THE MONSTROUS MOONSHINE PICTURE

LIEVEN LE BRUYN

1. Conway's Big Picture

In [3] John H. Conway introduced a picture that makes it easier to understand groups commensurable with the modular group $\Gamma = PSL_2(\mathbb{Z})$, in particular the discrete groups appearing in Monstrous Moonshine [2]. We follow Conway's notation and terminology whenever possible.

Let $L_1 = \langle \vec{e}_1, \vec{e}_2 \rangle = \mathbb{Z}\vec{e}_1 \oplus \mathbb{Z}\vec{e}_2$ be a fixed 2-dimensional integral lattice. A lattice L is said to be *commensurable* with L_1 if their intersection has finite index in both of them. Two such lattices L and L' are in the same *projective class* if there is a non-zero rational number λ such that $L' = \lambda.L$.

Any lattice L commensurable with L_1 is in the same projective class as a unique lattice in *standard form*

$$L_{M,\frac{g}{h}} = \langle M\vec{e}_1 + \frac{g}{h}\vec{e}_2, \vec{e}_2 \rangle = \mathbb{Z}(M\vec{e}_1 + \frac{g}{h}\vec{e}_2) \oplus \mathbb{Z}\vec{e}_2$$

with M a strictly positive rational number and $\frac{g}{h}$ is a proper fraction in its least terms, that is $0 \leq g < h$ with $(g,h) = 1$. If $M \in \mathbb{N}_+$ we omit the comma and write $L_{M\frac{g}{h}}$ and call the lattice *number-like*. If in addition $g = 0$ we write L_M and call the lattice a *number-lattice*. With $M, \frac{g}{h}$ we denote teh projective class of $L_{M,\frac{g}{h}}$.

Swapping the roles of $\vec{e}_1$ and $\vec{e}_2$ the lattices $L_{M,\frac{g}{h}}$ can be shown to be in the same class as the lattice in *reverse standard form*

$$L_{(\frac{1}{h^2M},\frac{g'}{h})} = \langle \frac{1}{h^2M}\vec{e}_2 + \frac{g'}{h}\vec{e}_1, \vec{e}_1 \rangle = \mathbb{Z}(\frac{1}{h^2M}\vec{e}_2 + \frac{g'}{h}\vec{e}_1) \oplus \mathbb{Z}\vec{e}_1$$

with g' the inverse of g modulo h, and we write $(\frac{1}{h^2M}, \frac{g'}{h})$ for the corresponding projective class. That is we have

$$M, \frac{g}{h} = (\frac{1}{h^2M}, \frac{g'}{h})$$

It will be convenient to associate to a projective class of lattices $M, \frac{g}{h}$ the matrices, corresponding to the two distinct standard forms

$$\alpha_{M,\frac{g}{h}} = \begin{bmatrix} M & \frac{g}{h} \\ 0 & 1 \end{bmatrix} \leftrightarrow M, \frac{g}{h} \leftrightarrow \begin{bmatrix} 1 & 0 \\ \frac{g'}{h} & \frac{1}{h^2M} \end{bmatrix} = \beta_{M,\frac{g}{h}}$$

For two classes $M, \frac{g}{h}$ and $N, \frac{i}{j}$ let a be the smallest positive rational number such that

$$a.\alpha_{M,\frac{g}{h}}.\alpha^{-1}_{N,\frac{i}{j}} \in GL_2(\mathbb{Z})$$

and call the determinant of this matrix (which is in $\mathbb{N}_+$) the *hyperdistance* between the classes. Conway showed in [3, p. 329] that the *log* of the hyperdistance is a proper distance function on the projective classes of all lattices commensurable with L_1.

Definition 1. Conway's Big Picture $\mathbb{B}$ *is the graph with* vertices *the projective classes of lattices commensurable with* L_1*, that is all* $M, \frac{g}{h} \in \mathbb{Q}_+ \times \mathbb{Q}/\mathbb{Z}$*, and with* edges *between pairs of classes at prime hyperdistance from each other.*

Conway showed that each class $M, \frac{g}{h}$ has exactly $p+1$ neighbours at hyperdistance p. These are the classes

$$\frac{M}{p}, \frac{g}{hp} + \frac{k}{p} \bmod 1 \quad \text{for } 0 \leq k < p \text{ and } \quad pM, \frac{pg}{h} \bmod 1$$

It is convenient to consider these p-neighbours as the classes corresponding to the matrices obtained by multiplying $\alpha_{M,\frac{g}{h}}$ *on the left* with the matrices

$$P_k = \begin{bmatrix} \frac{1}{p} & \frac{k}{p} \\ 0 & 1 \end{bmatrix} \text{ for } 0 \leq k < p, \text{ and } P_p = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$$

The classes at hyperdistance a p-power from L_1 form a $p+1$-valent tree, and the big picture $\mathbb{B}$ itself 'factorizes' as a product of these p-adic trees, see [3, p. 332] or section 2 for a precise statement.

A discrete group is said to be *commensurable* with the modular group Γ if their intersection has finite index in both groups. With the aim of understanding groups commensurable with Γ, Conway specifies certain finite subgraphs of $\mathbb{B}$. The idea being that many interesting groups are described as the point- or set-wise stabiliser groups of sets of classes of lattices. It is more convenient to work with subgroups of $GL_2(\mathbb{Q})$, even though we intend their images in $PGL_2(\mathbb{Q})$. This should cause no unnecessary confusion.

1.1. **Snakes.** Elements of $GL_2(\mathbb{Q})$ act on elements of the lattice $\mathbb{Q} \otimes L_1$ by basechange, that is, *right multiplication*. A subgroup $G \subset GL_2(\mathbb{Q})$ is said to *stabilise* a class $M, \frac{g}{h}$ if $\alpha_{M,\frac{g}{h}} G \alpha_{M,\frac{g}{h}}^{-1} \subset SL_2(\mathbb{Z})$. Therefore, the full stabiliser of $M, \frac{g}{h}$ is $\alpha_{M,\frac{g}{h}}^{-1} SL_2(\mathbb{Z}) \alpha_{M,\frac{g}{h}}$. The *joint stabiliser* of the classes of number-lattices 1 and N is the group (note that we use the notation for the image in Γ)

$$\Gamma_0(N) = \{ \begin{bmatrix} a & b \\ cN & d \end{bmatrix} \mid ad - Nbc = 1 \}$$

which Conway denotes as $\Gamma_0(N|1)$ to reveal the symmetry. Likewise, $\Gamma_0(X|Y)$ will denote the joint stabiliser of the classes $X, Y \in \mathbb{B}$.

In fact, by [3, Theorem p.336] all classes stabilised by $\Gamma_0(N|1)$ are number-like and consists exactly of the classes $M\frac{g}{h}$ where h is a divisor of 24 such that $h^2|N$ and $1|M|\frac{N}{h^2}$. The subgraph of $\mathbb{B}$ on this set of lattices Conway calls the $(N|1)$-*snake*.

The $(N|1)$-snake is important in describing the normalizer of $\Gamma_0(N)$ in $PSL_2(\mathbb{Q})$. Note that all groups occurring in monstrous moonshine contain a specific $\Gamma_0(N)$ and are contained in its normalizer.

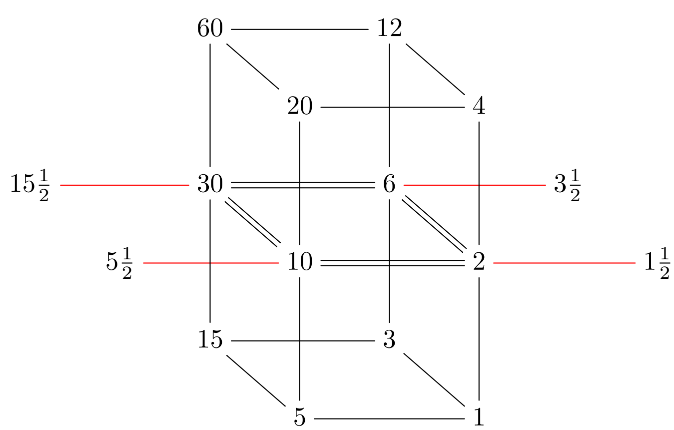
1.2. **Threads.** The number-classes M contained in the $(N|1)$-snake, that is, the case when $h = 1$, are exactly the divisors of N. The subgraph of $\mathbb{B}$ on the divisors of N Conway calls $(N|1)$-*thread*. If $h|n$, then he likewise call the $(n|h)$-*thread* the subgraph on alle number-classes e with $h|e|n$, because the subgroup $\Gamma_0(n|h)$ is the conjugate $\alpha_h^{-1} \Gamma_0(\frac{n}{h}) \alpha_h$ of $\Gamma_0(\frac{n}{h})$.

The $(N|1)$-thread has a symmetry group of order 2^k where k is the number of distinct prime divisors of N. These symmetries are called *Atkin-Lehner involutions* and there is one such involution W_e for every exact divisor e of N, that is $(e, \frac{N}{e}) = 1$.

$\Gamma_0(N|1)+$ is the group generated by $\Gamma_0(N|1)$ and all its Atkin-Lehner involutions W_e. $\Gamma_0(N|1)+$ fixes the $)N|1)$-thread setwise. Similarly, the $(n|h)$-thread is fixed pointwise by $\Gamma_0(n|h)$ and setwise by a group called $\Gamma_0(n|h)+$ which is a conjugate of $\Gamma_0(\frac{n}{h})+$.

1.3. **Spines.** If h is the largest divisor of 24 such that $h^2|N$, then Conway calls the *spine* of the $(N|1)$-snake the subgraph on all classes whose hyperdistance to the periphery is equal to h. For $n = \frac{N}{h}$, the spine of the $(N|1)$-snake is equal to the $(n|h)$-thread. The upshot of this terminology is that the normalizer of $\Gamma_0(N)$ fixes the $(N|1)$-snake setwise and must then also fix the spine setwise, so must be equal to $\Gamma_0(n|h)+$, which is the Atkin-Lehner theorem.

Example 1. *Let us illustrate these concepts in the example of the* $(60|1)$*-snake. As* $60 = 2^2.3.5$ *the only possible values for* h *are* 1 *or* 2*. If* $h = 1$ *we have the classes* M *for all divisors of* 60*, if* $h = 2$ *we have the classes* $M\frac{1}{2}$ *with* M *a divisor of* $\frac{60}{4} = 15$*. Adding the edges gives the graph below. Here the black-edges form the* $(60|1)$*-thread and the double edges form the spine of the* $(60|1)$*-snake, which is the* $(30|2)$*-thread.*



2. Factorization of $\mathbb{B}$ and roots of unity

All threads are finite subgraphs of the *Big Cell* $\mathbb{D}$ which is the restriction of $\mathbb{B}$ to the classes of all number-lattices, that is, the Hasse diagram of $\mathbb{N}_+$ partially ordered under divisibility. Clearly, the vertices of $\mathbb{D}$ correspond to elements of the monoid generated by the commuting operations P_p for all prime numbers p and $\mathbb{D}$ can therefore be viewed as the rooted product $\mathbb{D} = *_p\ A_\infty^{(p)}$ of line-graphs of type A_∞

$$A_\infty^{(p)} = \ 1 \text{———} p \text{———} p^2 \text{———} \dots$$

We have seen that the subgraph of $\mathbb{B}$ of all classes at hyperdistance a p-power from 12 is a free $p+1$-valent tree T_p. Here, the class X at hyperdistance p^k from 1 corresponds to a unique product $P_{i_b}.\dots.P_{i_2}.P_{i_1}P_p^a$ of the matrices P_i with $0 \leq ip$ and P_p introduced before with $a+b=k$. Here, p^a is the maximal number-class on the unique path from X to 1. That is, the matrices P_i with $0 \leq i < p$ generated a free monoid and for all $0 \leq i < p$ we have $P_p.P_i = id$. In factoring $\mathbb{B} = *_p\ T_p$ we have to take into account that the matrices P_i and Q_j corresponding to different prime nmbers p and q do not necessarily commute. Still, they satisfy a *meta-commutation relation* similar to that of prime in the Hurwitz quaternions, see [5, §5.2].

Lemma 1. *For distinct primes p and q and for all $0 \leq i < p$ and $0 \leq j < q$ there exist unique $0 \leq k < p$ and $0 \leq l < q$ such that*

$$P_i.Q_j = Q_l.P_k \quad \textit{and} \quad P_p.Q_j = Q_a.P_p \ \textit{with } a = pi \ \textit{mod } q$$

Proof.

$$P_i.Q_j = \begin{bmatrix} \frac{1}{pq} & \frac{iq+j}{pq} \\ 0 & 1 \end{bmatrix}$$

and as $0 \leq iq + j < pq$ we have unique $0 \leq k < p$ and $0 \leq l < q$ such that $iq + j = lp + k$. □

Lemma 2. *Any class $M, \frac{g}{h} \in \mathbb{B}$ at hyperdistance $N = p^k.q^l.\dots.r^s$ from 1 can be identified uniquely with a product*

$$P_{i_1}P_{i_2}\dots P_{i_b}Q_{j_1}Q_{j_2}\dots Q_{j_d}\dots R_{z_1}R_{z_2}\dots R_{z_v}P_p^aQ_q^c\dots R_r^u$$

for unique $0 \leq i_a < p$, $0 \leq j_b < q, \dots, 0 \leq z_u < r$ and with $a + b = k, c + d = l, \dots, u + v = s$. Here $K = P_p^aQ_q^b\dots R_r^u$ is the unique number-class in $\mathbb{D}$ of minimal hyperdistance from $M, \frac{g}{h}$. For a number-like class $M\frac{g}{h}$ we have $K = Mh$ and $N = Mh^2$.

Proof. A path from 1 to $M, \frac{g}{h}$ of minimal length can be viewed as a product (left multiplication) $X_lX_{l-1}\dots X_2X_1$ with each X_i one of the matrices $P_a, Q_b, \dots, R_u$. The claim follows from using the meta-commutation relations of the previous lemma. □

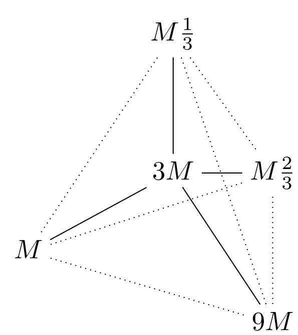
It will be convenient to view a number-like class $M\frac{g}{h}$ as a *primitive h-th root of unity centered at Mh.* Note that all classes $M\frac{e}{d}$ with $d|h$ are at hyperdistance h from Mh and can be thought of as a copy of $\boldsymbol{\mu}_h$, the cyclic group of h-th roots of unity.

If we switch to the second standard form, there's another copy of $\boldsymbol{\mu}_h$ of classes centered at $Mh = (\frac{1}{Mh}, 0)$ consisting of the classes $(\frac{1}{h^2M}, \frac{e}{d})$ with $d|h$. The primitive h-th roots of unity $M\frac{g}{h}$ are the intersection of these two copies.

Example 2. *For $h = 3$ the neighbours in the 3-tree of $3M$ are the 4-classes (with represenations in the two standard forms)*

$$M \leftrightarrow (\frac{1}{M}, 0), \quad M\frac{1}{3} \leftrightarrow (\frac{1}{9M}, \frac{1}{3}), \quad M\frac{2}{3} \leftrightarrow (\frac{1}{9M}, \frac{2}{3}), \quad 9M \leftrightarrow (\frac{1}{9M}, 0)$$

It is convenient to view these classes as the vertices of a tetrahedron with center of gravity $3M \leftrightarrow (\frac{1}{3M}, 0)$.



The two sets of 3*-rd roots of unity, centered at* $3M$ *consist of the classes*

$$\{M, M\frac{1}{3}, M\frac{2}{3}\} \quad \textit{and} \quad \{9M, M\frac{1}{3}, M\frac{2}{3}\}$$

Power maps in the first copy of $\boldsymbol{\mu}_3$ *correspond to rotations with pole vertex* $9M$ *and power maps in the second copy are rotations with pole vertex* M*.*

3. The moonshine picture

Monstrous moonshine assigns to each conjugacy class of the monster simple groep $\mathbf{M}$ a group commensurable with Γ, see [2]. There are exactly 171 such groups which are all of the form

$$(n|h) + e, f, \ldots$$

where n is the order of the monster element, h is a divisor of 24 and a divisor of n and where $e, f, \ldots$ are exact divisors of $\frac{n}{h}$.

We have already described the group $\Gamma_0(n|h)+$ in section 1 which is a conjugate of $\Gamma_0(\frac{n}{h})+$ which in turn is obtained by adjoining to $\Gamma_0(\frac{n}{h})$ all Atkin-Lehner involutions W_e for all exact divisors e of $\frac{n}{h}$. The subgroup generated by $\Gamma_0(\frac{n}{h})$ and a selection $W_e, W_f, \ldots$ of these involutions is denoted by $\Gamma_0(\frac{n}{h}) + e, f, \ldots$ and its conjugate under α_h is denoted by $\Gamma_0(n|h) + e, f, \ldots$.

The *moonshine group* $(n|h) + e, f, \ldots$ is a specific subgroup of $\Gamma_0(n|h) + e, f, \ldots$ of index h. Let $N = n.h$ then we have seen that $\Gamma_0(n|h)+$ is the normalizer of $\Gamma_0(N)$ and so $\Gamma_0(N)$ is a normal subgroup of $\Gamma_0(n|h) + e, f, \ldots$. In [4] it is shown that for all moonshine groups there exists a one-dimensional representation

$$\lambda \ : \ \Gamma_0(n|h) \longrightarrow \mathbb{C}^*$$

such that

1. (1) $\lambda = 1$ for elements of $\Gamma_0(N)$,
2. (2) $\lambda = 1$ for all Atkin-Lehner involutions W_e of $\Gamma_0(N)$ for which every prime divisor of e also divides $\frac{n}{h}$,
3. (3) $\lambda = e^{-2\pi i/h}$ for $\Gamma_0(N)$-cosets containing $x = \begin{bmatrix} 1 & \frac{1}{h} \\ 0 & 1 \end{bmatrix}$,
4. (4) $\lambda = e^{\pm 2\pi i/h}$ for $\Gamma_0(N)$-cosets containing $y = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$ with $+$ sign if $\begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}$ is present, and $-$ if not.

The group $(n|h) + e, f, \ldots$ is the kernel of λ.

As λ is trivial on $\Gamma_0(N)$ and on the relevant Atkin-Lehner involutions it suffices in order to describe $(n|h) + e, f, \ldots$ to know the structure of the finite group

$$\Gamma_0(n|h)^* = \Gamma_0(n|h)/\Gamma_0(N)$$

This group is generated by the action of x and y on the classes in the $(N|1)$-snake. The relevance of the introduction of the two sets of roots of unity centered at a number-class is that x will act as a power-map on the first set, whereas y acts as a power-map on the second set.

Example 3. *Let us describe the moonshine group* $(3|3)$ *corresponding to conjugacy class 3C, see also* [6, example 2.9.1]*. The* $(9|1)$*-snake is given in example 2 with*

$M = 1$. the action of $x = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{bmatrix}$ on a class $M\frac{g}{h}$ is given by right-multiplication of $\alpha_{M\frac{g}{h}}$. As

$$\alpha_1.x = \alpha_{1\frac{1}{3}}, \quad \alpha_{1\frac{1}{3}}.x = \alpha_{1\frac{2}{3}}, \quad \alpha_{1\frac{2}{3}}.x = \alpha_1, \quad \text{and} \quad \alpha_9.x = \alpha_9$$

That is, x can be viewed as rotation with pole the class 9. To study the action of $y = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ it is best to use the second standard form and then the action of y is given b right-multiplication of $\beta_{M\frac{g}{h}} = \begin{bmatrix} 1 & 0 \\ \frac{g'}{h} & \frac{1}{h^2 M} \end{bmatrix}$.

$$\beta_9.y = \beta_{1\frac{1}{3}}, \quad \beta_{1\frac{1}{3}}.y = \beta_{1\frac{2}{3}}, \quad \beta_{1\frac{2}{3}}.y = \beta_9, \quad \text{and} \quad \beta_1.y = \beta_1$$

That is, y is a rotation with pole the class 1. Clearly, both rotations generate the full rotation symmetry group of the tetrahedron A_4. That is $\Gamma_0(3|3)^* = \Gamma_0(3|3)/\Gamma_0(9) \simeq A_4$ and as this group has a unique subgroup of index 3 generated by $x.y$ and $y.x$ it follows that the moonshine group $(3|3)$ is generated by

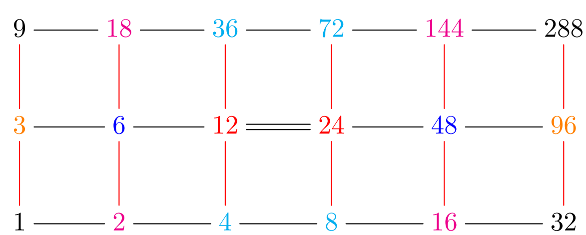
$$(3|3) = \langle \Gamma_0(9), \begin{bmatrix} 2 & \frac{1}{3} \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \frac{1}{3} \\ 3 & 2 \end{bmatrix} \rangle$$

Definition 2. *The* Monstrous Moonshine Picture $\mathbb{M}$ *is the subgraph of the Big Picture $\mathbb{B}$ on the classes contained in all snakes needed to describe the 171 moonshine groups. That is, the union of all $(N|1)$-snakes with $N = n.h$ when $(n|h) + e, f, \ldots$ is the group associated to a conjugacy class of order n in the monster group $\mathbf{M}$.*

Before giving our results on the structure of the moonshine picture we will outline our strategy by dissecting the largest snake in $\mathbb{M}$.

3.1. **The big anaconda.** The largest snake contained in $\mathbb{M}$ determines the moonshine group $(24|12)$ which is associated to conjugacy class 24J of the monster $\mathbf{M}$. It contains 70 lattices, about one third of the total number of lattices in $\mathbb{M}$.

The corresponding $(288|1)$-snake we will call the *anaconda*. It's backbone is the $(288|1)$ thread below (edges in the 2-tree are black, those in the 3-tree red and coloured numbers are symmetric with respect to the $(24|12)$-spine and have the same local structure in the $(288|1)$-snake.



Apart from these number classes, which are all divisors of 288, we have to determine the number-like classes of the $(288|1)$-snake. These are all classes $M\frac{g}{h}$ with h a divisor of 24 such that h^2 divides 288 and such that $(g, h) = 1$. As $288 = 2^5.3^2$ we have $h = 1, 2, 4, 6$ or 12.

$h = 1$ gives $M = 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 32, 36, 48, 72, 96, 144, 288$.

$h = 2$ gives the classes $M\frac{1}{2}$ for M a divisor of 72 :$1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72$.

$h = 3$ gives the classes $M\frac{1}{3}$ and $M\frac{2}{3}$ for M a divisor of 32 : $1, 2, 4, 8, 16, 32$.

$h = 4$ gives the classes $M\frac{1}{4}$ and $M\frac{3}{4}$ for M a divisor of 18 : $1, 2, 3, 6, 9, 18$.

$h = 6$ gives the classes $M\frac{1}{6}$ and $M\frac{5}{6}$ for M a divisor of $8 : 1, 2, 4, 8$.

$h = 12$ gives the classes $M\frac{1}{12}, M\frac{5}{12}, M\frac{7}{12}$ and $M\frac{11}{12}$ for $M = 1, 2$.

This gives a total of 70 classes $M\frac{g}{h}$ for M a divisor of 288 and $\frac{g}{h}$ a primitive h-th root of unity, summarized in the following table

M	h	M	h
1	$1, 2, 3, 4, 6, 12$	18	$1, 2, 4$
2	$1, 2, 3, 4, 6, 12$	24	$1, 2$
3	$1, 2, 4$	32	$1, 3$
4	$1, 2, 3, 6$	36	$1, 2$
6	$1, 2, 4$	48	1
8	$1, 2, 3, 6$	72	$1, 2$
9	$1, 2, 4$	96	1
12	$1, 2$	144	1
16	$1, 3$	288	1

Next, we will focus on the center $C = M.h$ of these primitive roots of unity, which will give us the number-classes having the same local structure in the $(288|1)$-snake.

C	h	C	h
1	1	18	$1, 2$
2	$1, 2$	24	$1, 2, 3, 4, 6, 12$
3	$1, 3$	32	1
4	$1, 2, 4$	36	$1, 2, 4$
6	$1, 2, 3, 6$	48	$1, 2, 3, 6$
8	$1, 2, 4$	72	$1, 2, 4$
9	1	96	$1, 3$
12	$1, 2, 3, 4, 6, 12$	144	$1, 2$
16	$1, 2$	288	1

Next, we will determine the local structures for each of these 6 different types.

For Type Black : $M = 1, 9, 32, 288$ there are no strict number-like neighbours in the $(288|1)$-snake.

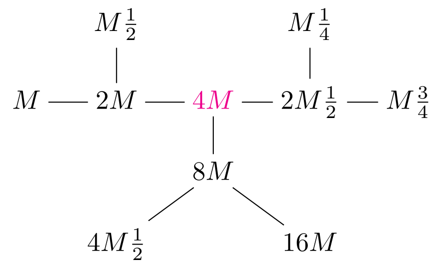
For Type Cyan : $2M = 2, 16, 18, 144$ the local structure involves all 2-nd roots of unity

$$M - 2M - M\frac{1}{2}$$

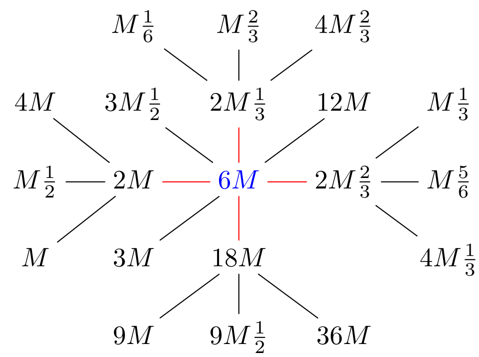
For Type Orange : $3M = 3, 96$ the local structure involves all 3-rd roots of unity

$$\begin{array}{ccccc} & & M\frac{1}{3} & & \\ & & | & & \\ M & - & 3M & - & M\frac{2}{3} \\ & & | & & \\ & & 9M & & \end{array}$$

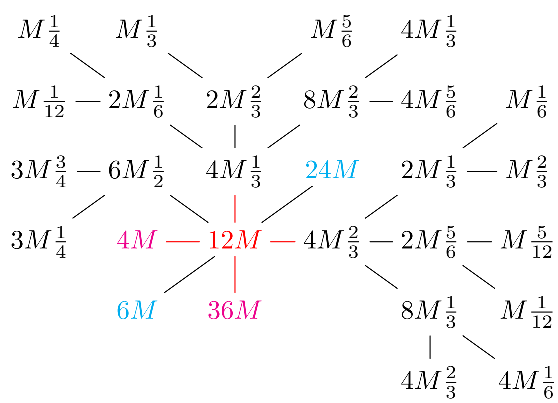
For Type Magenta : $4M = 4, 8, 36, 72$ the local structure involves all 2-nd and 4-th roots of unity



For Type Blue : $6M = 6, 48$ the local structure involves all 2-nd, 3-rd and 6-th roots of unity



For Type Red : $12M = 12, 24$ the local structure involves all 2-nd, 3-rd, 4-th, 6-th amd 12-th roots of unity



Here we simplified the picture a bit by indicating the local type of the intersection of the the number-classes $4M, 6M, 24M$ and $36M$ with the neighbourhood of $12M$..

3.2. **Local structure of $\mathbb{M}$.** Starting from the description of the moonshine groups $(n|h) + e, f, \ldots$ given in [2, p. 327-329] we determine for each of them all classes contained in the $(N|1)$-snake where $N = n.h$.

Theorem 1. *The Monstrous Moonshine Picture* $\mathbb{M}$ *is the subgraph of* $\mathbb{B}$ *on exactly* 218 *classes. Of them, exactly* 97 *are number-classes*

$$N = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26,$$

$27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42, 44, 45, 46, 47, 48, 50, 51, 52, 54, 55,$
$56, 57, 59, 60, 62, 63, 64, 66, 68, 69, 70, 71, 72, 78, 80, 84, 87, 88, 90, 92, 93, 94, 95, 96, 104,$
$105, 110, 112, 117, 119, 120, 126, 136, 144, 160, 168, 171, 176, 180, 208, 224, 252, 279, 288$
$360, 416$

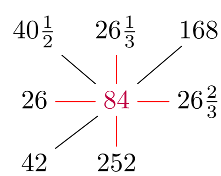
The remaining 121 *classes are all number-like of the form* $M\frac{g}{h}$ *where* $1 \leq g < h$ *and* $(g, h) = 1$ *for the following pairs* (M, h) *with* $h > 1$

M	h	M	h	M	h	M	h
1	1, 2, 3, 4, 6, 8, 12	25	1	52	1, 2	94	1
2	1, 2, 3, 4, 6, 12	26	1, 2, 4	54	1	95	1
3	1, 2, 4	27	1	55	1	96	1
4	1, 2, 3, 4, 6	28	1, 2, 3	56	1, 2	104	1, 2
5	1, 2, 3, 4, 6	29	1	57	1	105	1
6	1, 2, 4	30	1, 2	59	1	110	1
7	1, 2, 3, 4, 6	31	1, 3	60	1	112	1
8	1, 2, 3, 6	32	1, 3	62	1	117	1
9	1, 2, 4	33	1	63	1, 2	119	1
10	1, 2, 3, 6	34	1, 2	64	1	120	1
11	1, 2, 4	35	1	66	1	126	1
12	1, 2	36	1, 2	68	1	136	1
13	1, 2, 3, 4	38	1	69	1	144	1
14	1, 2, 3, 4	39	1	70	1	160	1
15	1, 2	40	1, 2, 3	71	1	168	1
16	1, 2, 3	41	1	72	1, 2	171	1
17	1, 2	42	1, 2	78	1	176	1
18	1, 2, 4	44	1, 2	80	1	180	1
19	1, 3	45	1, 2	84	1	208	1
20	1, 2, 3	46	1	87	1	224	1
21	1, 2	47	1	88	1	252	1
22	1, 2	48	1	90	1, 2	279	1
23	1	50	1	92	1	288	1
24	1, 2	51	1	93	1	360	1
						416	1

To determine the local structure of $\mathbb{M}$ we consider the class $M\frac{g}{h}$ as a primitive h-th root of unity, centered at the number-class $C = M.h$. For each C we then collect which roots of unity centered at C belong to $\mathbb{M}$. This data is contained in Figure 3.2 below.

We note that, apart from the six local types already encountered in the dissection of the green anaconda, there are two further types of local behaviour each concentrated in a single number-class:

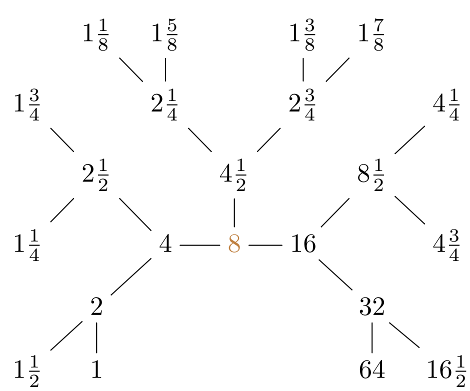
Type Purple : $M = 84$ the local structure involves all 2-nd and 3-rd roots of unity



C	h	C	h	C	h	C	h
1	1	25	1	52	1, 2, 4	94	1
2	1, 2	26	1, 2	54	1	95	1
3	1, 3	27	1	55	1	96	1, 3
4	1, 2, 4	28	1, 2, 4	56	1, 2, 4	104	1, 2, 4
5	1	29	1	57	1, 3	105	1
6	1, 2, 3, 6	30	1, 2, 3, 6	59	1	110	1
7	1	31	1	60	1, 2, 3, 6	112	1, 2
8	1, 2, 4, 8	32	1, 2	62	1	117	1
9	1	33	1	63	1	119	1
10	1, 2	34	1, 2	64	1	120	1, 3
11	1	35	1	66	1	126	1, 2
12	1, 2, 3, 4, 6, 12	36	1, 2, 4	68	1, 2	136	1
13	1	38	1	69	1	144	1, 2
14	1, 2	39	1, 3	70	1	160	1
15	1, 3	40	1, 2	71	1	168	1
16	1, 2, 4	41	1	72	1, 2, 4	171	1
17	1	42	1, 2, 3, 6	78	1	176	1
18	1, 2	44	1, 2, 4	80	1, 2	180	1, 2
19	1	45	1	84	1, 2, 3	208	1, 2
20	1, 2, 4	46	1	87	1	224	1
21	1, 3	47	1	88	1, 2	252	1
22	1, 2	48	1, 2, 3, 6	90	1, 2	279	1
23	1	50	1	92	1	288	1
24	1, 2, 3, 4, 6, 12	51	1	93	1, 3	360	1
						416	1

FIGURE 1. The local types of $\mathbb{M}$ at number classes

Type Brown : $M = 8$ the local structure involves all 2-nd, 4-th and 8-th roots of unity.

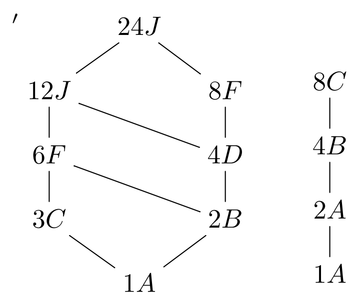


We can now collect all local types of the Moonshine Picture $\mathbb{M}$ in all of the 97 number-classes.

Type	Number classes
1	$1, 5, 7, 9, 11, 13, 17, 19, 23, 25, 27, 29, 31, 33, 35, 38, 41, 45, 46, 47, 50, 51, 54, 55, 59, 62, 63, 64, 66, 69, 70, 71, 78, 87, 92, 94, 95, 105, 110, 117, 119, 136, 160, 168, 171, 176, 224, 252, 279, 288, 360, 416$
$1, 2$	$2, 10, 14, 18, 22, 26, 32, 34, 40, 68, 80, 88, 90, 112, 126, 144, 180, 208$
$1, 2, 4$	$4, 16, 20, 28, 36, 44, 52, 56, 72, 104$
$1, 3$	$3, 15, 21, 39, 57, 93, 96, 120$
$1, 2, 3, 6$	$6, 30, 42, 48, 60$
$1, 2, 3, 4, 6, 12$	$12, 24$
$1, 2, 4, 8$	8
$1, 2, 3$	84

4. The monster determines the moonshine threads

The 'anaconda' conjugacy class $24J$ appears to play a special role in moonshine, as does the conjugacy class $8C$. Powers of these elements belong to the conjugacy classes



Theorem 2. *Let X be a conjugacy class of the monster $\mathbf{M}$ of order n and let Y be a conjugacy class of powers of $24J$ or $8C$ of maximal order k such that X belongs to the power-up classes of Y, see [1]. Then, for all but 12 counter-examples the thread of the moonshine group corresponding to X is*

$$(n|\frac{k}{2}) \text{ if } k \text{ is even,} \quad (n|3) \text{ if } Y = 3C, \text{ and } (n|1) \text{ if } Y = 1A.$$

Proof. This follows from comparing the list of all 171 moonshine groups, given in [2, p. 327-329] with the values obtained by this procedure, given in figure 4 below. Here, we write $!t$ for the correct value when it differs from that of the statement of the theorem. □

We observe that all counter-examples are conjugacy classes of order a multiple of 8 and that the intended value differs from the computed by a factor 2 in every case. The faulty conjugacy classes are:

$$8B, 8D, 16A, 24A, 24D, 24E, 24H, 32B, 40B, 40C, 40D, 48A, 88A, 88B$$

This set consists exactly of the power-up classes, see [1], of

$8B$	$24A, 24E, 40B, 88A, 88B$
$8D$	$24D, 24H, 40C$
$16A$	$48A$
$16C$	$32B$

$1A$	1	$10B$	1	$18B$	1	$26A$	1	$39A$	1	$60C$	1
$2A$	1	$10C$	1	$18C$	1	$26B$	1	$39B$	3	$60D$	1
$2B$	1	$10D$	1	$18D$	1	$27A$	1	$39DC$	1	$60E$	1
$3A$	1	$10E$	1	$18E$	1	$27B$	1	$40A$	4	$60F$	6
$3B$	1	$11A$	1	$19A$	1	$28A$	2	$40B$	1(!2)	$62AB$	1
$3C$	3	$12A$	1	$20A$	1	$28B$	1	$40CD$	1(!2)	$66A$	1
$4A$	1	$12B$	1	$20B$	2	$28C$	1	$41A$	1	$66B$	1
$4B$	2	$12C$	2	$20C$	1	$28D$	2	$42A$	1	$68A$	2
$4C$	1	$12D$	3	$20D$	2	$29A$	1	$42B$	1	$69AB$	1
$4D$	2	$12E$	1	$20E$	2	$30A$	1	$42C$	3	$70A$	1
$5A$	1	$12F$	2	$20F$	1	$30B$	1	$42D$	1	$70B$	1
$5B$	1	$12G$	2	$21A$	1	$30C$	1	$44AB$	1	$71AB$	1
$6A$	1	$12H$	1	$21B$	1	$30D$	1	$45A$	1	$78A$	1
$6B$	1	$12I$	1	$21C$	3	$30E$	3	$46AB$	1	$78BC$	1
$6C$	1	$12J$	6	$21D$	1	$30F$	1	$46CD$	1	$84A$	2
$6D$	1	$13A$	1	$22A$	1	$30G$	1	$47AB$	1	$84B$	2
$6E$	1	$13B$	1	$22B$	1	$31AB$	1	$48A$	1(!2)	$84C$	3
$6F$	3	$14A$	1	$23AB$	1	$32A$	1	$50A$	1	$87AB$	1
$7A$	1	$14B$	1	$24A$	1(!2)	$32B$	1(!2)	$51A$	1	$88AB$	1(!2)
$7B$	1	$14C$	1	$24B$	1	$33A$	1	$52A$	2	$92AB$	1
$8A$	1	$15A$	1	$24C$	1	$33B$	1	$52B$	2	$93AB$	3
$8B$	1(!2)	$15B$	1	$24D$	1(!2)	$34A$	1	$54A$	1	$94AB$	1
$8C$	4	$15C$	1	$24E$	3(!6)	$35A$	1	$55A$	1	$95AB$	1
$8D$	1(!2)	$15D$	3	$24F$	4	$35B$	1	$56A$	1	$104AB$	4
$8E$	1	$16A$	1(!2)	$24G$	4	$36A$	1	$56B$	4	$105A$	1
$8F$	4	$16B$	1	$24H$	1(!2)	$36B$	1	$57A$	3	$110A$	1
$9A$	1	$16C$	1	$24I$	1	$36C$	2	$59AB$	1	$119AB$	1
$9B$	1	$17A$	1	$24J$	12	$36D$	1	$60A$	2		
$10A$	1	$18A$	1	$25A$	1	$38A$	1	$60B$	1		

FIGURE 2. Threads of Moonshine groups

Therefore, we obtain

Theorem 3. *The thread of the moonshine group corresponding to conjugacy class X of order n is $(n|t)$ where t is the value obtained from theorem 2, unless X is in the power-up classes of $8B, 8D, 16A$ or $16C$ in which case we have the thread $(n|2t)$ except when $X = 16C$.*

As a consequence, the monster group $\mathbf{M}$ determines the Monstrous Moonshine picture $\mathbb{M}$. In fact, for $\mathbb{M}$ we only need the correct values for $32B$ and $88AB$, so we can suffice with proper power-ups of $8B$ and $16C$.

REFERENCES

[1] Atlas, *the Monster*, `http://brauer.maths.qmul.ac.uk/Atlas/v3/spor/M/`

[2] John H. Conway, Simon P. Norton, *Monstrous moonshine*, Bull. London Math. Soc. 11 (1979) 308-339

[3] John H. Conway, *Understanding groups like $\Gamma_0(N)$*,. In: *Groups, Difference Sets and the Monster, Edited by K.T. Arasu & co*, Walter de Gruyter, Berlin New-York 1996, 327-343.

[4] John H. Conway, John McKay, and Abdellah Sebbar, *On the discrete groups of moonshine*, Proc. AMS 132 (2004) 2233-2240

[5] John H. Conway and Derek A. Smith, *On quaternions and octonions, their geometry, arithmetic and symmetry*, A.K. Peters, Natick, Massachusetts (2003)

[6] John F. Duncan, *Arithmetic groups and the affine E_8 Dynkin diagram*, Groups and symmetries, 135-163, CRM Proc. Lecture Notes 47, AMS, Providence R.I. (2009)

Department of Mathematics, University of Antwerp, Middelheimlaan 1, B-2020 Antwerp (Belgium), lieven.lebruyn@uantwerpen.be