq i \leq l_{\alpha-1}$

$\psi_m(a)_{1,\ k} = m\,.\,a$

and every other entry will be zero. It is easily verified that ψ_m is again an algebra morphism satisfying the requirements of the theorem.

Definition 4.2 : Let Σ be a central simple K-algebra. $m \in End_K(\Sigma)$ is called inner, if there are elements a_i, $a_i' \in \Sigma$ such that $m(a) = \sum_{i=1}^{n} a_i\, a\, a_i'$ for all a in Σ.

Theorem 4.3 : For all $m \in H_K(\Sigma,\Sigma)$: m is inner.

proof.

By a theorem of Konstant, $H_K(\Sigma,\Sigma) = KG \# H^1$ with G the group like elements of $H_K(\Sigma,\Sigma)$ and H^1 the pointed irreducible component of 1. The group-like elements are precisely the K-automorphisms and they are inner by the Noether-Skolem theorem. Therefore it will be sufficient to prove that every m in H^1 is inner. If $m \in C_n(H^1)$ then $m' = m - \varepsilon(m)\ 1_\Sigma \in C_n^+(H^1)$, thus we can find a natural number k and an algebra morphism $\psi : \Sigma \hookrightarrow M_k(\Sigma)$ with :

$$\psi(a) = \begin{bmatrix} a & & m'(a) \\ & \ddots \; n_{ij}(a) & \\ \underline{0} & & a \end{bmatrix} \quad \text{with } n_{ij} \in C_1^+(H^1),\ 1 < n$$

Now, ψ is an isomorphism between Σ (imbedded diagonaly in $M_k(\Sigma)$) and $\psi(\Sigma)$, two simple subalgebras of the simple Artinian ring $M_k(\Sigma)$. Furthermore, since n_{ij} and m' are in $C^+(H^1)$, ψ leaves K elementswise fixed, so by the Noether-Skolem theorem there exists an invertible $(\lambda_{ij}) \in M_k(\Sigma)$ with : $\psi(a)\ \lambda_{ij} = \lambda_{ij}\ a$ for all $a \in \Sigma$.
For all $a \in \Sigma$: $a\lambda_{ni} = \lambda_{ni}a$, thus $\lambda_{ni} \in K$ for every i.
Since (λ_{ij}) is invertible, we can find an idex j : $\lambda_{nj} \neq 0$.
Computing on both sides the product entry (1,j) gives us :

$$\alpha\lambda_{1j} + \sum_{\alpha=2}^{k-1} n_{1\alpha}(a)\lambda_{\alpha j} + m'(a)\lambda_{nj} = \lambda_{1j}\, a \quad \text{, or}$$

$$m'(a) = \lambda_{nj}^{-1}\ (\lambda_{1j}\, a - a\,\lambda_{1j} - \sum_{\alpha=2}^{k-1} n_{1\alpha}(a)\ \lambda_{\alpha j})$$

Now, apply induction : $C_1^+(H^1)$ consists of derivations, hence they are inner, so we may assume all $n_{1\alpha}$ to be inner and thus m' is inner too. Finally, $m = m' + \varepsilon(m)\ 1_\Sigma$ and therefore m is inner.

<u>Theorem 4.4</u> : Let D be a Dedekind ring such that $\cap\{P : ht(P) = 1\} = 0$, K its field of fractions, Σ a central simple K-algebra and Λ a D-order in K. For all but a finite number $P \in Spec(D)$ we have :
$H_{D_P}(\Lambda_P, \Lambda_P)$ is a D_P-order in $H_K(\Sigma,\Sigma)$.

<u>proof</u>.

Let $G(H_K(\Sigma,\Sigma)) = \{\varphi_1,\ldots,\varphi_n\}$, each φ_i is of the form $\varphi_i(x) = a_i \, x \, a_i^{-1}$ with a_i in Σ. We can find elements d_i, d_i' in D, λ_i in Λ such that : $a_i = (d_i/d_i') \, \lambda_i$.

Now, put $d = \prod_{i=1}^{n} d_i \, d_i' \neq 0$, hence all but a finite number of prime ideals P of D exclude d. Thus, for all but a finite number $P \in \mathrm{Spec}(D)$ we have : $\varphi_i(\Lambda_P) \subset \Lambda_P$ for all i, hence $G \subset H_{D_P}(\Lambda_P,\Lambda_P)$. By theorem 2.10, $H_K^1(\Sigma,\Sigma) \subset H_{D_P}(\Lambda_P,\Lambda_P) \otimes_{D_P} K$ and the foregoing yields $KG \subset H_{D_P}(\Lambda_P,\Lambda_P) \otimes_{D_P} K$. Konstant's theorem now finishes the proof.

<u>Acknowledgement</u>.

I wish to thank Prof. F. Van Oystaeyen for suggesting the problem and for his continuous advice.

References.

[1] M. Artin : "Integral homomorphisms of PI-rings", Proceedings Durham, July 1979.

[2] L. Béqueri : "Schéma d'automorphismes. Application à l'étude d'extensions finies radicielles" Bull.Sc.math., 93, 1969, p. 89 à 111.

[3] N. Bourbaki :"Elém.de Math:Algèbre I" Hermann, Paris 1970.

[4] S. Chase-M.E. Sweedler : "Hopf Algebras and Galois Theory" Springer, LNM 97, 1969.

[5] De Meyer-Ingraham : "Separable Algebras over Commutative Rings", Springer LNM 181, 1971.

[6] Janutz : "Separable Algebras over Commutative Rings", Trans. Amer.Math.Soc. 122 (1966), 461-479.

[7] I. Reiner : "Maximal Orders" Academic Press, London, 1975.

[8] M.E. Sweedler : "The Hopf Algebra of an Algebra Applied to Field Theory" Journal of Alg. 8, 262-276 (1968)

[9] M.E. Sweedler : "Hopf Algebras", Benjamin, New York, 1969.

[10] F. Van Oystaeyen-A. Verschoren : "Introduction to Noncommutative Algebraic Geometry", monograph, to appear.